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Characterizations of Multivariate Normality. I. Through Independence of some Statistics

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It is established that a vector variable $(X_1, ..., X_k)$ has a multivariate normal distribution if for each X_i the regression on the rest is linear and the conditional distribution about the regression does not depend on the rest of the variables, provided the regression coefficients satisfy some mild conditions. The result is extended to the case where X_i themselves are vector variables.

1. INTRODUCTION

In a previous paper (Khatri and Rao [4]), the authors considered some characterizations of the multivariate normal distribution based on properties of linear functions of independent vector variables. In the present paper, we study similar problems considering linear functions of dependent vector variables. The problem arose out of recent work by one of the authors (Rao [6, 7]) on characterizations of prior distributions. More specifically, Rao considered the problem of characterizing the distribution of vector variables $\mathbf{X}_1$, $\mathbf{X}_2$ such that $\mathbf{X}_1 + \mathbf{A}_{12}\mathbf{X}_2$ and $\mathbf{X}_2$ are i.d. (independently distributed) and $\mathbf{X}_2 + \mathbf{A}_{21}\mathbf{X}_1$ and $\mathbf{X}_1$ are i.d.

We consider problems of the above type but involving more than two vector variables. We note that some of these problems have been partially investigated by Fisk [1], but unfortunately the statements of his theorems seem to be incorrect due to an error in the solution of the functional equation that arises in the problem. The correct statements are given in Theorems 1–5 of Section 3 of our paper.

One important result established is that a vector variable $(X_1, ..., X_k)$ has an m.n.d. (multivariate normal distribution) if the regression of X_i on the rest of

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the variables is linear and the conditional distribution of X_i about the regression does not depend on the rest of the variables, provided the regression coefficients satisfy some mild conditions. Other results relate to cases where X_i themselves are vector variables.

2. Preliminary Lemmas

We consider some lemmas that are used in the proofs of the Main Theorems of Section 3 of the paper.

Lemma 1. *Let X and Y be a pair of r.v.'s (random variables) such that the marginal distribution of Y is degenerate. Then, X and Y are i.d. (independently distributed).*

Lemma 2. *Let X and Y be independent r.v.'s such that $X + Y$ and Y are i.d. or $E(X + Y \mid Y) = \text{constant}$ or $E(Y \mid X + Y) = \text{constant}$. Then, Y is degenerate.*

Lemma 3. *Let X and (Y, Z) be i.d. If $X + Y$ and Z are i.d., and the c.f. of X does not vanish anywhere, then Y and Z are i.d.*

(Note that the notation (Y, Z) is used when we want to consider the r.v.'s Y and Z jointly).

In Lemmas 1, 2, and 3, the r.v.'s can take values in any vector space. The results are easy to prove and the proofs are omitted.

Lemma 4. *Let $\mathbf{A} = (a_{ij})$ be an $n \times n$ matrix such that the first $(n - 1)$ principal subdeterminants are not zero. Then, $\mathbf{A}$ admits the decomposition $\mathbf{A} = \mathbf{TS}$, where $\mathbf{T} = (t_{ij})$ is lower triangular with $t_{ii} \neq 0$ except possibly for $i = n (t_{nn} = 0$ iff $| \mathbf{A} | = 0)$, and $\mathbf{S} = (s_{ij})$ is upper triangular with $s_{ii} = 1$ for all i. Further, $(a_{1r}, a_{2r}, \ldots, a_{r-1,r})$ is a nonnull vector iff $(s_{1r}, \ldots, s_{r-1,r})$ is a nonnull vector, and $(a_{1r}, \ldots, a_{r-1,r}, a_{r+1,r}, \ldots, a_{nr})$ is a nonnull vector iff $(s_{1r}, \ldots, s_{r-1,r}, t_{r+1,r}, \ldots, t_{nr})$ is a nonnull vector for $r = 1, \ldots, n$.*

Lemma 5. *Let $\mathbf{A} = (\mathbf{A}_{ij})$ be a matrix with k partitions of rows and columns such that the determinant*

$$\begin{vmatrix} \mathbf{A}_{11} & \cdots & \mathbf{A}_{1r} \\ \cdots & \cdots & \cdots \\ \mathbf{A}_{r1} & \cdots & \mathbf{A}_{rr} \end{vmatrix} \neq 0, \qquad r = 1, 2, \ldots, k - 1. \tag{2.1}$$

Then, $\mathbf{A}$ admits the decomposition $\mathbf{A} = \mathbf{TS}$, where $\mathbf{T} = (\mathbf{T}_{ij})$ is lower triangular *in partitions with* $\mathbf{T}_{ii}$ *as nonsingular except possibly for* $i = k$($\mathbf{T}_{kk}$ *is singular if* $\mathbf{A}$ *is singular*), *and* $\mathbf{S} = (\mathbf{S}_{ij})$ *is upper triangular with* $\mathbf{S}_{ii} = \mathbf{I}$ *for all* i (*of appropriate orders*). *Further,*

$$\begin{pmatrix} \mathbf{A}_{1r} \\ \mathbf{A}_{2r} \\ \vdots \\ \mathbf{A}_{jr} \end{pmatrix} = \begin{pmatrix} \mathbf{T}_{11} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{T}_{21} & \mathbf{T}_{22} & \cdots & \mathbf{0} \\ \cdot & \cdot & \cdots & \cdot \\ \mathbf{T}_{j1} & \mathbf{T}_{j2} & \cdots & \mathbf{T}_{jj} \end{pmatrix} \begin{pmatrix} \mathbf{S}_{1r} \\ \mathbf{S}_{2r} \\ \vdots \\ \mathbf{S}_{jr} \end{pmatrix}, \tag{2.2}$$

for $j \leqslant r - 1$, *and*

$$\begin{pmatrix} \mathbf{A}_{1r} \\ \vdots \\ \mathbf{A}_{r-1,r} \\ \mathbf{A}_{r+1,r} \\ \vdots \\ \mathbf{A}_{jr} \end{pmatrix} = \left(\begin{array}{ccc|ccc} \mathbf{T}_{11} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \cdot & \cdots & \cdot & & \cdots & \\ \mathbf{T}_{r-1,1} & \cdots & \mathbf{T}_{r-1,r-1} & \mathbf{0} & \cdots & \mathbf{0} \\ \hline \mathbf{T}_{r+1,1} & \cdots & \mathbf{T}_{r+1,r-1} & \mathbf{I} & \cdots & \mathbf{0} \\ \cdot & \cdots & & & \cdots & \\ \mathbf{T}_{j1} & \cdots & \mathbf{T}_{j,r-1} & \mathbf{0} & \cdots & \mathbf{I} \end{array}\right) \begin{pmatrix} \mathbf{S}_{1r} \\ \vdots \\ \mathbf{S}_{r-1,r} \\ \mathbf{T}_{r+1,r} \\ \vdots \\ \mathbf{T}_{jr} \end{pmatrix}, \tag{2.3}$$

for $j \geqslant r + 1$.

