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A NOTE ON THE UNIVERSAL OPTIMALITY CRITERION FOR FULL RANK MODELS

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Abstract: The aim of this note is to suggest a revised formulation of the universal optimality criterion for full rank models as stated in Kiefer (1975). We have presented the relevant results with indications of some possible applications.

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1. Introduction

In this note we point out an error in one of the universal optimality results in Kiefer (1975) and suggest a way to rectify it. We shall throughout refer to this paper by (K) and use the same notations as in (K). In (K), the Proposition 1 is meant for the case of singular models whereas the Proposition 1′ relates to a full rank model. It is observed that the conditions (a)–(c) of (2.1) in (K) (vide Section 2), as they stand, do *not* by themselves lead to the validity of Proposition 1′ [1]. As a matter of fact, the condition (b) (defining the class of optimality criteria Φ) is useless for the full rank set-up and a revised condition is needed (unless, of course, the condition (c) (of permutation invariance) is changed (to one of orthogonal invariance). Vide Kiefer (1971).) We state below the precise conditions and results. Before that, we simply point out by means of a counter-example that Proposition 1′ in (K) is not valid *in general*.

[1] Let $\mathcal{D}_v^*$ ($\subset \mathcal{D}_v$) consist of all C_d matrices satisfying $\sum\sum_{i\neq j} c_{d_{ij}}=0$. Then it is easy to argue out the validity of Proposition 1′ (with (a)–(c) of (2.1) in (K) defining the class of criteria Φ) as regards the subclass $\mathcal{D}_v^*$

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It is known (Sinha (1971, 1972)) that for estimating the total weight of a set of n (≥ 3) objects in exactly n weighing operations in an unbiased chemical balance, retaining the individual estimability of the objects, the (unique) optimum design is $X(n \times n) = E_n - I_n$, where E_n is a $(n \times n)$ matrix with all elements unity and I_n is the identity matrix of order n. Certainly X is better than H_n (the Hadamard matrix of order n, whenever it exists). Referring to Proposition 1′ in (K), H_n plays the role of d^* which is, however, not Φ-optimum where Φ is defined as $\Phi(C) = \mathbf{1}'C^{-1}\mathbf{1}$ with $\mathbf{1} = (1, 1, \ldots, 1)'$. It is easy to check that $\Phi(C)$, so defined, satisfies (a)–(c) of (2.1) in (K).

Certainly, plenty of examples of this type may be cited. One could also take $\Phi(C) = (\mathbf{1}'C\mathbf{1})^{-1}$ and verify the validity of the statement made above with respect to this Φ as well.

2. A modified version of universal optimality proposition for full rank models

For matrices belonging to the class $\mathscr{B}_n$ (of all $n \times n$ non-negative definite matrices in a given context) let $\Phi : \mathscr{B}_n \to (-\infty, \infty]$ be redefined as[2]

(a) Φ is convex, $\Phi(C) = +\infty$ whenever $\operatorname{rank}(C) < n$,

(b)′ $\Phi(\alpha I_n + \beta E_n) \geq \Phi(a I_n)$, whenever $a \geq \alpha + \beta$,

(c) Φ is permutation invariant, i.e. $\Phi(G_g' C G_g) = \Phi(C)$, $\forall C \in \mathscr{B}_n$, $\forall g \in \mathscr{P}_n$, the symmetric group of all $n!$ permutations.

Then the following holds:

Proposition 1″. (A revised version of Proposition 1′ in (K).) *If a class of matrices $\mathscr{C} = \{C_d \mid d \in \mathscr{D}\}$ contains a C_{d^*} which is a multiple of I_n and which maximizes $\operatorname{tr}(C_d)$ among all $d \in \mathscr{D}$, then d^* is universally optimal in $\mathscr{D}$. [Here 'universal optimality' refers to minimization of any criterion of the type Φ satisfying the conditions* (a), (b)′ *and* (c) *above.]*

We omit the proof which follows essentially along the same line of arguments as in the case of Proposition 1 in (K).

Remark 1. The universal optimality (under the modified version) of chemical balance weighing designs based on Hadamard matrices is now evident.

Remark 2. One can readily check that the condition (b)′ is satisfied by the commonly employed criteria like A−, D−, E− and Φ_p-optimality criteria (vide (K)) as also the Φ_f-criteria of both types (Cheng (1980)). An optimality functional of a different kind is given by $\Phi(C) = \max_i (C^{-1})_{ii}$. (This example is due to Kiefer.) Cheng (1980) has discussed various interesting optimality results (for weighing

[2] In (K), the condition (b) is stated as: $\Phi(bC)$ is nonincreasing in the scalar $b \geq 0$.

problems) relevant to the situations where the Hadamard matrices do not exist. In the next section, we present some further observations along this direction. This becomes possible in view of the revised concept of universal optimality for full rank models.

3. Some applications

It is now possible to propose refined statements concerning some of the specific optimality results in the context of weighing designs and factorial designs. The latter topic has been taken up separately by one of the authors (Mukerjee (1981)). We will use the notations as in Cheng (1980). We present the results without proofs.

(i) The $[N,1,0]_n$ design, whenever it exists, is universally optimal within the subclass of all designs d satisfying $\operatorname{tr}(C_d) \le n(N-1)$.

(ii) Within the subclass $\mathscr{D}'_{N,n}$ (of designs having complete symmetry (vide (K)) of the C-matrices), the designs $[N,1,0]_n$ and $\{[N,0,\lambda]_n \mid \lambda = \pm 1, \pm 2, \dots\}$ *formally* constitute a complete class of designs with respect to any optimality functional satisfying (a), (b)' and (c). Further, replacing (b)' by (b)", we immediately have ($X_1 > X_2$ implying X_1 is better than X_2) $[N,0,1]_n > [N,0,2]_n > \cdots$ and $[N,0,-1]_n > [N,0,-2]_n > \cdots$ where (b)" is stated as

(b)" $\Phi(\alpha' I_n + \beta' E_n) \ge \Phi(\alpha I_n + \beta E_n)$ whenever $\alpha + \beta \ge \alpha' + \beta'$, $0 \le \beta \le \beta'$ or $\beta' \le \beta \le 0$.

It is not difficult to verify that the Φ_f-criteria of Cheng (1980) (and hence all the commonly employed criteria) satisfy (b)".

Remark 3. The optimality result in (i) above is, in a sense, supplementary to the results in Cheng (1980). A typical situation (e.g., when *not* all the objects may be weighed simultaneously) would necessarily imply $\operatorname{tr}(C) \le n(N-1)$.

Remark 4. Within the subclass $\mathscr{D}'_{N,n}$, the three outstanding members are $[N,1,0]_n$, $[N,0,\pm 1]_n$. As is well known (vide Raghavarao (1971)), they do not exist simultaneously. Cheng (1980) furnished optimality results concerning them. We simply add that the $[n,1,0]_n$ design is Φ_p-optimal in $\mathscr{D}'_{n,n}$ for all $p \ge p_0$ where p_0 (strictly positive) satisfies

$$1 = \left(1 - \frac{1}{n}\right)\left(\frac{n-1}{n-2}\right)^{p_0} + \frac{1}{n}\left(\frac{n-1}{3n-2}\right)^{p_0}.$$

Calculations yield $p_0 \simeq 0.15$ for all $n \ge 6$. Thus the D-optimum design $[n,0,2]_n$ is only marginally better than $[n,1,0]_n$.

Remark 5. An earlier version of this paper is to be found as a Technical Report (No. 3/81) in the Stat–Math Division of the Indian Statistical Institute, Calcutta.

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