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# ON COMPLETE CLASSES OF EXPERIMENTS FOR CERTAIN INVARIANT PROBLEMS OF LINEAR INFERENCE

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*Abstract:* For certain non-singularly estimable full-rank *appropriately invariant* problems of linear inference in the setting of block designs, some complete classes of experiments have been characterized through the relation between the relevant $C$-matrices and their g-inverses (of the Moore–Penrose type) in regard to the specific *invariance criterion* discussed here. It follows that for a $\mathcal{G}$-invariant non-singularly estimable full-rank problem, a complete class of experiments *formally* consists *only* of $\mathcal{G}$-invariant designs.

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## 1. Introduction

The object of the present article is to combine the statistical concept of invariance of linear inferential problems (Sinha (1970, 1971, 1972)) with the mathematical aspects of generalized inverses (g-inverses) of matrices, in order to establish some results on complete classes of experiments for inferring about certain invariant linear inferential problems in block-design settings.

First, in Section 2, we intend to derive various results, mostly of mathematical interest, relating to g-inverses and the specific invariance criterion, as applied to them.

Then, in Section 3, we would take up the statistical aspect of the problem and display some special results (involving the g-inverses) fitting into the context of analysis of block designs. Our main aim is, however, to undertake, in Section 4, the problem of exploring the relevance of the considerations of invariant g-inverse matrices in the search for some complete classes of experiments for certain invariant problems of linear inference in block-design settings.

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## 2. The invariance criterion

At the very outset, we refer to Rao and Mitra (1971) for the definitions and properties of g-inverses of matrices including, in particular, the Moore-Penrose (M-P) g-inverse.

In Sinha (1970, 1971, 1972), the statistical concept of invariance was introduced with respect to linear inferential problems. We just take out the mathematical aspect of that concept and apply here in defining invariant matrices in general terms.

Let $A$ $(m \times n)$ be a matrix of real (or complex) elements. To the $n$ columns of the matrix $A$, we associate $n$ numbers, say, $(1, 2, \ldots, n)$ and consider the symmetric group $S_n$ of all possible $n!$ permutations of these numbers. Similarly, we associate $m$ numbers, $(1, 2, \ldots, m)$ to the $m$ rows of $A$ and consider $S_m$, the symmetric group of all the $m!$ permutations of these numbers. Members of $S_n$ will be denoted by the letters $\sigma, \sigma_1, \sigma_2, \ldots$ etc. Very often we will deal with matrix representations of these permutations, like, $G_\sigma, G_{\sigma_1}, G_{\sigma_2}, \ldots$ for members of $S_n$, and $G_{\bar\sigma}, G_{\bar\sigma_1}, G_{\bar\sigma_2}, \ldots$ for members of $S_m$. To unify the operations with these matrices, it will always be understood that a permutation matrix $G_\sigma$ $(G_{\bar\sigma})$ is an $n \times n$ $(m \times m)$ matrix, representing the permutation of the columns of the identity matrix $I_n$ $(I_m)$ according to $\sigma$ $(\bar\sigma)$. Thus, $G'_\sigma = G_{\sigma^{-1}}$ = inverse permutation of $\sigma$ and, similarly, $G'_{\bar\sigma} = G_{\bar\sigma^{-1}}$.

We now introduce the following notion.

**Definition 2.1.** The matrix $A$ $(m \times n)$ is $\sigma$-invariant (i.e. invariant with respect to the permutation $\sigma$ of its columns) if and only if there exists a permutation $\bar\sigma$ $(\in S_m)$, to be called the *neutralizer* of $\sigma$, such that

$$G'_{\bar\sigma} A G_\sigma = A. \tag{2.1}$$

**Note 2.1.** Suppose $A$ is $\sigma_1$-invariant as well as $\sigma_2$-invariant. Then it may well happen that the same matrix $G_{\bar\sigma}$ satisfies $G'_{\bar\sigma} A G_{\sigma_i} = A$ $(i = 1, 2)$. The neutralizers of $\sigma_1$ and $\sigma_2$ need not be different. We might write $\bar\sigma(\sigma)$ for $\bar\sigma$ to show its dependence on $\sigma$.