The proofs of Lemmas 4 and 5 follow on standard lines (see Khatri [3]).

We quote an important lemma from Khatri and Rao [4] (also given by Kagan, Linnik, and Rao [2, p. 472, Theorem A. 4.2.]), which plays an important role in the proofs of Main Theorems of the present paper.

LEMMA 6. *Let* ϕ_i *be a complex valued and continuous function defined on a real Euclidean space of dimension* p_i, *and let* $\mathbf{C}_i$ *be an* $m \times p_i$ *matrix,* $i = 1,..., s$ *such that*

$$\phi_1(\mathbf{C}_1'\mathbf{t}) + \cdots + \phi_s(\mathbf{C}_s'\mathbf{t}) = P_k(\mathbf{t}),$$
$$R(\mathbf{C}_i : \mathbf{C}_j) = R(\mathbf{C}_i) + R(\mathbf{C}_j), \qquad i = 1, 2,..., r \qquad \text{and} \qquad j = r + 1,..., s,$$

where P_k *is a polynomial of degree* k *and* $R(\mathbf{A})$ *denotes rank of* $\mathbf{A}$. *Then*

$$\phi_1(\mathbf{C}_1'\mathbf{t}) + \cdots + \phi_r(\mathbf{C}_r'\mathbf{t}) \qquad \text{and} \qquad \phi_{r+1}(\mathbf{C}_{r+1}'\mathbf{t}) + \cdots + \phi_s(\mathbf{C}_s'\mathbf{t}),$$

are polynomials in $\mathbf{t}$ *of degree utmost* $\max(k, s - 2)$.

3. THE MAIN THEOREMS

THEOREM 1. *Let* X_1, X_2, U_1, U_2 *be r.v.'s defined on* R^1 *such that with respect to suitable constants* a_{12} *and* a_{21} *the following conditions* (a) *and* (b) *hold:*

(a) $Z_1 = X_1 + a_{12}X_2 + U_1$ *and* (X_2, U_1, U_2) *are i.d.*

(b) $Z_2 = a_{21}X_1 + X_2 + U_2$ *and* (X_1, U_1, U_2) *are i.d.*

Then:

(i) *Z_1 is degenerate and Z_2 is arbitrary if $a_{12} = 0$ and $a_{21} \neq 0$, while Z_2 is degenerate and Z_1 is arbitrary if $a_{21} = 0$ and $a_{12} \neq 0$.*

(ii) *(Z_1, Z_2) has a b.n.d. (bivariate normal distribution) if $a_{12} \neq 0$, $a_{21} \neq 0$.*

(iii) *In any case, (Z_1, Z_2) and (U_1, U_2) are i.d.*

Proof. The proof follows on the same lines as in Rao [6, 7]. Note that under the conditions of the theorem, $Z_2 = a_{21}Z_1 + (1 - a_{12}a_{21}) X_2 + U_2 - a_{21}U_1$ and $(Z_1 - a_{12}X_2, U_1, U_2)$ are i.d. Also, Z_1 and (X_2, U_1, U_2) are i.d.

If $a_{21} \neq 0$ and $a_{12} = 0$, applying Lemma 2, Z_1 degenerate and Z_2 is arbitrary. A similar result holds when $a_{12} \neq 0$ and $a_{21} = 0$, which proves (i).

If $(1 - a_{12}a_{21}) = 0$, $a_{12} \neq 0$, $a_{21} \neq 0$, repeated applications of Lemma 2 show that $U_2 - a_{21} U_1$ and Z_1, and hence, Z_2 are all degenerate, which is a special case of a b.n.d. If $1 - a_{12}a_{21} \neq 0$, $a_{12} \neq 0$, $a_{21} \neq 0$,

$$a_{21} Z_1 + W \tag{3.1}$$

$$\frac{1 - a_{12}a_{21}}{a_{12}} Z_1 - W \tag{3.2}$$

are i.d., where $W = (1 - a_{12}a_{21}) X_2 + U_2 - a_{21}U_1$. Since Z_1 and W are i.d., the Darmois–Skitovic theorem shows that Z_1 and W are normal variables. Hence, $(Z_1, Z_2 = a_{21}Z_1 + W)$ has a b.n.d.

The results (i) and (ii) imply (iii) by using Lemma 3. Theorem 1 is established.

Note 1. Since (3.1) and (3.2) are i.d., we must have

$$\frac{a_{21}(1 - a_{12}a_{21})}{a_{12}} \sigma_1^2 - \sigma_2^2 = 0, \tag{3.3}$$

where σ_1^2 and σ_2^2 are variances of Z_1 and W, respectively. The Eq. (3.3) shows that (Z_1, Z_2) is necessarily degenerate if $a_{12}a_{21} < 0$, or $a_{12}a_{21} \geqslant 1$.

THEOREM 2. *Let $\mathbf{Z}$, $\mathbf{X}$, and $\mathbf{U}$ be k-vector variables such that $\mathbf{Z} = \mathbf{AX} + \mathbf{U}$, where the diagonal elements of $\mathbf{A}$ are all unities and Z_i is distributed independently of $(X_1, ..., X_{i-1}, X_{i+1}, ..., X_k, \mathbf{U})$ for $i = 1, 2, ..., k$, where Z_i and X_i are the ith components of $\mathbf{Z}$ and $\mathbf{X}$, respectively. Then,* (a) *$\mathbf{Z}$ has an m.n.d., and* (b) *$\mathbf{Z}$ and $\mathbf{U}$ are i.d. if every column of $\mathbf{A}$ has at least two nonnull elements.*

Proof. Applying Theorem 1 for $Z_i = x_i + a_{ij}x_j + \eta_i$, and $Z_j =$

$x_j + a_{ji}x_i + \eta_j$ (η_i and η_j being linear functions of the elements of $\mathbf{X}$ and $\mathbf{U}$ except X_i and X_j), we get for each $i \neq j = 1, 2, \ldots, k$,

(i) $a_{ij}Z_j$ is normally distributed, and (3.4)

(ii) (Z_i, Z_j) and $(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_{j-1}, X_{j+1}, \ldots, X_k, \mathbf{U})$, are i.d. (3.5)

Hence, (3.4) along with every column of $\mathbf{A}$ having two nonnull elements implies that Z_j, $j = 1, 2, \ldots, k$, are normally distributed. For their joint normality, we can proceed step by step as follows:

$$Z_2 = (a_{21}Z_1 + w_1) \qquad \text{is normal, and } Z_1 \text{ and } w_1 \text{ are i.d.}$$

$\Rightarrow w_1$ is normally distributed by Cramer's theorem.

