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Representations of Best Linear Unbiased Estimators in the Gauss–Markoff Model with a Singular Dispersion Matrix

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In the general Gauss-Markoff model $(\mathbf{Y}, \mathbf{X\beta}, \sigma^2\mathbf{V})$, when $\mathbf{V}$ is singular, there exist linear functions of $\mathbf{Y}$ which vanish with probability 1 imposing some restrictions on $\mathbf{Y}$ as well as on the unknown $\boldsymbol{\beta}$. In all earlier work on linear estimation, representations of best-linear unbiased estimators (BLUE's) are obtained under the assumption: "$\mathbf{L}'\mathbf{Y}$ is unbiased for $\mathbf{X\beta} \Rightarrow \mathbf{L}'\mathbf{X} = \mathbf{X}$." Such a condition is not, however, necessary. The present paper provides all possible representations of the BLUE's some of which violate the condition $\mathbf{L}'\mathbf{X} = \mathbf{X}$. Representations of $\mathbf{V}$ for given classes of BLUE's are also obtained.

1. Introduction

Let us consider the general Gauss–Markoff Model (GGM),

$$(\mathbf{Y}, \mathbf{X\beta}, \sigma^2\mathbf{V}) \tag{1.1}$$

where $\mathbf{Y}$ is a vector of random variables, $\mathrm{E}(\mathbf{Y}) = \mathbf{X\beta}$ and $\mathrm{D}(\mathbf{Y}) = \sigma^2\mathbf{V}$. The operators E and D stand for expectation and dispersion respectively. The parameters $\boldsymbol{\beta}$ and σ^2 are unknown. The matrices $\mathbf{X}$ and $\mathbf{V}$ are known, but $\mathbf{X}$ may be deficient in rank implying that only certain linear combinations of $\boldsymbol{\beta}$ are estimable (or identifiable) and $\mathbf{V}$ may be singular implying that the random variables are linearly dependent. In all earlier work of the author, as well as of the other writers on the subject, the BLUE (Best Linear Unbiased Estimator) of a parametric function $\mathbf{p}'\boldsymbol{\beta}$ is defined as a linear function $\mathbf{L}'\mathbf{Y}$ where $\mathbf{L}$ is such that $\mathbf{L}'\mathbf{X} = \mathbf{p}'$ and $\mathbf{L}'\mathbf{VL}$ is a minimum. It has been pointed out by the author [9], that the condition $\mathbf{L}'\mathbf{X} = \mathbf{p}'$ is not necessary for unbiasedness when $\mathbf{V}$ is singular. This is due to the existence of linear functions of $\mathbf{Y}$ which are zero with prob-

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ability] that can be added to any estimator without altering its value but violating the condition $\mathbf{L'X} = \mathbf{p'}$. The existence of such zero functions was recognized by various writers (see, for instance Goldman and Zelen [1] who were the first to consider the case of singular **V**), but the consequences have not been followed up, quite rightly perhaps. However, as the author has shown [9], the earlier approaches which implicitly involved the necessity of the condition $\mathbf{L'X} = \mathbf{p'}$ do provide BLUE's although it does not answer the wider problem of providing all *representations* of the BLUE's, which is of some theoretical interest. The object of the present paper is to consider this wider problem, and also to provide more rigorous statements of theorems proved in earlier work on the subject.

2. The GGM Model

The following notations are used.

$\mathbf{A'}$ = Transpose of **A**. $R(\mathbf{A})$ = Rank of **A**. $\mathcal{M}(\mathbf{A})$ is the linear space generated by the columns of **A**. $\mathbf{A}^{\perp}$ is any matrix of maximum rank such that $\mathbf{A'A}^{\perp} = \mathbf{0}$. $\mathbf{A}^{-}$ is a g-inverse of **A**. Matrices **A** and **B** are said to be disjoint if the intersection of $\mathcal{M}(\mathbf{A})$ and $\mathcal{M}(\mathbf{B})$ consists of the null vector only. $(\mathbf{A} : \mathbf{B})$ is a partitioned matrix with **A** and **B** as the partitions.

Some results in matrix algebra used in the paper are stated below.

A(2.1). Let **V** be an nnd (nonnegative definite) matrix, and **X** be any matrix with the same number of rows as in **V**. Then **X** and **VZ**, where $\mathbf{Z} = \mathbf{X}^{\perp}$ are disjoint.

A(2.2). One representation of $\mathbf{X}^{\perp}$ is

$$\mathbf{I} - (\mathbf{X'})^{-}\mathbf{X'} \tag{2.1}$$

for any choice of the g-inverse.

A(2.3). The projection operator $P_{\mathbf{X}}$ on $\mathcal{M}(\mathbf{X})$ has the representation

$$P_{\mathbf{X}} = \mathbf{X}(\mathbf{X'}\Lambda\mathbf{X})^{-}\mathbf{X'}\Lambda \tag{2.2}$$

where Λ is the pd (positive definite) matrix defining the inner product of vectors $(\mathbf{x}, \mathbf{y}) = \mathbf{y'}\Lambda\mathbf{x}$.

A(2.4). If $\mathbf{T}^{-}$ is a g-inverse of **T** then

$$\mathbf{TT}^{-}\mathbf{C} = \mathbf{C} \Leftrightarrow \mathcal{M}(\mathbf{C}) \subset \mathcal{M}(\mathbf{T}) \tag{2.3}$$

For further results on g-inverse reference may be made to Rao and Mitra [11].