**Note 2.2.** A consequence of (2.1) is that

$$\begin{aligned} &AG_\sigma = G_{\bar\sigma} A; \quad AG'_\sigma = G'_{\bar\sigma} A; \quad G_{\bar\sigma} A G'_\sigma = A; \\ &G_{\bar\sigma}^{i\prime} A G_\sigma^i = A; \quad G_{\bar\sigma}^i A G_\sigma^{i\prime} = A; \quad i = 0, 1, 2, \ldots. \end{aligned} \tag{2.2}$$

Immediately, we have the following result.

**Theorem 2.1.** *$\mathscr{G} = \{\sigma \mid G'_{\bar\sigma(\sigma)} A G_\sigma = A$ for some $\bar\sigma(\sigma) \in S_m\}$ is a subgroup of $S_n$.*

The proof is easy and hence omitted.

We are interested in examining the status of the g-inverses of the matrix $A$ regarding this concept of invariance.

Let $A^-$ be any g-inverse of $A$. Observe that $A^-$ is of order $n \times m$ so that we may define $\bar{\sigma}$-invariance of $A^-$ (for some $\bar{\sigma} \in S_m$) as the property of $A^-$ satisfying the condition $G'_{\bar{\sigma}} A^- G_\sigma = A^-$ for some $\sigma \in S_n$. We might write $\sigma(\bar{\sigma})$ for $\sigma$.

When $A$ is square and non-singular, the relation $G'_{\bar{\sigma}} A G_\sigma = A$ implies and is also implied by $G_\sigma^{-1} A^{-1} (G'_{\bar{\sigma}})^{-1} = A^{-1}$, i.e. $G'_\sigma A^{-1} G_{\bar{\sigma}} = A^{-1}$ so that "$A$ is $\sigma$-invariant" $\Leftrightarrow$ "$A^{-1}$ is $\bar{\sigma}$-invariant", $\sigma$ and $\bar{\sigma}$ being the neutralizers of each other. We want to investigate how far this invariance property is retained by the g-inverses of $A$, in case $A$ is rectangular or square singular.

Towards this, we present the following results.

**Theorem 2.2.** *Whenever the matrix $A$ ($m \times n$) is $\sigma$-invariant for some $\sigma \in S_n$, there exists at least one $\bar{\sigma}$-invariant g-inverse of $A$, and further, there exists at least one reflexive $\bar{\sigma}$-invariant g-inverse of $A$ ($\bar{\sigma}$ being the neutralizer of $\sigma$).*

**Proof.** See Appendix.

The converse of Theorem 2.2 is embodied in the following.

**Theorem 2.3.** *The property of $\bar{\sigma}$-invariance of an arbitrary g-inverse of $A$ does not necessarily imply $\sigma$-invariance of $A$. Nor the $\bar{\sigma}$-invariance of an arbitrary reflexive g-inverse of $A$ implies $\sigma$-invariance of $A$ (here $\sigma$ is the neutralizer of $\bar{\sigma}$).*

**Proof.** Through counterexamples.

However, with the Moore–Penrose g-inverse $A^+$ (vide Rao and Mitra (1971)), the results are quite in order as we can see from the following theorem.

**Theorem 2.4.** *Whenever $A$ is $\sigma$-invariant, the M–P g-inverse $A^-$ is $\bar{\sigma}$-invariant and vice versa (here $\sigma$ and $\bar{\sigma}$ are the neutralizers of each other).*

**Proof.** It can be verified easily that $(AG_\sigma)^+ = G'_\sigma A^+$ and $(G_{\bar{\sigma}} A)^+ = A^+ G'_{\bar{\sigma}}$. Hence whenever $A$ is $\sigma$-invariant with $\bar{\sigma}$ as the neutralizer, from (2.2), we have $AG_\sigma = G_{\bar{\sigma}} A$, i.e. $(AG_\sigma)^+ = (G_{\bar{\sigma}} A)^+$, i.e. $G'_\sigma A^+ = A^+ G'_{\bar{\sigma}}$, i.e. $G'_\sigma A^+ G_{\bar{\sigma}} = A^+$. This proves $\bar{\sigma}$-invariance of $A^+$ with $\sigma$ as its neutralizer. Changing the role of $\sigma$ and $\bar{\sigma}$ and remembering that $(A^+)^+ = A$, we have that $G'_\sigma A^+ G_{\bar{\sigma}} = A^+$ implies $G'_{\bar{\sigma}} A G_\sigma = A$. This proves the converse. Hence the theorem.