This proves that

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} 1 \\ a_{21} \end{pmatrix} Z_1 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} w_1$$

is jointly normal. This shows that any two Z_i's are jointly normal. Now, note that

$$\begin{pmatrix} Z_2 \\ Z_3 \end{pmatrix} = \begin{pmatrix} a_{21} \\ a_{31} \end{pmatrix} Z_1 + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \qquad \text{and} \qquad Z_1 \text{ and } \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \text{ are i.d.} \tag{3.6}$$

$\Rightarrow \binom{w_1}{w_2}$ is normally distributed by Cramer's theorem.

This proves that

$$\begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix} = \begin{pmatrix} 1 \\ a_{21} \\ a_{31} \end{pmatrix} Z_1 + \begin{pmatrix} 0 \\ w_1 \\ w_2 \end{pmatrix}, \tag{3.7}$$

is normally distributed.

This shows that any three Z_i's are jointly normal. Extending this procedure, result (a) of Theorem 2 is established.

To prove independence of $\mathbf{Z}$ and $\mathbf{U}$, we note by Theorem 1, (Z_1, Z_2) and $(X_3, \ldots, X_k, U)$ are i.d. Observe that Z_3 and $(v_2, X_4, \ldots, X_k, \mathbf{U})$ are i.d., and

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix} Z_3 + v_2 \qquad \text{and} \qquad (X_4, \ldots, X_k, \mathbf{U})$$

are i.d., where v_2 depends on $(X_1, X_2, X_4, \ldots, X_k, \mathbf{U})$. Then, noting that the c.f. of $(a_{13}, a_{23})'Z_3$ is nonvanishing anywhere, we conclude from Lemma 3 that v_2 and $(X_4, \ldots, X_k, \mathbf{U})$ are i.d., and therefore, (v_2, Z_3) and $(X_4, \ldots, X_k, \mathbf{U})$ are i.d. This shows the independence of (Z_1, Z_2, Z_3) and $(X_4, \ldots, X_k, \mathbf{U})$. Continuing the process, we establish result (b). This proves Theorem 2.

Note 2. The conditions on independence of variables given in Theorem 2 imply that

$$\operatorname{cov}(Z_i, Z_j) = a_{ji} V(Z_i) = a_{ij} V(Z_j), \tag{3.8}$$

so that the dispersion matrix of **Z** is **AK**, where **K** is a diagonal matrix with $V(Z_i)$ as the ith diagonal element.

Note 3. If $(X_1, ..., X_k, \mathbf{U})$ has a nonsingular distribution or any $(k-1)$ of Z_i, $i = 1, 2, ..., k$, are nondegenerate, then, all the principal minors of order i from **A** are nonzero for $i = 1, 2, ..., k$. (A random variable **X** is said to have a nonsingular distribution if no linear function of **X** has a degenerate distribution.)

Proof. Let us suppose that the principal subdeterminants do not vanish up to $r-1$ and the rth vanishes. Applying Theorem 2 for $Z_1, ..., Z_{r-1}$, we have

$$(Z_1, ..., Z_{r-1}) \quad \text{and} \quad (X_r, ..., X_k, \mathbf{U}), \text{ are i.d.}$$

Further, there exist constants $d_1, ..., d_{r-1}$ such that

$$Z_r = d_1 Z_1 + \cdots + d_{r-1} Z_{r-1} + V, \tag{3.9}$$

where V depends on $X_{r+1}, ..., X_k, \mathbf{U}$ through linear functions. Also, there exist constants $b_1, ..., b_{r-1}$ such that

$$Z_r \quad \text{and} \quad (Z_1 - b_1 X_r, ..., Z_{r-1} - b_{r-1} X_r, X_{r+1}, ..., X_k, \mathbf{U}), \text{ are i.d.} \tag{3.10}$$

This implies that

$$Z_r = X + V \quad \text{and} \quad (X - cX_r, V), \text{ are i.d.}, \tag{3.11}$$

where $X = d_1 Z_1 + \cdots + d_{r-1} Z_{r-1}$ and $c = d_1 b_1 + \cdots + d_{r-1} b_{r-1}$. Then, using Lemmas 1 and 2, Z_r is degenerate, which is contrary to assumption. The assertion made in the note is established.

Note 4. The result of Theorem 2 that **Z** has an m.n.d. implies that the conditional distribution of **X** given **U** is m.n. if **A** is nonsingular.

Note 5. Suppose that only the first s principal subdeterminants are nonzero, and the $(s+1)$th is zero for any ordering of the variables $X_1, ..., X_k$. Then, the r.v.'s $Z_{s+1}, ..., Z_k$ are necessarily degenerate, while $Z_1, ..., Z_s$ has an s-variate singular normal distribution.

Note 6. Let $(X_1, ..., X_k)$ be a k-vector variable with a nonsingular distribution such that the regression of X_i on the rest of the variables is linear and the conditional distribution of X_i depends on the rest of the variables only through the mean value. If the coefficient of X_i in at least one regression function is nonzero for $i = 1, 2, \ldots, k$, then $(X_1, \ldots, X_k)$ has an m.n.d.

Note 7. If in Theorem 2, $a_{ji} = 0$ for $j = 1, \ldots, i-1, i+1, \ldots, n$, then Z_i will have an arbitrary distribution and $a_{ij}Z_j$, $j = 1, 2, \ldots, i-1, i+1, \ldots, n$, are all degenerate.