A(2.5). Let $\mathbf{V}$ be an nnd matrix and $\mathbf{U}$ be any matrix such that $\mathbf{T} = \mathbf{V} + \mathbf{XUX}'$ has the same rank as $(\mathbf{V} : \mathbf{X})$. Further let $\mathbf{Z} = \mathbf{X}^{\perp}$. Then

$$\mathscr{M}(\mathbf{VZ} : \mathbf{X}) = \mathscr{M}(\mathbf{V} : \mathbf{X}) = \mathscr{M}(\mathbf{T}), \tag{2.4}$$

$$\mathscr{M}[(\mathbf{VZ})^{\perp}] = \mathscr{M}(\mathbf{T}^{-}\mathbf{X} : \mathbf{I} - \mathbf{T}^{-}\mathbf{T}). \tag{2.5}$$

The result (2.4) is easily proved. To establish (2.5), consider a vector $\boldsymbol{\lambda}$ such that

$$\boldsymbol{\lambda}'\mathbf{T}^{-}\mathbf{X} = \mathbf{0}, \qquad \boldsymbol{\lambda}'(\mathbf{I} - \mathbf{T}^{-}\mathbf{T}) = \mathbf{0}. \tag{2.6}$$

$\boldsymbol{\lambda}'\mathbf{T}^{-}\mathbf{X} = \mathbf{0} \Rightarrow \boldsymbol{\lambda}'\mathbf{T}^{-} = \boldsymbol{\mu}'\mathbf{Z}'$ for some choice of vector $\boldsymbol{\mu}$. Substituting in $\boldsymbol{\lambda}' = \boldsymbol{\lambda}'\mathbf{T}^{-}\mathbf{T}$, we have $\boldsymbol{\lambda}' = \boldsymbol{\mu}'\mathbf{Z}'\mathbf{T} = \boldsymbol{\mu}'\mathbf{Z}'\mathbf{V}$. Hence $\boldsymbol{\lambda}'(\mathbf{VZ})^{\perp} = \mathbf{0} \Rightarrow$

$$\mathscr{M}[(\mathbf{VZ})^{\perp}] \subset \mathscr{M}(\mathbf{T}^{-}\mathbf{X} : \mathbf{I} - \mathbf{T}^{-}\mathbf{T}). \tag{2.7}$$

To prove the other way, consider $\boldsymbol{\lambda}'(\mathbf{VZ})^{\perp} = \mathbf{0} \Rightarrow \boldsymbol{\lambda}' = \boldsymbol{\mu}'\mathbf{Z}'\mathbf{V} = \boldsymbol{\mu}'\mathbf{Z}'\mathbf{T}$. Then $\boldsymbol{\lambda}'\mathbf{T}^{-}\mathbf{X} = \boldsymbol{\mu}'\mathbf{Z}'\mathbf{TT}^{-}\mathbf{X} = \boldsymbol{\mu}'\mathbf{Z}'\mathbf{X} = \mathbf{0}$ using (2.3), $\mathbf{TT}^{-}\mathbf{X} = \mathbf{X}$ since $\mathscr{M}(\mathbf{X}) \subset \mathscr{M}(\mathbf{T})$. Also $\boldsymbol{\lambda}'(\mathbf{I} - \mathbf{T}^{-}\mathbf{T}) = \boldsymbol{\mu}'\mathbf{Z}'\mathbf{T}(\mathbf{I} - \mathbf{T}^{-}\mathbf{T}) = \mathbf{0}$. Thus

$$\mathscr{M}[(\mathbf{VZ})^{\perp}] \supset \mathscr{M}(\mathbf{T}^{-}\mathbf{X} : \mathbf{I} - \mathbf{T}^{-}\mathbf{T}) \tag{2.8}$$

and the proposition is proved.

A(2.6). The columns of the matrices $(\mathbf{X} : \mathbf{VZ})^{\perp}$, $\mathbf{Z}(\mathbf{Z}'\mathbf{V})^{\perp}$ and $\mathbf{K}(\mathbf{K}'\mathbf{X})^{-}$ where $\mathbf{K} = \mathbf{V}^{\perp}$ generate the same space.

We state and prove a number of results concerning the GGM (General Gauss-Markoff) Model (1.1). It may be noted [8] that the GGM model includes the case of restrictions on parameters.

Lemma 2.1. *Let* $(\mathbf{Y}, \mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{V})$ *be a GGM model. Then*

$$\mathbf{Y} \in \mathscr{M}(\mathbf{V} : \mathbf{X}) \quad \textit{with probability } 1 \tag{2.9}$$

where $(\mathbf{V} : \mathbf{X})$ *denotes the partitioned matrix.*

The result (2.9) follows, since $\mathbf{L}'(\mathbf{V} : \mathbf{X}) = \mathbf{0} \Rightarrow E(\mathbf{L}'\mathbf{Y}) = 0$ and $V(\mathbf{L}'\mathbf{Y}) = 0$, i.e., $\mathbf{L}'\mathbf{Y} = 0$ with probability 1. We say the model $(\mathbf{Y}, \mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{V})$ is consistent if $\mathbf{L}'\mathbf{V} = 0$, $\mathbf{L}'\mathbf{X} = \mathbf{0} \Rightarrow \mathbf{L}'\mathbf{Y} = 0$.

Lemma 2.1 specifies that $\mathbf{Y}$ belongs to the space generated by the columns of $\mathbf{V}$ and $\mathbf{X}$, which is the only statement that can be made when $\mathbf{Y}$ is not observed. However, when we have an observation on $\mathbf{Y}$ we have the necessary information to determine the subspace of $\mathscr{M}(\mathbf{V} : \mathbf{X})$ to which the random variable $\mathbf{Y}$ belongs. The answer is given in Lemma 2.2.

LEMMA 2.2. *Let* $\mathbf{K} = \mathbf{V}^{\perp}$. *Then*

$$\mathbf{K'Y} = \mathbf{d} \quad \textit{with probability } 1, \tag{2.10}$$

$$\mathbf{K'X}\boldsymbol{\beta} = \mathbf{d}, \tag{2.11}$$

where d *is a constant vector.*

The results (2.10) and (2.11) follow since $\mathbf{K'V} = \mathbf{0} \Rightarrow D(\mathbf{K'Y}) = \mathbf{0} \Rightarrow \mathbf{K'Y} = E(\mathbf{K'Y})$ with probability 1, where D denotes the dispersion and E the expectation operators. The expressions (2.10) and (2.11) are in the nature of restrictions on the random variable $\mathbf{Y}$ and the unknown parameter $\boldsymbol{\beta}$. However, they are *known* only when we have an observation on $\mathbf{Y}$. If $\mathbf{V}$ is $n \times n$ matrix, then $R(\mathbf{K}) = n - R(\mathbf{V})$.