**Corollary 2.1.** *Let*

$$\mathcal{G} = \{\sigma \mid \sigma \in S_n, \quad G'_{\bar{\sigma}} A G_\sigma = A \quad \text{for some } \bar{\sigma} \in S_m\}$$

*and*

$$\bar{\mathcal{G}} = \{\bar{\sigma} \mid \bar{\sigma} \in S_m, \quad G'_\sigma A^+ G_{\bar{\sigma}} = A^+ \text{ for some } \sigma \in S_n\}.$$

*Then $\sigma \in \mathcal{G} \Leftrightarrow \bar{\sigma} \in \bar{\mathcal{G}}$ ($\sigma$ and $\bar{\sigma}$ being the neutralizers of each other).*

In the next section, we specialize to the setting of block designs and display similar results involving the $C$-matrices.

## 3. Specific results for C-matrices

In the setting of block designs (underlying linear models) with, for example, one-way heterogeneity, we have to deal with matrices, simply known as $C$-matrices, of the following type:

$$C(v \times v) = r^{\delta} - Nk^{-\delta}N' \tag{3.1}$$

where $N(v \times b)$ is the incidence matrix of the design;

$$N\mathbf{1} = (r_1, r_2, \ldots, r_v)', \qquad N'\mathbf{1} = (k_1, k_2, \ldots, k_b)'$$

($\mathbf{1}$ being a column vector of 1's of suitable order); $r_i$ = number of replications of the $i$th variety, $k_j$ = $j$th block size, $r^{\delta} = \mathrm{Diag}(r_1, r_2, \ldots, r_v)$ and $k^{-\delta} = (k^{\delta})^{-1}$, $k^{\delta} = \mathrm{Diag}(k_1, k_2, \ldots, k_b)$. The $C$-matrix has the following properties in regard to its elements: $C_{ii} \geq 0$, $1 \leq i \leq v$; $C_{ij} \leq 0$, $1 \leq i \neq j \leq v$; $\sum_j C_{ij} = 0$, $1 \leq i \leq v$. $C$ is symmetric p.s.d. of rank at the most $v-1$. When rank $(C) = v-1$, we call $C$ a connected matrix (and the corresponding design is called a connected design). All treatment contrasts are estimable if and only if the design is connected. Althroughout, it will be assumed that the design is connected. An immediate consequence of this is that $C_{ii} > 0$ for all $i$. Hence we derive the following basic result.

**Proposition.** *If $C$ is $\sigma$-invariant (according to Definition 2.1), then, necessarily, $\bar{\sigma} = \sigma$ (i.e. $\sigma$ itself is its neutralizer).*

One may easily provide a proof of this proposition by considering a general representation of $\sigma$ in the form of a product of disjoint cyclic permutations and remembering that only the diagonal elements of the $C$-matrix are positive and the rest are all negative or zeroes.

**Note 3.1.** It must be noted that the g-inverses of the $C$-matrices need not lend themselves to such algebraic properties simply because a g-inverse of a $C$-matrix may not even be symmetric.

The following modified versions of the Theorems of section 2 may easily be verified in this context:

**Theorem 3.1.** *The class $\mathcal{G} = \{\sigma \mid \sigma \in S_v,\ G'_\sigma C G_\sigma = C\}$ forms a subgroup of $S_v$.*

**Theorem 3.2.** *Whenever the matrix $C$ is $\sigma$-invariant (for some $\sigma \in S_v$), there exists at least one $\sigma$-invariant g-inverse of $C$, and, further, there exists at least one reflexive $\sigma$-invariant g-inverse of $C$.*

**Theorem 3.3.** *The property of $\sigma$-invariance of an arbitrary g-inverse of C does not necessarily imply $\sigma$-invariance of C. Nor the $\sigma$-invariance of an arbitrary reflexive g-inverse of C implies $\sigma$-invariance of C.*

**Theorem 3.4.** *Whenever C is $\sigma$-invariant, the Moore–Penrose g-inverse $C^+$ of C is also $\sigma$-invariant and vice versa.*

Finally, we also have the following corollary.