THEOREM 3. *Let* $\mathbf{X}_1, \mathbf{X}_2, \mathbf{U}_1, \mathbf{U}_2$ *be vector variables and define*

$$\mathbf{Z}_1 = \mathbf{X}_1 + \mathbf{A}_{12}\mathbf{X}_2 + \mathbf{U}_1, \tag{3.12}$$

$$\mathbf{Z}_2 = \mathbf{A}_{21}\mathbf{X}_1 + \mathbf{X}_2 + \mathbf{U}_2. \tag{3.13}$$

If $\mathbf{Z}_1$ *and* $(\mathbf{X}_2, \mathbf{U}_1, \mathbf{U}_2)$ *are i.d., and* $\mathbf{Z}_2$ *and* $(\mathbf{X}_1, \mathbf{U}_1, \mathbf{U}_2)$ *are i.d., then*

(i) $(\mathbf{A}_{21}\mathbf{Z}_1, \mathbf{A}_{12}\mathbf{Z}_2)$ *has an m.n.d., and*

(ii) $(\mathbf{Z}_1, \mathbf{Z}_2)$ *and* $(\mathbf{U}_1, \mathbf{U}_2)$ *are i.d.*

Proof. From (3.12) and (3.13),

$$\mathbf{Z}_2 = \mathbf{A}_{21}\mathbf{Z}_1 + \mathbf{T}_{22}\mathbf{X}_2 + \mathbf{U} \quad \text{and} \quad (\mathbf{Z}_1 - \mathbf{A}_{12}\mathbf{X}_2, \mathbf{U}) \tag{3.14}$$

are i.d., where $\mathbf{T}_{22} = \mathbf{I} - \mathbf{A}_{21}\mathbf{A}_{12}$ and $\mathbf{U} = \mathbf{U}_2 - \mathbf{A}_{21}\mathbf{U}_1$. Representing the s.c.f.'s of $\mathbf{Z}_1$ and $(\mathbf{X}_2, \mathbf{U})$ by f and g (where s.c.f. stands for the second characteristic function or the logarithm of the c.f. defined in the neighborhood of the origin), we obtain from the condition (3.14), the functional equation

$$f(\mathbf{B}_1'\mathbf{t}) + g(\mathbf{B}_2'\mathbf{t}) + g_1(\mathbf{B}_3'\mathbf{t}) + g_2(\mathbf{B}_4'\mathbf{t}) = 0, \tag{3.15}$$

where g_1 and g_2 are suitably defined functions. The matrices $\mathbf{B}_i$ are as follows:

$$\mathbf{B}_1 = \begin{pmatrix} \mathbf{A}_{21} \\ \mathbf{A}_{21} \\ \mathbf{0} \end{pmatrix}, \quad \mathbf{B}_2 = \begin{pmatrix} \mathbf{T}_{22} & \mathbf{I} \\ -\mathbf{A}_{21}\mathbf{A}_{12} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}, \quad \mathbf{B}_3 = \begin{pmatrix} \mathbf{I} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \quad \mathbf{B}_4 = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}, \tag{3.16}$$

where the identity matrices are of suitable orders. It may be verified that

$$R(\mathbf{B}_1 : \mathbf{B}_j) = R(\mathbf{B}_1) + R(\mathbf{B}_j), \qquad j = 2, 3, 4, \tag{3.17}$$

where R stands for the rank of a matrix. Using Lemma 6, we find that, under the condition (3.17), f is a polynomial, i.e., $\mathbf{A}_{21}\mathbf{Z}_1$ has an m.n.d. Similarly, $\mathbf{A}_{12}\mathbf{Z}_2$ has an m.n.d. The joint normality of $\mathbf{A}_{21}\mathbf{Z}_1$ and $\mathbf{A}_{12}\mathbf{Z}_2$ follows from the relation

$$\mathbf{A}_{12}\mathbf{Z}_2 = \mathbf{A}_{12}(\mathbf{A}_{21}\mathbf{Z}_1) + \mathbf{A}_{12}(\mathbf{T}_{22}\mathbf{X}_2 + \mathbf{U}), \tag{3.18}$$

and the condition that $\mathbf{Z}_1$ and $\mathbf{T}_{22}\mathbf{X}_2 + \mathbf{U}$ are i.d., which implies that $\mathbf{A}_{12}(\mathbf{T}_{22}\mathbf{X}_2 + \mathbf{U})$ has also an m.n.d. The result (i) is established.

To prove (ii), let us observe that

(a) $\mathbf{Z}_2 = \mathbf{A}_{21}\mathbf{Z}_1 + \mathbf{W}$ and $(\mathbf{U}_1, \mathbf{U}_2)$ are i.d.,

(b) $\mathbf{Z}_1$ and $(\mathbf{W}, \mathbf{U}_1, \mathbf{U}_2)$ are i.d., and

(c) the c.f. of $\mathbf{A}_{21}\mathbf{Z}_1$ does not vanish anywhere,

where $\mathbf{W} = \mathbf{T}_{22}\mathbf{X}_2 + \mathbf{U}$ as in (3.14). Applying Lemma 3, $\mathbf{W}$ and $(\mathbf{U}_1, \mathbf{U}_2)$ are i.d. Then, $(\mathbf{Z}_1, \mathbf{W})$ and $(\mathbf{U}_1, \mathbf{U}_2)$ are i.d. $\Rightarrow (\mathbf{Z}_1, \mathbf{Z}_2)$ and $(\mathbf{U}_1, \mathbf{U}_2)$ are i.d. Theorem 3 is proved.

Note 8. If $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ are dispersion matrices of $\mathbf{A}_{12}\mathbf{Z}_2$ and $\mathbf{A}_{21}\mathbf{Z}_1$, respectively, then it is easily seen that the covariance of $\mathbf{A}_{12}\mathbf{Z}_2$ and $\mathbf{A}_{21}\mathbf{Z}_1$ is $\mathbf{A}_{12}\mathbf{\Sigma}_2 = \mathbf{\Sigma}_1\mathbf{A}'_{21}$. From this, it follows that the characteristic roots of $\mathbf{A}_{12}\mathbf{A}_{21}$ are all real and lie in [0, 1]. The joint distribution of $\mathbf{A}_{21}\mathbf{Z}_1$ and $\mathbf{A}_{12}\mathbf{Z}_2$ is necessarily singular when a characteristic root is $\geqslant 1$ or < 0.

Note 9. In Theorem 3, while $(\mathbf{A}_{21}\mathbf{Z}_1, \mathbf{A}_{12}\mathbf{Z}_2)$ has an m.n.d., $(\mathbf{I} - \mathbf{A}_{12}^{-}\mathbf{A}_{12})'\mathbf{Z}_2$ and $(\mathbf{I} - \mathbf{A}_{21}^{-}\mathbf{A}_{21})'\mathbf{Z}_1$ may have arbitrary distributions, where $\mathbf{A}^{-}$ denotes a generalized inverse of $\mathbf{A}$.