LEMMA 2.3. *Let* $\mathbf{N} = \mathbf{Kd}^{\perp}$. *Then*

$$\mathbf{N'Y} = \mathbf{0} \quad \textit{with probability } 1, \tag{2.12}$$

$$\mathbf{N'X}\boldsymbol{\beta} = \mathbf{0}. \tag{2.13}$$

The results (2.12) and (2.13) are consequences of (2.10) and (2.11). They show that the random variable $\mathbf{Y}$ is, in fact, confined to a subspace and that the singularity of $\mathbf{V}$ induces some natural restrictions on the parameter $\boldsymbol{\beta}$. It may be seen that $R(\mathbf{N}) = n - R(\mathbf{V}) - 1$ if $\mathbf{d} \neq \mathbf{0}$ and $R(\mathbf{N}) = n - R(\mathbf{V})$ if $\mathbf{d} = \mathbf{0}$.

LEMMA 2.4. *Let* $\mathbf{S} = (\mathbf{X'N})^{\perp}$. *Then*

$$\mathbf{Y} \in \mathscr{M}(\mathbf{V} : \mathbf{S}). \tag{2.14}$$

The result (2.14) follows, since $\mathbf{L'V} = \mathbf{0}$, $\mathbf{L'XS} = \mathbf{0} \Rightarrow E(\mathbf{L'Y}) = 0$, $V(\mathbf{L'Y}) = 0 \Rightarrow \mathbf{L'Y} = 0$. Thus the knowledge of an observation on $\mathbf{Y}$ enables us to specify the particular subspace of (2.9) to which the random variable belongs.

LEMMA 2.5. *If* $\mathbf{L'Y}$ *is unbiased for* $\mathbf{p}'\boldsymbol{\beta}$, *then it is necessary and sufficient that*

$$\mathbf{X'L} - \mathbf{p} \in \mathscr{M}(\mathbf{X'N}) \Leftrightarrow \mathbf{S'X'L} = \mathbf{S'p} \tag{2.15}$$

where $\mathbf{S} = (\mathbf{X'N})^{\perp}$ *or, there exists a vector* $\boldsymbol{\lambda}$ *such that*

$$\mathbf{X'}(\mathbf{L} - \mathbf{N}\boldsymbol{\lambda}) = \mathbf{p}. \tag{2.16}$$

If $E(\mathbf{L'Y}) = \mathbf{p}'\boldsymbol{\beta}$, then $\mathbf{L'X}\boldsymbol{\beta} = \mathbf{p}'\boldsymbol{\beta}$ when $\boldsymbol{\beta}$ is subject to $\mathbf{N'X}\boldsymbol{\beta} = \mathbf{0}$, i.e., there exists a vector λ such that $L'X - p' = \lambda'N'X$ which proves (2.15) and (2.16). Conversely (2.16) $\Rightarrow L'X\beta = p'\beta$, i.e., $L'y$ is unbiased for $p'\beta$.

We note that the usual condition of unbiasedness employed in all earlier work on linear estimation is $X'L = p$, which is only sufficient.

LEMMA 2.6. *The condition* $X'L = p$ *is necessary and sufficient for* $L'Y$ *to be unbiased for* $p'\beta$ *iff* $N'X = 0$.

The result follows from (2.16). It may be noted that $N'X$ can vanish without V being nonsingular. However, $R(V : X)$ can utmost be $R(V) + 1$.

LEMMA 2.7. $L'Y$ *is unbiased for zero iff*

$$L \in \mathcal{M}(N : Z) = \mathcal{M}[(XS)^{\perp}] \tag{2.17}$$

where $Z = X^{\perp}$ *and* $S = (X'N)^{\perp}$.

The condition $E(L'Y) = 0 \Rightarrow$ that there exists a λ such that $L'X = \lambda'N'X \Rightarrow$ (2.17). The converse easily follows.

From the results of Lemmas 2.4–2.7, it follows that the linear function $L'Y$ is unbiased for the whole set of parametric functions $(L'X + \lambda'N'X)\beta$ where λ is arbitrary, which reduces to a unique function if $N'X = 0$. The set of vectors $\{p\}$ such that $E(L'Y) = p'\beta$ for given L is represented by $\mathscr{P}_L$, or more explicitly by $\mathscr{P}_L(V)$.

LEMMA 2.8. *If* $L'Y$ *is an unbiased estimator of* $p'\beta$, *then there exists a vector* M *such that*

$$\begin{aligned} L'Y &= M'Y \text{ with probability } 1, \\ M'X &= p. \end{aligned} \tag{2.18}$$

Further if $A'Y$ *is unbiased for zero, then there exists a vector* B *such that*

$$\begin{aligned} A'X &= B'Y \text{ with probability } 1, \\ B'X &= 0. \end{aligned} \tag{2.19}$$

From (2.16), $p = X'(L - N\lambda)$. Then choose $M = L - N\lambda$, which establishes (2.18). Similarly (2.19) is proved.

Lemma 2.7 is important. It establishes that to obtain the BLUE of $p'\beta$, we need only determine L such that $L'VL$ is a minimum subject to $L'X = p$, although it is not a necessary condition. Such an approach *does provide* a formula for computing the BLUE of an estimable function $p'\beta$, which was the object of all earlier work, but does not necessarily give *all representations* of the BLUE.

We give a *wider* definition of the BLUE and denote it by BLUE(W), retaining the abbreviation BLUE for the traditional type investigated in all earlier work.

DEFINITION 1. $\mathbf{L'Y}$ is said to be the BLUE(W) of $\mathbf{p'\beta}$ iff $\mathbf{L'VL}$ is a minimum subject to the condition $(\mathbf{X'L} - \mathbf{p}) \in \mathscr{M}(\mathbf{X'N})$.

DEFINITION 2. $\mathbf{L'Y}$ is said to be the BLUE of $\mathbf{p'\beta}$ iff $\mathbf{L'VL}$ is a minimum subject to the condition $\mathbf{X'L} = \mathbf{p}$.

We note that if $\mathbf{L}_1'\mathbf{Y}$ and $\mathbf{L}_2'\mathbf{Y}$ are two representations of the BLUE or BLUE(W) of the same parametric function, then $(\mathbf{L}_1 - \mathbf{L}_2)'\mathbf{Y} = 0$ with probability 1; as the minimum variance linear unbiased estimator is unique. The set of all vectors $\mathbf{L}$ giving the BLUE of $\mathbf{p'\beta}$ is represented by $\mathscr{L}_p^{\mathbf{V}}$ or simply by $\mathscr{L}_p$ when $\mathbf{V}$ is understood, and the set giving the BLUE(W)'s of $\mathbf{p'\beta}$ by $\mathscr{L}_p^{\mathbf{V}}(\mathrm{W})$ or simply by $\mathscr{L}_p(\mathrm{W})$. Also $\mathscr{L}^{\mathbf{V}}$ stands for the set of all vectors providing BLUE's or BLUE(W)'s of some parametric function or other.