**Corollary 3.1.** *If*

$$\mathcal{G}=\{\sigma \mid \sigma\in S_v,\ G_\sigma' C G_\sigma = C\} \quad \text{and} \quad \mathcal{G}^+=\{\sigma \mid \sigma\in S_v,\ G_\sigma' C^+ G_\sigma = C^+\},$$

*then $\mathcal{G}=\mathcal{G}^+$.*

**Note 3.2.** Theorem 3.4 essentially points out the fact that the study of properties of invariance of $C$-matrices (in the sense defined here) can be brought down to an equivalent study of such properties of their g-inverses of the Moore–Penrose type ($C^+$-matrices). In the next section, we intend to explore the relevance of this concept of *invariant* $C^+$-matrices (and the demonstrated implication of *invariance* of the corresponding $C$-matrices) in the *formal* determination of some complete classes of experiments for some suitably chosen invariant problems of linear inference.

## 4. Invariant problems of linear inference and some complete classes of experiments

In this section, we intend to apply the result reflected by Corollary 3.1 in the derivation of a *formal* general result on some complete classes of experiments for certain suitably chosen *invariant* linear inferential problems. The key papers for the present discussion are (a) a series of classical papers of the late Professor Kiefer (1958, 1959, 1974, 1975) in which, among other things, he has developed various optimality criteria and established a striking result on universal optimality of the BIBD's (and more generally of the BBD's) in the setting of block designs with one-way heterogeneity; (b) those of Sinha (1970, 1971, 1972, 1975) in which optimality of the BIBD's has been established for certain suitably chosen invariant problems of linear inference.

It is for the type of problems treated in Sinha with respect to various optimality criteria formulated by Kiefer that we want to construct *formally* certain complete classes of experiments.

Consider the following *non-singular* linear inferential problem:

$$\pi:\quad \eta(i\times 1)=L(i\times v)\tau(v\times 1),\quad L\mathbf{1}=0,\ \operatorname{rank}(L)=i. \tag{4.1}$$

Suppose $\pi$ is $\mathcal{G}$-invariant for some subgroup $\mathcal{G}$ of $S_v$ in the sense that for every $\sigma\in\mathcal{G}$, there exists one $\theta(\sigma)\in S_i$ such that

$$G_\theta' L G_\sigma = L \quad \text{(and hence)} \quad G_\theta L G_\sigma' = L. \tag{4.2}$$

For given $b$, $v$ and $k$, let now $\mathcal{D}$ be the class of all possible designs which provide inference on $\pi$ (in the sense of estimability of $\eta$). Using the design $d \in \mathcal{D}$, we get

$$\hat{\eta}_d = LC_d^+ Q_d$$

where

$Q_d$ = vector of 'adjusted treatment totals' based on observations under $d$.

Further, $D(\hat{\eta}_d) = \sigma^2 (LC_d^+ L')$ so that the information matrix is given by

$$\mathcal{J}_d = (1/\sigma^2)(LC_d^+ L')^{-1}.$$

Let now $\sigma$ ($\in \mathcal{G}$) modify $d$ to $d_\sigma$ by providing $N_{d_\sigma}$ (the incidence matrix under $d_\sigma$) = $G_\sigma' N_d$ where $N_d(v \times b)$ is the incidence matrix under $d$. Clearly, then, $C_{d_\sigma} = G_\sigma' C_d G_\sigma$ and, further,

$$C_{d_\sigma}^+ = G_\sigma' C_d^+ G_\sigma .$$

Hence, because of (4.2), we derive

$$\sigma^2 \mathcal{J}_d = (LC_d^+ L')^{-1} = \{(G_\theta L G_\sigma') C_d^+ (G_\sigma L' G_\theta')\}^{-1}$$
$$= [G_\theta \{L(G_\sigma' C_d^+ G_\sigma) L'\} G_\theta']^{-1} = G_\theta (LC_{d_\sigma}^+ L')^{-1} G_\theta'.$$

That is,

$$\mathcal{J}_{d_\sigma} = (1/\sigma^2) \cdot (LC_{d_\sigma}^+ L')^{-1} = (G_\theta' \mathcal{J}_d G_\theta) \tag{4.3}$$

whatever $\sigma \in \mathcal{G}$ with $\theta(\sigma)$ as its neutralizer.