Note 10. In Theorem 3, suppose that (3.13) is replaced by

$$\mathbf{Z}_2 = \mathbf{A}_{21}\mathbf{X}_1 + \mathbf{A}_{22}\mathbf{X}_2 + \mathbf{U}_2, \tag{3.19}$$

other conditions remaining the same. If $\mathbf{A}_{22}$ is nonsingular, then we have the same results as in Theorem 3. Let $\mathbf{A}_{22}$ be singular, but $\mathbf{T}_{22} = \mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{12}$ is nonsingular. Then, following the arguments of Theorem 3, the following can be established:

(a) $(\mathbf{A}_{21}\mathbf{Z}_1, \mathbf{A}_{12}\mathbf{T}_{22}^{-1}Z_2)$ has an m.n.d., and

(b) $(\mathbf{Z}_1, \mathbf{Z}_2)$ and $(\mathbf{U}_1, \mathbf{U}_2)$ are i.d.

Note 11. Suppose in Theorem 3, that $\mathbf{Z}_1$ or $\mathbf{Z}_2$ is a nondegenerate random variable. Then, $\mathbf{I} - \mathbf{A}_{21}\mathbf{A}_{12}$ and $\mathbf{I} - \mathbf{A}_{12}\mathbf{A}_{21}$ are nonsingular matrices.

Note 12. Suppose that in Theorem 3, $\mathbf{T}_{22} = \mathbf{I} - \mathbf{A}_{21}\mathbf{A}_{12}$ is nonsingular. Then, it is easy to establish from the independence of $\mathbf{Z}_2 = \mathbf{A}_{21}\mathbf{Z}_1 + \mathbf{W}$ and $\mathbf{Z}_1 - \mathbf{A}_{12}\mathbf{T}_{22}^{-1}\mathbf{W}$, where $\mathbf{Z}$ and $\mathbf{W}$ are i.d., that

$$f_1(\mathbf{A}'_{21}\mathbf{t}_2 + \mathbf{t}_1) - f_1(\mathbf{t}_1) \quad \text{and} \quad f_2(\mathbf{A}'_{12}\mathbf{t}_1 + \mathbf{t}_2) - f_2(\mathbf{t}_2) \tag{3.20}$$

are polynomials of the second degree in $\mathbf{t}_1$ and $\mathbf{t}_2$, where f_1 and f_2 are s.c.f's of $\mathbf{Z}_1$ and $\mathbf{Z}_2$, respectively.

THEOREM 4. *Let* $\mathbf{X}_1$, $\mathbf{X}_2$, *and* $\mathbf{X}_3$ *be vector variables such that*

$$\mathbf{Z}_i = \mathbf{X}_i + \mathbf{A}_{ij}\mathbf{X}_j + \mathbf{A}_{ik}\mathbf{X}_k \qquad \textit{and} \qquad (\mathbf{X}_j, \mathbf{X}_k), \textit{ are i.d.} \tag{3.21}$$

for $(i, j, k) = (1, 2, 3), (2, 3, 1)$, *and* $(3, 2, 1)$. *Then,*

$$(\mathbf{A}_1\mathbf{Z}_1, \mathbf{A}_2\mathbf{Z}_2, \mathbf{A}_3\mathbf{Z}_3), \qquad \textit{has an m.n.d.}, \tag{3.22}$$

where $\mathbf{A}_i' = (\mathbf{A}_{ji}' : \mathbf{A}_{ki}')$, *if*

(a) *at least one of the matrices* $(\mathbf{I} - \mathbf{A}_{21}\mathbf{A}_{12})$, $(\mathbf{I} - \mathbf{A}_{13}\mathbf{A}_{31})$ *and* $(\mathbf{I} - \mathbf{A}_{32}\mathbf{A}_{23})$ *is nonsingular, or*

$$\text{(b)} \quad \mathbf{A} = \begin{pmatrix} \mathbf{I} & \mathbf{A}_{12} & \mathbf{A}_{13} \\ \mathbf{A}_{21} & \mathbf{I} & \mathbf{A}_{23} \\ \mathbf{A}_{31} & \mathbf{A}_{32} & \mathbf{I} \end{pmatrix} \tag{3.23}$$

is nonsingular.

Proof. Using Theorem 3, we find the condition (3.21) alone implies that

(a) each of $\mathbf{A}_{21}\mathbf{Z}_1$, $\mathbf{A}_{31}\mathbf{Z}_1$, $\mathbf{A}_{12}\mathbf{Z}_2$, $\mathbf{A}_{32}\mathbf{Z}_2$, $\mathbf{A}_{13}\mathbf{Z}_3$, $\mathbf{A}_{23}\mathbf{Z}_3$, has an m.n.d. (3.24)

(b) $(\mathbf{Z}_1, \mathbf{Z}_2)$ and $\mathbf{X}_3$ are i.d., $(\mathbf{Z}_1, \mathbf{Z}_3)$ and $\mathbf{X}_2$ are i.d., and $(\mathbf{Z}_2, \mathbf{Z}_3)$ and $\mathbf{X}_1$ are i.d. (3.25)

Now, suppose that $\mathbf{I} - \mathbf{A}_{21}\mathbf{A}_{12}$ is nonsingular. Then, an application of Note 12 together with (3.24) gives that $\mathbf{A}_1\mathbf{Z}_1$ and $\mathbf{A}_2\mathbf{Z}_2$ have m.n.d.'s .Further, on account of (3.25),

$$\begin{pmatrix} \mathbf{Z}_1 \\ \mathbf{Z}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix} + \mathbf{A}_3\mathbf{X}_3 \qquad \text{and} \qquad \mathbf{X}_3, \text{ are i.d.,}$$

$$\mathbf{Z}_3 = (\mathbf{A}_{31}\mathbf{X}_1 + \mathbf{A}_{32}\mathbf{X}_2) + \mathbf{X}_3 \qquad \text{and} \qquad (\mathbf{X}_1, \mathbf{X}_2), \text{ are i.d.}$$

Then, applying Theorem 3 and Note 10, $\mathbf{A}_3\mathbf{Z}_3$ has an m.n.d. The joint normality of $\mathbf{A}_1\mathbf{Z}_1$, $\mathbf{A}_2\mathbf{Z}_2$ and $\mathbf{A}_3\mathbf{Z}_3$ is established as in Theorem 3.

Now, suppose that $\mathbf{A}$ as defined in (3.23) is nonsingular. Then, using (3.25) and Note 10, it is easily shown that $\mathbf{A}_1\mathbf{Z}_1$, $\mathbf{A}_2\mathbf{Z}_2$ and $\mathbf{A}_3\mathbf{Z}_3$ have m.n.d.'s. Then, as in Theorem 3, their joint normality can be established. Theorem 4 is proved.