3. REPRESENTATIONS OF BLUE'S AND BLUE (W)'S

The following theorem, which has been repeatedly used in all earlier work of the author since 1945, is basic in the theory of linear estimation.

THEOREM 3.1. *The linear function* $\mathbf{L'Y}$ *is the BLUE(W) of* $\mathbf{p'\beta}$, $(\mathbf{p} \in \mathscr{P}_L)$ *iff*

$$\mathbf{L'VZ} = 0 \tag{3.1}$$

where $\mathbf{Z} = \mathbf{X}^{\perp}$ provided $\sigma^2 > 0$.

The result (3.1) follows by applying Theorem 1, Section 5a.2 in Chapter 5 of Rao [5]: a statistic is the minimum variance unbiased estimator of its expected value if it is uncorrelated with statistics unbiased for zero. The origin of the theorem quoted in [5] can be traced to Fisher [12] who established it in the context of consistency and efficiency. The proof is the same for unbiasedness and minimum variance.

Using (2.17), $\mathbf{M'Y}$ is unbiased for zero iff $\mathbf{M} = \mathbf{Z}\boldsymbol{\alpha} + \mathbf{N}\boldsymbol{\delta}$ for some $\boldsymbol{\alpha}$ and $\boldsymbol{\delta}$. Then

$$\begin{aligned}\mathrm{cov}(\mathbf{L'Y}, \mathbf{M'Y}) &= \sigma^2\mathbf{L'VM}\\ &= \sigma^2\mathbf{L'V}(\mathbf{Z}\boldsymbol{\alpha} + \mathbf{N}\boldsymbol{\delta}) = \sigma^2\mathbf{L'VZ}\boldsymbol{\alpha} = 0.\end{aligned} \tag{3.2}$$

Since (3.2) holds for all $\boldsymbol{\alpha}$, we must have (3.1) if $\sigma^2 > 0$.

Note 1. The basic result (3.1) used in all earlier work is the same whether we are searching for a BLUE or BLUE(W).

Note 2. A necessary and sufficient condition for a set of linear functions $\mathbf{C'Y}$ to be BLUE(W)'s is

$$\mathbf{C'VZ} = \mathbf{0}, \tag{3.3}$$

which can be written in the alternative forms

$$\mathscr{M}(\mathbf{C}) \subset \mathscr{M}(\mathbf{VZ})^{\perp} \quad \text{or} \quad \mathscr{M}(\mathbf{VC}) \subset \mathscr{M}(\mathbf{X}). \tag{3.4}$$

We shall use (3.3) in obtaining representations of $\mathbf{C}$ given $\mathbf{V}$ and of $\mathbf{V}$ given $\mathbf{C}$.

COROLLARY 3.2. *Let $\mathscr{L}^{\mathbf{V}}$ denote the set of vectors* $\{\mathbf{L}\}$ *such that* $\mathbf{L'Y}$ *is the BLUE(W) of some parametric function. Then*

$$\mathscr{L}^{\mathbf{V}} = \mathscr{M}[(\mathbf{VZ})^{\perp}] \tag{3.5}$$

$$= \mathscr{M}(\mathbf{T^{-}X} : \mathbf{I} - \mathbf{T^{-}T}) \tag{3.6}$$

where $\mathbf{T} = (\mathbf{V} + \mathbf{XUX'})$, $\mathbf{U}$ *is such that* $R(\mathbf{T}) = R(\mathbf{V} : \mathbf{X})$, *and* $\mathbf{T}^{-}$ *is any g-inverse.*

The result (3.5) follows from (3.1) and the result (3.6) from the equivalence result established in (2.5).

COROLLARY 3.3. $\mathbf{L'Y}$ *is the BLUE of* $\mathbf{p}'\boldsymbol{\beta}$ *iff*

$$\mathbf{X'L} = \mathbf{p}, \qquad \mathbf{Z'VL} = \mathbf{0} \tag{3.7}$$

and is the BLUE(W) of $\mathbf{p}'\boldsymbol{\beta}$ *iff*

$$\mathbf{S'X'L} = \mathbf{S'p}, \qquad \mathbf{Z'VL} = \mathbf{0} \tag{3.8}$$

where $\mathbf{S} = (\mathbf{X'N})^{\perp}$.

COROLLARY 3.4. *The BLUE of* $\mathbf{p}'\boldsymbol{\beta}$ *has a unique representation iff* $R(\mathbf{V} : \mathbf{X}) = n$, *the order of* $\mathbf{V}$. *The BLUE(W) of* $\mathbf{p}'\boldsymbol{\beta}$ *has a unique representation iff* $\mathbf{X'N} \quad 0$ *in addition to* $R(\mathbf{V} : \mathbf{X}) = n$.

The results follow from those of Corollary 3.1 using the result $R(\mathbf{VZ} : \mathbf{X}) = R(\mathbf{V} : \mathbf{X})$, and in fact $\mathscr{M}(\mathbf{VZ} : \mathbf{X}) = \mathscr{M}(\mathbf{V} : \mathbf{X})$, which is easily established. It is interesting to note that unique representations of BLUE's are possible even when $\mathbf{V}$ is singular.

COROLLARY 3.5. *Let* $\mathscr{L}_{\mathbf{p}}$ *denote the set of all vectors* $\mathbf{L}$ *such that* $\mathbf{L'Y}$ *is the BLUE of* $\mathbf{p}'\boldsymbol{\beta}$. *Then* $\mathbf{L} \in \mathscr{L}_{\mathbf{p}}$ *can be written in two alternate forms*

$$\mathbf{L} = \mathbf{L}_0 + \mathbf{Z}(\mathbf{Z'V})^{\perp}\boldsymbol{\lambda}, \tag{3.9}$$

$$= \mathbf{T^{-}X}(\mathbf{X'T^{-}X})^{-}\mathbf{p} + (\mathbf{I} - \mathbf{T^{-}T})\boldsymbol{\nu} \tag{3.10}$$

where L_0 *is a particular* $L \in \mathscr{L}_p$, T *and* T^- *are as defined in Corollary* 3.2, *and* λ, ν *are arbitrary.*

We observe that L satisfies Eqs. (3.7), $X'L = p$, $Z'VL = 0$, a general solution of which can be written in the forms (3.9) and (3.10) using the result (2.5).