Speaking in terms of the $C$-matrices, define $\mathcal{C} = \{C_d \mid d \in \mathcal{D}\}$ and extend $\mathcal{C}$ to $\bar{\mathcal{C}}$ as $\bar{\mathcal{C}} = \{C(v \times v) \mid C$ is real psd, $C\mathbf{1} = \mathbf{0}\}$. Consider now the '$\mathcal{G}$-invariant' subclass $\tilde{\mathcal{C}}$ of $\bar{\mathcal{C}}$ defined as $\tilde{\mathcal{C}} = \{C(v \times v) \mid C \in \bar{\mathcal{C}}$, $C$ is $\mathcal{G}$-invariant, i.e. $G_\sigma' C G_\sigma = C \ \forall \sigma \in \mathcal{G}\}$. (The proof of Theorem 2.2 indicates how one can get into $\mathcal{G}$-invariant matrices starting with arbitrary matrices.) Restricting to the class of *non-singularly estimable full-rank* $\mathcal{G}$-invariant problems, i.e. to problems $\pi$ as in (4.1) with $i = v - 1$ and satisfying (4.2), we deduce below a Complete Class Theorem which states that for every member $C_d \in \mathcal{C}$ there is a better member $C_{d_0} \in \tilde{\mathcal{C}}$ with respect to *any* convex symmetric criterion for inferring on $\pi$.

We proceed as follows. Define simply

$$\mathcal{J}_{d_0} = \sum_{\sigma \in \mathcal{G}} \mathcal{J}_{d_\sigma} / \#\mathcal{G}. \tag{4.4}$$

Then it is known (Kiefer (1975)) that $C_{d_0}$ is better than $C_d$ with respect to any convex symmetric criterion (which includes the commonly used A-, D- and E-optimality criteria) where $C_{d_0}$ is defined through

$$\mathcal{J}_{d_0} = (1/\sigma^2)(LC_{d_0}^+ L')^{-1}, \quad C_{d_0}(\text{symmetric})\mathbf{1} = \mathbf{0}. \tag{4.5}$$

It is easy to show the existence of $C_{d_0}$ satisfying (4.5) in general terms. However, it becomes unique whenever $i = v - 1$.

We show below that $C_{d_0}$ is $\mathscr{G}$-invariant, i.e.

$$G'_\sigma C_{d_0} G_\sigma = C_{d_0} \quad \forall \sigma \in \mathscr{G}.$$

For this, in view of Corollary 3.1, it is enough to establish the validity of

$$G'_\sigma C^+_{d_0} G_\sigma = C^+_{d_0} \quad \forall \sigma \in \mathscr{G}. \tag{4.6}$$

Towards this, we first take up the following lemma.

**Lemma 4.1.** *For every $\sigma \in \mathscr{G}$, $G'_{\bar\sigma} \mathscr{J}_{d_0} G_{\bar\sigma} = \mathscr{J}_{d_0}$ where $\bar\sigma$ satisfies $G'_{\bar\sigma} L G_\sigma = L$.*

**Proof.** Let $\sigma^*$ be a particular member of $\mathscr{G}$ with $\bar\sigma^*$ as its neutralizer so that we have $G'_{\bar\sigma^*} L G_{\sigma^*} = L$. Then, of course, $\sigma^*\mathscr{G} = \mathscr{G}$ and, further, $\bar\sigma\bar\sigma^* = \overline{\sigma\sigma^*}$ since for any $\sigma \in \mathscr{G}$,

$$\begin{aligned} L = G'_{\bar\sigma} L G_\sigma \;&\Rightarrow\; L = G'_{\bar\sigma^*} G'_{\bar\sigma} L G_\sigma G_{\sigma^*} \\ &\Rightarrow\; L = (G_{\bar\sigma} G_{\bar\sigma^*})' L (G_\sigma G_{\sigma^*}) = G'_{\bar\sigma\bar\sigma^*} L G_{\sigma\sigma^*}. \end{aligned}$$

Hence, using (4.4), for every $\sigma^* \in \mathscr{G}$,

$$\begin{aligned} G'_{\bar\sigma^*} \mathscr{J}_{d_0} G_{\bar\sigma^*} &= \sum_\sigma (G'_{\bar\sigma^*} G'_{\bar\sigma} \mathscr{J}_d G_{\bar\sigma} G_{\bar\sigma^*}) / \#\mathscr{G} = \sum_\sigma (G'_{\bar\sigma\bar\sigma^*} \mathscr{J}_d G_{\bar\sigma\bar\sigma^*}) / \#\mathscr{G} \\ &= \sum_{\sigma\sigma^*} (G'_{\overline{\sigma\sigma^*}} \mathscr{J}_d G_{\overline{\sigma\sigma^*}}) / \#\mathscr{G} = \sum_\sigma (G'_{\bar\sigma} \mathscr{J}_d G_{\bar\sigma}) / \#\mathscr{G} = \mathscr{J}_{d_0}. \end{aligned}$$

Hence the lemma.