Note 13. It is seen that to establish (3.22) from (3.24) and (3.25), conditions such as (a) or (b) of Theorem 4 on the matrices $\mathbf{A}_{ij}$ are assumed. Thus, the result (3.22) may not hold only if the matrices

$$\mathbf{A}, \quad \mathbf{I} - \mathbf{A}_{21}\mathbf{A}_{12}, \quad \mathbf{I} - \mathbf{A}_{31}\mathbf{A}_{13}, \quad \text{and} \quad \mathbf{I} - \mathbf{A}_{32}\mathbf{A}_{23} \tag{3.26}$$

are all singular. It is worth examining whether (3.22) can be established without any conditions on the matrices $\mathbf{A}_{ij}$.

Note 14. If in Theorem 4, it is assumed that any two of $\mathbf{Z}_1$, $\mathbf{Z}_2$, and $\mathbf{Z}_3$ have nonsingular distributions, then, it immediately follows that the matrices (3.26) are all nonsingular. In such a case, $(\mathbf{A}_1\mathbf{Z}_1, \mathbf{A}_2\mathbf{Z}_2, \mathbf{A}_3\mathbf{Z}_3)$ and $(\mathbf{A}_1\mathbf{X}_1, \mathbf{A}_2\mathbf{X}_2, \mathbf{A}_3\mathbf{X}_3)$ have m.n.d.'s. Further, if $R(\mathbf{A}_i)$ is equal to the number of elements in $\mathbf{X}_i$, $i = 1, 2, 3$, then $(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3)$ has an m.n.d.

Note 15. In Theorem 4, if it is assumed that $\mathbf{A}$ and any one of $\mathbf{I} - \mathbf{A}_{21}\mathbf{A}_{12}$, $\mathbf{I} - \mathbf{A}_{31}\mathbf{A}_{13}$, and $\mathbf{I} - \mathbf{A}_{32}\mathbf{A}_{23}$ is nonsingular and $f_i(\mathbf{t})$ is the s.c.f. of $\mathbf{Z}_i$, $i = 1, 2, 3$, then

$$f_i(\mathbf{A}_{i\cdot}'\mathbf{t}) - f_i(\mathbf{t}_i) \tag{3.27}$$

is a second degree polynomial in $\mathbf{t}$, where $\mathbf{t}' = (\mathbf{t}_1', \mathbf{t}_2', \mathbf{t}_3')$ and $\mathbf{A}_{i\cdot}$ is the ith column partition of $\mathbf{A}$.

THEOREM 5. *Let* $\mathbf{X}_1, \ldots, \mathbf{X}_k$ *and* $\mathbf{U}_1, \ldots, \mathbf{U}_k$ *be vector variables and define*

$$\mathbf{Z}_i = \mathbf{A}_{i1}\mathbf{X}_1 + \cdots + \mathbf{A}_{ik}\mathbf{X}_k + \mathbf{U}_i, \tag{3.28}$$

with $\mathbf{A}_{ii} = \mathbf{I}$ *(of suitable order),* $\mathbf{A} = (\mathbf{A}_{ij}; i, j = 1, \ldots, k)$. $\mathbf{A}_j$ *is the* j*th column partition of* $\mathbf{A}$ *with* $\mathbf{A}_{jj}$ *omitted,* $j = 1, \ldots, k$, *and* $\mathbf{A}_{(j)}$ *is the principal submatrix formed by first* j *row partitions and first* j *column partitions of* $\mathbf{A}$. *Furtehr, let*

$$\mathbf{Z}_i \quad \text{and} \quad (\mathbf{X}_1, \ldots, \mathbf{X}_{i-1}, \mathbf{X}_{i+1}, \ldots, \mathbf{X}_k, \mathbf{U}), \ \text{be i.d.}, \ i = 1, \ldots, k, \tag{3.29}$$

where $\mathbf{U}$ *represents all* $\mathbf{U}_i$. *Then*:

$$\text{(i)} \quad (\mathbf{A}_1\mathbf{Z}_1, \ldots, \mathbf{A}_k\mathbf{Z}_k), \quad \text{has an m.n.d., and} \tag{3.30}$$

$$\text{(ii)} \quad (\mathbf{Z}_1, \ldots, \mathbf{Z}_k) \text{ and } (\mathbf{U}_1, \ldots, \mathbf{U}_k), \quad \text{are i.d.} \tag{3.31}$$

if (a) *the matrices* $\mathbf{A}_{(1)}, \ldots, \mathbf{A}_{(k)}$ *are nonsingular, or* (b) *the matrices* $\mathbf{I} - \mathbf{A}_{ij}\mathbf{A}_{ji}$ *for all* $i \neq j$ *are nonsingular.*

Proof. First, we shall prove the result under the sufficient condition (a). By Lemma 5, there exists a decomposition $\mathbf{A} = \mathbf{TS}$ where $\mathbf{S} = (\mathbf{S}_{ij})$ is upper triangular in partitions with $\mathbf{S}_{ii} = \mathbf{I}$, and $\mathbf{T} = (\mathbf{T}_{ij})$ is lower triangular in partitions with $\mathbf{T}_{ii}$ nonsingular. Let $\mathbf{X}' = (\mathbf{X}_1', \ldots, \mathbf{X}_k')$, $\mathbf{U}' = (\mathbf{U}_1', \ldots, \mathbf{U}_k')$, and $\mathbf{W}' = (\mathbf{W}_1',\dots,\mathbf{W}_k') = \mathbf{U}'(\mathbf{T}')^{-1} + \mathbf{X}'\mathbf{S}'$. Then, under the conditions of Theorem 5, $\mathbf{Z}_i = \mathbf{T}_{i1}\mathbf{W}_1 + \cdots + \mathbf{T}_{ii}\mathbf{W}_i$ and $(\mathbf{W}_1 - \mathbf{S}_{1i}\mathbf{W}_i,\dots,\mathbf{W}_{i-1} - \mathbf{S}_{i-1,i}\mathbf{W}_i, \mathbf{W}_{i+1},\dots,\mathbf{W}_k, \mathbf{U})$ are i.d. for $i = 1,\dots,k$. Let $f_i(\mathbf{t})$ be the s.c.f. of $\mathbf{W}_i$, $i = 1,\dots,k$. Then, considering $i = 1$ and 2 and applying Theorem 3 and Note 12,

$$f_1(\mathbf{T}_{21}'\mathbf{t}_2 + \mathbf{T}_{11}'\mathbf{t}_1) - f_1(\mathbf{T}_{11}'\mathbf{t}_1) \quad \text{and} \quad f_2(\mathbf{S}_{12}'\mathbf{t}_1 + \mathbf{T}_{22}'\mathbf{t}_2) - f_2(\mathbf{T}_{22}'\mathbf{t}_2). \tag{3.32}$$

are polynomials of the second degree in $(\mathbf{t}_1, \mathbf{t}_2)$. Further, by applying Lemma 3,

$$\mathbf{W}_1, \mathbf{W}_2, (\mathbf{W}_3,\dots,\mathbf{W}_k, \mathbf{U}), \quad \text{are i.d.} \tag{3.33}$$