Note 1. The BLUE of $p'\beta$, using the formula (3.10) is

$$Y'T^-X(X'T^-X)^-p \tag{3.11}$$

observing that the second term $Y'(I - T^-T)\nu = 0$ with probability 1 since, by Lemma 2.1, $Y \in \mathscr{M}(V : X) = \mathscr{M}(T)$. The expression (3.11) can also be written as

$$p'(X'(T')^-X)^-X'(T')^-Y \tag{3.12}$$

which is derived as a suitably defined least-squares estimator in [8] using an asymmetric matrix T and in [10] using a symmetric matrix T.

Note 2. If

$$\begin{pmatrix} V & X \\ X' & 0 \end{pmatrix}^- = \begin{pmatrix} C_1 & C_2 \\ C_3 & -C_4 \end{pmatrix} \tag{3.13}$$

is one choice of a g-inverse, then L in (3.9) can also be written as

$$L = C_2p + (I - C_1V - C_2X')\lambda_1 + C_1X\lambda_2, \tag{3.14}$$

where λ_1, λ_2 are arbitrary, and C_2p corresponds to a particular solution. The equivalence of (3.9) and (3.14) is easily established using the definition of a generalized inverse. Further the BLUE of $p'\beta$ is simply $p'C_2'Y$, since the contribution by the other terms in (3.14) in zero, which is derived in [8] as the inverse partitioned matrix (IPM) approach to linear estimation.

Note 3. The representations (3.9) and (3.10) consist of two parts

$$L = L_0 + L_1, \tag{3.15}$$

where $L_1'V = 0$, $L_1'X = 0 \Rightarrow L_1'Y = 0$ with probability 1, and $L_0'X = p'$.

COROLLARY 3.6. *Let* $\mathscr{L}_p(W)$ *denote the set of vectors* L *such that* $L'Y$ *is the* BLUE(W) *of* $p'\beta$. *Then* $L \in \mathscr{L}_p(W)$ *has the representation*

$$L = L_0 + L_1 + N\mu \tag{3.16}$$

where L_0 *and* L_1 *are the same as in* (3.9) *or* (3.10), N *is as defined in* (2.12) *and* μ *is arbitrary.*

The set (3.16) is precisely the set of all solutions of Eqs. (3.8), $S'X'L = S'p$, $Z'VL = 0$, defining the BLUE(W) of $p'\beta$.

Note that in (3.16), $L = L_0 + L_1 + L_2$, where $L_2'Y = 0$ with probability 1 and $L_2'X$ is not necessarily zero, $L_1Y = 0$ with probability 1 and $L_1'X = 0$, and $L_0'X = p'$. The sets of solutions (3.9) or (3.10) and (3.16) are the same if $N'X = 0$.

COROLLARY 3.7. *The set of all matrices* {C} *such that* $C'Y$ *is the BLUE of* $X\beta$ *is given by*

$$C = I + Z\Lambda, \tag{3.17}$$

where Λ *is an arbitrary solution of* $-Z'V = Z'VZ\Lambda$, *or*

$$\begin{aligned} C &= C_0 + Z(Z'V)^{\perp} M, \\ &= C_0 + (V : X)^{\perp} M, \end{aligned} \tag{3.18}$$

where C_0 *is a particular solution and* M *is arbitrary, or*

$$C = T^- X(X'T^- X)^- X' + (I - T^- T)F, \tag{3.19}$$

where F *is arbitrary*, T *and* T^- *being as defined in* (3.10).

If $C'Y$ is the BLUE of $X\beta$, then C satisfies the equations

$$\begin{aligned} X'C &= X' \\ Z'VC &= 0 \end{aligned} \tag{3.20}$$

the general solution of which can be represented in the alternative forms mentioned in the theorem.

COROLLARY 3.8. *The set of all matrices* {C} *such that* $C'Y$ *is the BLUE(W) of* $X\beta$ *is obtained by adding* NB, *where* B *is arbitrary, to any of the expressions* (3.17), (3.18) *or* (3.19).

The result of Corollary 3.8 is easily established.

Note 1. The equation $Z'VZ\Lambda = -Z'V$ (of Corollary 3.7) admits solutions for Λ and a particular solution is

$$\Lambda = -(Z'VZ)^- Z'V, \tag{3.21}$$

so that a particular choice C_0 in (3.18) is, using (3.17),

$$I - Z(Z'VZ)^- Z'V. \tag{3.22}$$

Then the BLUE of $\mathbf{X\beta}$ can be written as

$$(\mathbf{I} - \mathbf{VZ}(\mathbf{Z'VZ})^{-}\mathbf{Z'})\mathbf{Y}. \tag{3.23}$$

We have also the alternative choice provided by (3.19)

$$\mathbf{X}(\mathbf{X'}(\mathbf{T'})^{-}\mathbf{X})^{-}\mathbf{X'}(\mathbf{T'})^{-}\mathbf{Y}. \tag{3.24}$$

The multiplying matrices in (3.23) and (3.24) are equal when the columns of $\mathbf{X}$ and $\mathbf{VZ}$ span the whole space, thus providing an interesting identity used in an earlier publication of the author [6]. The expressions (3.23) and (3.24) have the same value although the multiplying matrices (of $\mathbf{Y}$) are not the same.