**Note 4.1.** Observe that this result holds in general terms whatever $i \le v-1$.

We establish (4.6) through the following lemma.

**Lemma 4.2.** *Whenever the problem $\pi$ is non-singularly estimable $\mathscr{G}$-invariant and of full rank,* (4.6) *holds.*

**Proof.** By Lemma 1, $\mathscr{J}_{d_0}$ is $\bar\sigma$-invariant for every $\sigma \in \mathscr{G}$. Thus, if $G'_{\bar\sigma} L G_\sigma = L$, then $G'_{\bar\sigma} \mathscr{J}_{d_0} G_{\bar\sigma} = \mathscr{J}_{d_0}$, i.e. $G'_{\bar\sigma}(L C^+_{d_0} L') G_{\bar\sigma} = L C^+_{d_0} L'$, i.e.

$$L(G'_\sigma C^+_{d_0} G_\sigma - C^+_{d_0}) L' = 0 \quad \text{whatever } \sigma \in \mathscr{G}. \tag{4.7}$$

Observe that $L\mathbf{1} = 0$ and further that $L$ is of order $(v-1) \times v$ and of rank $v-1$. Hence, $L$ can be completed to a full rank square matrix by adjoining to it the row vector $\mathbf{1}'$. Since $(G'_\sigma C^+_{d_0} G_\sigma - C^+_{d_0})\mathbf{1} = 0$ (recall (4.5)), (4.7) is equivalent to

$$\begin{pmatrix} L \\ \mathbf{1}' \end{pmatrix} (G'_\sigma C^+_{d_0} G_\sigma - C^+_{d_0})(L' \mid \mathbf{1}) = 0 \quad \text{whatever } \sigma \in \mathscr{G} \tag{4.8}$$

which leads directly to (4.6).

**Note 4.2.** The crucial point in getting into this result is, of course, the observation made in the course of the demonstration of this lemma which puts (4.7) and (4.8) as equivalent. For $i<v-1$, $(L' \mid 1)$ is not a square matrix even though (4.7) and (4.8) are still equivalent. Hence the argument breaks down and (4.6) may not hold necessarily.

**Note 4.3.** Thus the invariance considerations developed here apply only to non-singularly estimable full-rank problems. See Sinha (1970) for other types of singularly estimable full-rank problems; see also Roy (1958) in this context.

**Note 4.4.** The result derived here does not, however, lead to the optimum design at any rate; it only indicates a way to look for the same within a smaller class of appropriately invariant $C$-matrices for such invariant problems. One may wonder regarding the arbitrariness of the members of the subclass $\tilde{\mathscr{C}}$ in general. However, it must be noted that they are not too arbitrary; each $C_{d_0}$ is derived from a particular $C_d$ which has a definite form. Hence, $C_{d_0}$ is *not* just *any* matrix in $\mathscr{C}$ with a $\mathscr{G}$-invariant form. For a specific optimality criterion, the matrices $C_{d_0}$ may be compared without much difficulty. In some cases, such a consideration nicely leads to the *optimum* design (the *best* among the $C_{d_0}$-matrices *but in* $\mathscr{C}$) as the following results demonstrate:

**Result 1** (Sinha, 1970, 1972). *Within the class of* ***binary****, proper,* ***equireplicate*** *connected designs for given b, v and k, a BIBD, if it exists, is optimal (with respect to any convex symmetric criterion) for any non-singularly estimable full-rank problem invariant under $S_{v-1}(i)$ (the symmetric subgroup of $(v-1)!$ permutations of all the treatments except for the treatment 'i').*

**Result 2** (Sinha, 1972). *Within the class of* ***binary****, proper, connected designs with $r_1$ (predetermined) as the number of replications of the first treatment, an IIGBBD (Inter- and Intra-Group Balanced Block Design) $\Delta(b, v, r_1, r', k, n_1=1, n_2=v-1, \lambda_2, \lambda_0)$, if it exists, is optimal (with respect to any convex symmetric criterion) for any non-singularly estimable full-rank problem invariant under $S_{v-1}(1)$.*

We end up this section by citing the following example of a non-trivial optimality result as a consequence of the complete-class result discussed here.