Similarly, taking $i = 3$, and applying Theorem 3 and Note 12 and the results (3.32) and (3.33), we find that

(a) $f_1(\mathbf{T}_{31}'\mathbf{t}_3 + \mathbf{T}_{21}'\mathbf{t}_2 + \mathbf{T}_{11}'\mathbf{t}_1) - f_1(\mathbf{T}_{11}'\mathbf{t}_1), f_2(\mathbf{T}_{32}'\mathbf{t}_3 + \mathbf{T}_{22}'\mathbf{t}_2 + \mathbf{S}_{12}'\mathbf{t}_1)$

$-f_2(\mathbf{T}_{22}'\mathbf{t}_2)$ and $f_3(\mathbf{T}_{33}'\mathbf{t}_3 + \mathbf{S}_{23}'\mathbf{t}_2 + \mathbf{S}_{13}'\mathbf{t}_1) - f_3(\mathbf{T}_{33}'\mathbf{t}_3)$,

are polynomials of the second degree in $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$, and (3.34)

(b) $\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3, (\mathbf{W}_4,\dots,\mathbf{W}_k, \mathbf{U})$, are i.d. (3.35)

Proceeding in the same manner, we establish that

(i) $f_j(\mathbf{T}_{kj}'\mathbf{t}_k + \cdots + \mathbf{T}_{j+1,j}'\mathbf{t}_{j+1} + \mathbf{T}_{jj}'\mathbf{t}_j + \mathbf{S}_{j-1,j}'\mathbf{t}_{j-1} + \cdots + \mathbf{S}_{1j}'\mathbf{t}_1)$

$-f_j(\mathbf{T}_{jj}'\mathbf{t}_j)$, is a polynomial of second degree in $\mathbf{t}_1,\dots,\mathbf{t}_k$, and (3.36)

(ii) $\mathbf{W}_1,\dots,\mathbf{W}_k$ and $\mathbf{U}$ are i.d. (3.37)

Since $\mathbf{Z}_1,\dots,\mathbf{Z}_k$ are functions of $\mathbf{W}_1,\dots,\mathbf{W}_k$, (3.37) proves the result (3.31) of Theorem 5. Taking

$$\mathbf{B}_j' = (\mathbf{S}_{1j}',\dots,\mathbf{S}_{j-1,j}', \mathbf{T}_{j+1,j}',\dots,\mathbf{T}_{kj}'), \tag{3.38}$$

(3.36) shows that

$$\mathbf{B}_1\mathbf{W}_1,\dots,\mathbf{B}_k\mathbf{W}_k, \quad \text{have independent m.n.d.'s.} \tag{3.39}$$

Noting that $\mathbf{A} = \mathbf{TS}$, it is easy to see that

$$\mathbf{A}_1\mathbf{T}_{11}\mathbf{W}_1,\dots,\mathbf{A}_k\mathbf{T}_{kk}\mathbf{W}_k, \quad \text{have independent m.n.d.'s.} \tag{3.40}$$

Observing that $\mathbf{Z}_i = \mathbf{T}_{i1}\mathbf{W}_1 + \cdots + \mathbf{T}_{ii}\mathbf{W}_i$, $i = 1,\dots,k$, the result (3.40) implies (3.30) of Theorem 5.

Now, we shall prove Theorem 5 under the sufficient condition (b), viz, that $\mathbf{I} - \mathbf{A}_{ij}\mathbf{A}_{ji}$ is nonsingular for every $i \neq j$. Let, as before, f_i be the s.c.f. of $\mathbf{Z}_i$, $i = 1,\ldots, k$. Considering (3.29) for two values i and j and applying Theorem 3 and Note 12,

$$f_i(\mathbf{A}'_{ji}\mathbf{t}_j + \mathbf{t}_i) - f_i(\mathbf{t}_i) \tag{3.41}$$

is a polynomial of the second degree in $\mathbf{t}_i$, $\mathbf{t}_j$ for every $i \neq j$. This shows that

$$f_i(\mathbf{A}'_{1i}\mathbf{t}_1 + \cdots + \mathbf{A}'_{i-1,i}\mathbf{t}_{i-1} + \mathbf{t}_i + \mathbf{A}'_{i+1,i}\mathbf{t}_{i+1} + \cdots + \mathbf{A}'_{ki}\mathbf{t}_k) - f_i(\mathbf{t}_i). \tag{3.42}$$

is a polynomial of the second degree in $\mathbf{t}_1,\ldots, \mathbf{t}_k$. Thus,

$$\mathbf{A}_i\mathbf{Z}_i \qquad \text{has an m.n.d. for each } i. \tag{3.43}$$

To prove the joint normality of $\mathbf{A}_i\mathbf{Z}_i$, it is obvious from Theorem 3 that any two of (3.43), say $\mathbf{A}_{k-1}\mathbf{Z}_{k-1}$ and $\mathbf{A}_k\mathbf{Z}_k$ are jointly normal. Then

$$\begin{pmatrix}\mathbf{A}_{k-1}\mathbf{Z}_{k-1}\\ \mathbf{A}_k\mathbf{Z}_k\end{pmatrix} = \begin{pmatrix}\mathbf{A}_{k-1}\mathbf{A}_{k-1,k-2}\\ \mathbf{A}_k\mathbf{A}_{k,k-2}\end{pmatrix}\mathbf{Z}_{k-2} + \mathbf{G}, \tag{3.44}$$

where $\mathbf{G}$ is a linear function of all $\mathbf{U}_i$, and all $\mathbf{X}_i$ except for $i = k - 2$, and hence, $\mathbf{Z}_{k-2}$ and $\mathbf{G}$ are i.d. Then, by Cramer's theorem, both the components on the right-hand side of (3.44) have m.n.d.'s. This shows that $(\mathbf{A}_{k-2}\mathbf{Z}_{k-2}$, $\mathbf{A}_{k-1}\mathbf{Z}_{k-1}$, $\mathbf{A}_k\mathbf{Z}_k)$ has an m.n.d. We proceed in the same way, step by step, to establish the result (3.30) of Theorem 5.