Note 2. For any $\mathbf{\Lambda}$ in (3.17)

$$\mathbf{VC} = \mathbf{V} + \mathbf{VZ\Lambda} = \mathbf{\Lambda'Z'VZ\Lambda}, \tag{3.25}$$

so that $\mathbf{VC}$ is symmetric. But $\mathbf{C'}$ is not idempotent unless $\mathbf{\Lambda}$ is chosen such that $R(\mathbf{C}) = R(\mathbf{X})$ as shown in Theorem 5.1 of [8]. Then we may ask for solutions of $\mathbf{C}$ satisfying the equations

$$\begin{aligned} R(\mathbf{C}) &= R(\mathbf{X}), \\ \mathbf{X'C} &= \mathbf{X'}, \\ \mathbf{Z'VC} &= \mathbf{0}. \end{aligned} \tag{3.25}$$

Any solution of (3.25) is of the form $\mathbf{C} = \mathbf{C}_0 + \mathbf{BX'}$ where

$$\mathbf{C}_0 = (\mathbf{T'})^{-}\mathbf{X}(\mathbf{X'}(\mathbf{T'})^{-}\mathbf{X})^{-}\mathbf{X'}, \tag{3.26}$$

and $\mathbf{B}$ is a solution of

$$\mathbf{TBX'} = \mathbf{0}, \tag{3.27}$$

or $\mathbf{BX'}$ is of the form

$$(\mathbf{I} - \mathbf{T}^{-}\mathbf{T})\,\mathbf{AX'}, \tag{3.28}$$

where $\mathbf{A}$ is arbitrary. Thus a general solution of (3.25) is the sum of (3.26) and (3.28), providing a characterization of the BLUE of $\mathbf{X\beta}$ in the restricted class of estimators $\mathbf{C'Y}$ with a minimum rank for $\mathbf{C}$.

We can also characterize the matrix $\mathbf{C'}$ of Corollary 3.7 as a projection operator in an extended sense.

DEFINITION. Let $\mathbf{U}$ and $\mathbf{W}$ be two matrices such that $\mathcal{M}(\mathbf{U})$ and $\mathcal{M}(\mathbf{W})$ are disjoint subspaces. Then $\mathbf{Y} \in \mathcal{M}(\mathbf{U} : \mathbf{W})$ has the unique decomposition $\mathbf{Y} =$

$Y_1 + Y_2$, $Y_1 \in \mathcal{M}(U)$ and $Y_2 \in \mathcal{M}(W)$. Then P is said to be a projector on $\mathcal{M}(U)$ along $\mathcal{M}(W)$ iff $PY = Y_1$ for all $Y \in \mathcal{M}(U : W)$.

It is easy to see that P is a projector on $\mathcal{M}(U)$ along $\mathcal{M}(W)$ iff

$$PU = U, \qquad PW = 0. \tag{3.29}$$

Let $G' = W^{\perp}$. Then a general solution for P is of the form $P = KG$ where K is any solution of $KGU = U$. One choice of P is

$$P = U(GU)^{-}G, \tag{3.30}$$

which is similar to the representation given in [11].

In general, P satisfying (3.29) need not be indempotent, although such choices exist as in (3.30).

Corollary 3.9. *$C'Y$ is the BLUE of $X\beta$ iff C' is a projector on $\mathcal{M}(X)$ along $\mathcal{M}(VZ)$ in the sense of the above definition.*

The corollary provides us with another representation of C' in addition to (3.17–3.19), through the formula (3.30). If $G' = (VZ)^{\perp}$, then

$$C' = KG \tag{3.31}$$

where K is any solution of $KGX = X$, and a particular choice is

$$C' = X(GX)^{-}G. \tag{3.32}$$

The formula (3.32) provides well known answers in particular cases. Thus, when $V = I$, $C' = X(X'X)^{-}X'$ and when $|V| \neq 0$, $C' = X(X'V^{-1}X)^{-}X'V^{-1}$.

4. Representation of V for Given Estimators

In Section 3, we examined the problem of obtaining best linear unbiased estimators when V is given. Now we consider the converse problem of determining V given a class of estimators. First we prove an algebraic lemma, which plays a key role in our study.

Lemma 4.1. *Let V be an nnd matrix satisfying the equations*

$$C'VZ = 0, \qquad VK = 0, \tag{4.1}$$

where C, Z and K are given. If $D = (C : K)^{\perp}$ and $X (= Z^{\perp})$ are disjoint then the general nnd solution of (4.1) is of the form

$$V = DU_1U_1'D' + X\Gamma U_2U_2'\Gamma'X, \tag{4.2}$$

where $\Gamma = (X'K)^{\perp}$, *and* U_1, U_2 *are arbitrary.*

First we show that

$$\mathcal{M}(V) \subset \mathcal{M}(D : X\Gamma). \tag{4.3}$$

Let λ be such that $\lambda'D = 0$, $\lambda'X\Gamma = 0$. Then $\lambda'D = 0 \Rightarrow \lambda' = \mu'C' + \nu'K'$ for some choice of vectors μ and ν giving

$$\lambda'X\Gamma = 0 \Rightarrow \mu'C'X\Gamma = 0. \tag{4.4}$$

Also $C'VZ = 0 \Rightarrow C'V = BX' \Rightarrow C'VK = BX'K = 0 \Rightarrow B = A\Gamma'$. Then

$$C'VC = BX'C = A\Gamma'X'C$$
$$C'VC\mu = A\Gamma'X'C\mu = 0 \quad \text{using (4.4)}$$
$$\Rightarrow \mu'C'V = 0 = \lambda'V$$

which proves (4.3).

Since V is nnd, $V = FF'$ where $F = DU_1 + X\Gamma U_2$. Then

$$V = (DU_1 + X\Gamma U_2)(DU_1 + X\Gamma U_2)'. \tag{4.5}$$

The desired representation (4.2) is obtained if $X\Gamma U_2U_1'D' = 0$. Using the condition $F'VZ = 0$ where $F = (C : K)$, we have from (4.5)

$$F'X\Gamma U_2U_1'D'Z = 0 \tag{4.6}$$

$\Rightarrow F'X\Gamma U_2U_1'D' = MX' \Rightarrow F'X\Gamma U_2U_1'D' = 0$ since X and D are disjoint. Further $F'X\Gamma U_2U_1'D' = 0 \Rightarrow X\Gamma U_2U_1'D' = DG \Rightarrow X\Gamma U_2U_1'D' = 0$, since D and X are disjoint. The result (4.2) is proved.

Corollary 4.1. *Let* $D = C^{\perp}$ *and* $X = Z^{\perp}$ *be disjoint. Then the nnd solutions of* $C'VZ = 0$ *are of the form*

$$V = DU_1U_1'D' + XU_2U_2'X' \tag{4.7}$$

where U_1 *and* U_2 *are arbitrary.*

The result (4.7) is a special case of (4.2) obtained by setting $K = 0$. It may be noted that the result (4.2) is true without the assumption that the columns of D and $X\Gamma$ span the entire space, and (4.7) is true without the assumption that the columns of D and X span the entire space.