**Example.** For estimating $\eta((v-1)\times 1)=(\tau_1-\tau_2, \tau_1-\tau_3, \ldots, \tau_1-\tau_v)'$ for given $b=40$, $k=3$ and $v=9$, the *unique A*-optimum design within the class of *binary* proper connected designs is an IIGBBD with the following parameters: $b=40$, $k=3$, $v=9$, $r_1=32$, $r_2=11$, $\lambda_0=8$, $\lambda_2=2$. The details of the calculations are to be found elsewhere (Sinha (1980)). It is not difficult to construct this design practically. (See Sinha and Sinha (1969) in this context.)

## 5. Concluding remarks

(a) Much yet, however, remains as undone towards finding optimum designs even for invariant problems simply because the 'no-design-related' subclass $\bar{\mathscr{C}} - \mathscr{C}$ may contain the 'best' $C$-matrix and, consequently, may complicate the whole search.

(b) Almost a decade ago, while the author was developing invariance considerations in the study of optimal designs, Professor S.K. Chatterjee of the Department of Statistics, Calcutta University initiated a similar consideration to the author concerning the role of invariance towards attaining simplicity in the analysis of block designs. The author persued on the latter topic only recently (Sinha (1982)); however, he always felt that invariance considerations on the $C^+$-matrices could be profitably used elsewhere also. It is only recently that those ideas are reflected through the general results and the specific results on g-inverses incorporated in this article. The author records his deep gratitude to Professor Chatterjee.

(c) Simplicity criterion in the analysis of block designs has taken a definite shape by now with the work initiated by Tocher in as early as 1952 and persued only in recent years by Calinski (1971) and Saha (1976) among others (detailed references are omitted here). However, invariance considerations in this connection have only been recently taken up (Sinha (1982)). Another look to the existing simplicity criterion has also been reported by the author (Saha and Sinha (1981)).

(d) An earlier draft of this article was presented (by title) at the 'Research Conference on Variance Components, g-Inverses and Their Applications' under the title 'Structurally Invariant g-Inverse Matrices and Their Uses in the Search for Optimal Designs', held at the Department of Statistics of Ohio State University in June 1979.

## 6. Appendix

**Proof of Theorem 2.2.** Let $G(n \times m)$ satisfy $AGA = A$. From $G$, we construct

$$G^{(i)} = G_\sigma^{i\prime} G G_{\bar\sigma}^{i}, \quad i = 0, 1, 2, \ldots, p-1, \tag{6.1}$$

where $p$ = l.c.m.(order of $\sigma$, order of $\bar\sigma$). Using (2.2), one can now show

$$AG^{(i)}A = A, \quad i = 0, 1, 2, \ldots, p-1, \tag{6.2}$$

so that every $G^{(i)}$ is a g-inverse of $A$. Now we form

$$G^* = \sum_i G^{(i)}/p \tag{6.3}$$

and claim that $G^*$ is a $\bar\sigma$-invariant g-inverse of $A$ with $\sigma$ as its neutralizer. The verification is left to the reader. The most general form of such a $\bar\sigma$-invariant g-inverse of $A$ would be given by

$$A^- = \frac{1}{p} \sum_i G_\sigma^{i\prime}(G + U - GAUAG)G_{\bar\sigma}^{i}. \tag{6.4}$$

Here $U(n \times m)$ is an arbitrary matrix (Rao and Mitra (1971)). It is now enough to form

$$G^{**} = G^{*}AG^{*} \tag{6.5}$$

with $G^{*}$ as in (6.3), to prove the existence of a reflexive $\sigma$-invariant g-inverse of $A$ with $\sigma$ as its neutralizer. And, yet, the most general form would be given by

$$G^{**} = A_1^{-}AA_2^{-} \tag{6.6}$$

where $A_j^{-}$ $(j = 1, 2)$ are matrices as in (6.4).

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