To prove the result (3.31), we note that by Theorem 3, $(\mathbf{Z}_1, \mathbf{Z}_2)$ and $(\mathbf{X}_3,\ldots, \mathbf{X}_k, \mathbf{U})$ are i.d. Observe that

$$\begin{pmatrix}\mathbf{Z}_1\\ \mathbf{Z}_2\end{pmatrix} = \begin{pmatrix}\mathbf{A}_{13}\\ \mathbf{A}_{23}\end{pmatrix}\mathbf{Z}_3 + \mathbf{K}_2 \qquad \text{and} \qquad (\mathbf{X}_4,\ldots, \mathbf{X}_k, \mathbf{U}), \text{ are i.d.}, \tag{3.45}$$

where $\mathbf{K}_2$ depends on $(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_4,\ldots, \mathbf{X}_k, \mathbf{U})$. Then $\mathbf{Z}_3$ and $(\mathbf{K}_2, \mathbf{X}_4,\ldots, \mathbf{X}_k, \mathbf{U})$ are i.d. Noting that the c.f. of $(\mathbf{A}_{13}\mathbf{Z}_3, \mathbf{A}_{23}\mathbf{Z}_3)$ is nonvanishing anywhere, we conclude from Lemma 3 that $\mathbf{K}_2$ and $(\mathbf{X}_4,\ldots, \mathbf{X}_k, \mathbf{U})$ are i.d. Therefore, $(\mathbf{Z}_3, \mathbf{K}_2)$ and $(\mathbf{X}_4,\ldots, \mathbf{X}_k, \mathbf{U})$ are i.d. This shows the independence of $(\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3)$ and $(\mathbf{X}_4,\ldots, \mathbf{X}_k, \mathbf{U})$. Continuing the process, we establish the desired result.

Note 16. Let $R(\mathbf{A}_i) = p_i$, the dimension of the vector variable $\mathbf{X}_i$, $i = 1,\ldots, k$, in Theorem 5. Then, $(\mathbf{Z}_1,\ldots, \mathbf{Z}_k)$ has an m.n.d., and

$$\operatorname{cov}(\mathbf{Z}_i, \mathbf{Z}_j) = \mathbf{A}_{ij}\mathbf{\Sigma}_j = \mathbf{\Sigma}_i\mathbf{A}'_{ji}, \tag{3.46}$$

where $\mathbf{\Sigma}_i = D(\mathbf{Z}_i)$, the variance covariance matrix of $\mathbf{Z}_i$. Hence, the dispersion matrix of $(\mathbf{Z}_1,\ldots, \mathbf{Z}_k)$ is of the form $\mathbf{AK}$, where $\mathbf{A} = (\mathbf{A}_{ij})$ and $\mathbf{K} = \operatorname{diag}(\mathbf{\Sigma}_1, \mathbf{\Sigma}_2,\ldots, \mathbf{\Sigma}_k)$.

Note 17. In addition to the conditions $R(\mathbf{A}_i)$ in Note 16, let $\mathbf{A} = (\mathbf{A}_{ij})$ be nonsingular. Then, the conditional distribution of $(\mathbf{X}_1, ..., \mathbf{X}_k)$ given $(\mathbf{U}_1, ..., \mathbf{U}_k)$ is m.n. and depends on $\mathbf{U}_1, ..., \mathbf{U}_k$ only through the conditional expectation.

Note 18. If in Note 16, $\mathbf{\Sigma}_i$ is nonsingular for each i, then $\mathbf{AK}$ is positive definite and

$$R(\mathbf{A}_{ij}\mathbf{A}_{ji}) = R(\mathbf{A}_{ji}\mathbf{A}_{ij}) = R(\mathbf{A}_{ij}) = R(\mathbf{A}_{ji}).$$

Note 19. In Theorem 5, we have stated one of the sufficient conditions as nonsingularity of the principal minors $\mathbf{A}_{(1)}, ..., \mathbf{A}_{(k)}$. An alternative condition to this is the nonsingularity of the individual distributions of $\mathbf{Z}_1, ..., \mathbf{Z}_{k-1}$. Actually, Theorem 5 can be proved under the condition that $\mathbf{A}_{(1)}, ..., \mathbf{A}_{(k-1)}$ are nonsingular, which is also implied by the nonsingularity of the individual distributions of $\mathbf{Z}_1, ..., \mathbf{Z}_{k-1}$. The proof is a little involved in this case.

Note 20. In Theorem 5, we assumed in the second alternative sufficient condition that all the matrices $\mathbf{I} - \mathbf{A}_{ij}\mathbf{A}_{ji}$ for $i \neq j$ are nonsingular. In Theorem 4, for the case $k = 3$, we needed only one such matrix to be nonsingular. To establish the result of Theorem 5 for a general k, we need nonsingularity of a much smaller number of appropriately chosen matrices of the form $\mathbf{I} - \mathbf{A}_{ij}\mathbf{A}_{ji}$. For example, when $k = 4$, it is sufficient to have two nonsingular matrices, $\mathbf{I} - \mathbf{A}_{21}\mathbf{A}_{12}$ and $\mathbf{I} - \mathbf{A}_{34}\mathbf{A}_{43}$. Again, the proof will be involveed.

Note 21. Let $\mathbf{X}_i$ be a p_i-vector variable, $i = 1, ..., k$, and $(\mathbf{X}_1, ..., \mathbf{X}_k)$ have a nonsingular distribution. If (i) the regression of $\mathbf{X}_i$ on the rest is linear (i.e., of the form $\mathbf{A}_{i1}\mathbf{X}_1 + \cdots + \mathbf{A}_{i,i-1}\mathbf{X}_{i-1} + \mathbf{A}_{i,i+1}\mathbf{X}_{i+1} + \cdots + \mathbf{A}_{ik}\mathbf{X}_k$), and (ii) the conditional distribution depends on the rest of the variables only through the mean value, and $R(\mathbf{A}_i') = p_i$, where

$$\mathbf{A}_i' = (\mathbf{A}_{1i}' \vdots \cdots \vdots \mathbf{A}_{i-1,i}' \vdots \mathbf{A}_{i+1,i}' \vdots \cdots \vdots \mathbf{A}_{ki}'),$$

then $(\mathbf{X}_1, ..., \mathbf{X}_k)$ has an m.n.d.

Note 22. Theorems 3–5 can be extended to random variables $\mathbf{X}_i$ defined on a more general vector space like Hilbert space H_i, $i = 1, ..., k$. In such a case, the matrices $\mathbf{A}_{ij}$ are replaced by linear bounded operators A_{ij} and the rank conditions are replaced by range conditions similar to those given by Kumar and Pathak [5].

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