Theorem 4.1. *Let* $(Y, X\beta, \sigma^2V)$ *be a GGM model where* V *is subject to the condition* $VK = 0$ *for given* K. *Further let* $C'Y$ *be the BLUE's of estimable para-*

metric functions $\mathbf{P}'\boldsymbol{\beta}$, *where* $\mathbf{C}'\mathbf{X} = \mathbf{P}'$ *and* $R(\mathbf{P}) = R(\mathbf{X})$. *Then it is necessary and sufficient that* $\mathbf{V}$ *is of the form*

$$\mathbf{V} = \mathbf{D}\boldsymbol{\Lambda}_1\mathbf{D}' + \mathbf{X}\boldsymbol{\Gamma}\boldsymbol{\Lambda}_2\boldsymbol{\Gamma}'\mathbf{X}', \tag{4.8}$$

where $\mathbf{D} = (\mathbf{C} : \mathbf{K})^{\perp}$, $\boldsymbol{\Gamma} = (\mathbf{X}'\mathbf{K})^{\perp}$, *and* $\boldsymbol{\Lambda}_1$ *and* $\boldsymbol{\Lambda}_2$ *are nnd matrices.*

The result (4.8) follows from Lemma 4.1, if it can be established that $\mathbf{D}$ and $\mathbf{X}$ are disjoint. Since $\mathbf{C}'\mathbf{Y}$ is the BLUE of $\mathbf{P}'\boldsymbol{\beta}$, $\mathbf{C}'\mathbf{X} = \mathbf{P}'$ and $\mathbf{C}'\mathbf{V}\mathbf{Z} = \mathbf{0}$ where $\mathbf{Z} = \mathbf{X}^{\perp}$. Also

$$\mathbf{C}'\mathbf{X} = \mathbf{P}', \qquad R(\mathbf{P}) = R(\mathbf{X}) \Rightarrow \mathbf{X} = \boldsymbol{\Lambda}\mathbf{P}' \quad \text{(for some } \boldsymbol{\Lambda}). \tag{4.9}$$

Now let $\mathbf{D}\mathbf{a} = \mathbf{X}\mathbf{b} \neq \mathbf{0}$.

$$\mathbf{D}\mathbf{a} = \mathbf{X}\mathbf{b} \Rightarrow \mathbf{0} = \mathbf{C}'\mathbf{D}\mathbf{a} = \mathbf{C}'\mathbf{X}\mathbf{b} = \mathbf{P}'\mathbf{b},$$
$$\mathbf{P}'\mathbf{b} = \mathbf{0} \Rightarrow \boldsymbol{\Lambda}\mathbf{P}'\mathbf{b} = \mathbf{0} = \mathbf{X}\mathbf{b},$$

which is a contradiction, i.e., $\mathbf{D}$ and $\mathbf{X}$ are disjoint. It is easy to see that if $\mathbf{V}$ is of the form (4.8), then $\mathbf{C}'\mathbf{Y}$ is the BLUE of $\mathbf{P}'\boldsymbol{\beta}$.

Note 1. If $\mathbf{K} = \mathbf{0}$, the necessary and sufficient condition (4.8) reduces to

$$\mathbf{V} = \mathbf{D}\boldsymbol{\Lambda}_1\mathbf{D}' + \mathbf{X}\boldsymbol{\Lambda}_2\mathbf{X}' \tag{4.10}$$

where $\mathbf{D} = \mathbf{C}^{\perp}$. The result (4.10) was established by Mitra and Moore [4]. The more general result (4.8) provides the class of dispersion matrices associated with random vectors $\mathbf{Y}$ such that $\mathbf{K}'\mathbf{Y}$ = constant with probability 1.

Note 2. Let $\mathbf{C} = \mathbf{X}$. Then it is necessary and sufficient that

$$\mathbf{V} = \mathbf{X}\mathbf{U}_1\mathbf{U}_1'\mathbf{X}' + \mathbf{Z}\mathbf{U}_2\mathbf{U}_2'\mathbf{Z}' = \mathbf{X}\boldsymbol{\Lambda}_1\mathbf{X}' + \mathbf{Z}\boldsymbol{\Lambda}_2\mathbf{Z}' \tag{4.11}$$

where $\boldsymbol{\Lambda}_1$ and $\boldsymbol{\Lambda}_2$ are nnd matrices, which includes the case $\mathbf{V} = \mathbf{I}$. The representation (4.11) which arises in a natural way in growth studies [2, 13] was first presented at the Fifth Berkeley Symposium in 1965, in complete generality, as an answer to the specific question: what is the class of dispersion matrices for which the simple least-squares estimators are optimum (see [7])? The work on representations of $\mathbf{V}$ was continued and extended to more general cases by Rao [8], Mitra and Rao [3], Rao and Mitra [11], and Mitra and Moore [4].

THEOREM 4.2. *Let* $\sigma^2\mathbf{V}$ *and* $\sigma^2\mathbf{V}_0$ *be two alternative choices of* $D(\mathbf{Y})$ *in the GGM model. If every representation* $\mathbf{C}'\mathbf{Y}$ *of the BLUE of* $\mathbf{X}\boldsymbol{\beta}$ *under* $\mathbf{V}_0$ *is also the*

BLUE of $\mathbf{X\beta}$ *under* $\mathbf{V}$, *then the following necessary and sufficient conditions are equivalent*:

$$\text{(i)}\quad \mathcal{M}(\mathbf{VZ}) \subset \mathcal{M}(\mathbf{V}_0\mathbf{Z}), \tag{4.12}$$

$$\text{(ii)}\quad \mathbf{V} = \mathbf{X}\Lambda_1\mathbf{X}' + \mathbf{V}_0\mathbf{Z}\Lambda_2\mathbf{Z}'\mathbf{V}_0, \tag{4.13}$$

where $\mathbf{Z} = \mathbf{X}^\perp$ *and* Λ_1, Λ_2 *are nnd matrices.*

It is shown in Corollary 3.7 that if $\mathbf{C'Y}$ is the BLUE of $\mathbf{X\beta}$ under $\mathbf{V}_0$, then

$$\mathbf{X'C} = \mathbf{X'}, \qquad \mathbf{Z'V}_0\mathbf{C} = \mathbf{0}. \tag{4.14}$$

If $\mathbf{C'Y}$ is also the BLUE under $\mathbf{V}$, then $\mathbf{Z'VC} = \mathbf{0}$. The theorem demands that

$$\mathbf{X'C} = \mathbf{X'}, \qquad \mathbf{Z'V}_0\mathbf{C} = 0 \Rightarrow \mathbf{Z'VC} = \mathbf{0} \tag{4.15}$$

for which it is necessary and sufficient that

$$\mathbf{Z'V}(\mathbf{V}_0\mathbf{Z})^\perp = \mathbf{0} \Rightarrow \mathcal{M}(\mathbf{VZ}) \subset \mathcal{M}(\mathbf{V}_0\mathbf{Z}), \tag{4.16}$$

which is the condition (i). Since $\mathbf{Z'V}(\mathbf{V}_0\mathbf{Z})^\perp = \mathbf{0}$, to establish the representation (ii) using Lemma 4.1, it is enough to show that $\mathbf{X}(= \mathbf{Z}^\perp)$ and $\mathbf{V}_0\mathbf{Z}$ are disjoint, which is trivially true. Of course (4.13) $\Rightarrow$ (4.12).

Note 1. The condition $\mathcal{M}(\mathbf{VZ}) \subset \mathcal{M}(\mathbf{V}_0\mathbf{Z})$, which can also be written in the equivalent forms

$$\mathbf{VZ} = \mathbf{V}_0\mathbf{ZK} \qquad \text{for some } \mathbf{K}, \tag{4.17}$$

$$\mathbf{V}(\mathbf{V}_0\mathbf{Z})^\perp = \mathbf{XS} \qquad \text{for some } \mathbf{S}, \tag{4.18}$$

was obtained in an earlier paper [8]. The result (4.12) constitutes a natural generalization of the author's earlier result [7] for the case $\mathbf{V}_0 = \mathbf{I}$, viz., $\mathbf{X'VZ} = \mathbf{0}$ which implies $\mathcal{M}(\mathbf{VZ}) \subset \mathcal{M}(\mathbf{Z})$ or $\mathcal{M}(\mathbf{VX}) \subset \mathcal{M}(\mathbf{X})$.

Note 2. The representation (4.13) was given in [8] under the additional assumption that the columns of $\mathbf{X}$ and $\mathbf{V}_0\mathbf{Z}$ span the entire space. The general form of the representation without this assumption is given by Mitra and Moore [4] using a different argument and not explicitly using the condition (4.12). Theorem 4.2, however, uses a different approach which shows that the representation is a direct consequence of the basic condition $\mathbf{Z'V}(\mathbf{V}_0\mathbf{Z})^\perp = \mathbf{0}$, or $\mathcal{M}(\mathbf{VZ}) \subset \mathcal{M}(\mathbf{V}_0\mathbf{Z})$ which is established without any assumptions.

Theorem 4.2 can also be stated in two other alternative forms.

THEOREM 4.3. *If every linear function of* $\mathbf{Y}$ *which is a BLUE under* $\mathbf{V}_0$ *is also a BLUE under* $\mathbf{V}$, *then the conditions in Theorem* 4.2 *are necessary and sufficient.*

The next Theorem 4.4 refers to the complete set of vectors providing BLUE's or BLUE(W)'s as considered in Corollary 3.2.

THEOREM 4.4. *Let* $\mathscr{L}^{V_0}$ *and* $\mathscr{L}^{V}$ *be the complete classes of vectors providing the best unbiased linear estimators under* $\mathbf{V}_0$ *and* $\mathbf{V}$ *respectively. If* $\mathscr{L}^{V_0} \subset \mathscr{L}^{V}$, *then the conditions given in Theorem* 4.2 *are necessary and sufficient.*

Note. Theorem 4.4 only demands that if $\mathbf{L} \in \mathscr{L}^{V_0}$ then it also belongs to $\mathscr{L}^{V}$ without requiring that the set of parametric functions estimated by any particular linear function $\mathbf{L'Y}$ under $\mathbf{V}_0$ is contained in the corresponding set under $\mathbf{V}$. Theorem 4.5 takes into account this requirement. We denote by $\mathscr{P}_L(\mathbf{V})$ the set of parametric functions $\mathbf{p}'\boldsymbol{\beta}$ estimated by $\mathbf{L'Y}$ under $\mathbf{V}$.

THEOREM 4.5. *Let* $\mathbf{N}$ *be a matrix of maximum rank such that* $\mathbf{N'Y} = \mathbf{0}$ *with probability* 1 *under* $\mathbf{V}_0$, *as defined in* (2.6). *Let*

$$\mathbf{L} \in \mathscr{L}^{V_0} \Rightarrow \mathbf{L} \in \mathscr{L}^{V}, \tag{4.19}$$

and

$$\mathscr{P}_L(\mathbf{V}_0) \subset \mathscr{P}_L(\mathbf{V}). \tag{4.20}$$

(i) *If* $\mathbf{N'X} = \mathbf{0}$, *then* (4.19) $\Rightarrow$ (4.20) *in which case* $\mathbf{V}$ *has the representation* (4.13).

(ii) *If* $\mathbf{N'X} \neq \mathbf{0}$, *then it is necessary that*

$$\mathbf{V} = \mathbf{D}\boldsymbol{\Lambda}_1\mathbf{D}' + \mathbf{X}\boldsymbol{\Gamma}\boldsymbol{\Lambda}_2\boldsymbol{\Gamma}'\mathbf{X}' \tag{4.21}$$

where $\mathbf{D} = \mathbf{V}_0\mathbf{Z}$, $\boldsymbol{\Gamma} = (\mathbf{N'X})^{\perp}$ *and* $\boldsymbol{\Lambda}_1$, $\boldsymbol{\Lambda}_2$ *are nnd matrices.*

It is already shown that (4.19) implies

$$\mathbf{Z'V}(\mathbf{V}_0\mathbf{Z})^{\perp} = \mathbf{0}. \tag{4.22}$$

If, in addition, (4.20) is also satisfied then it is *necessary and sufficient* that

$$\mathbf{N'Y} = \mathbf{0} \qquad \text{with probability 1 under } \mathbf{V}, \tag{4.23}$$

for which it is *necessary* that

$$\mathbf{VN} = \mathbf{0}. \tag{4.24}$$

The condition (4.24) is included in (4.22) if $\mathbf{N'X} = \mathbf{0}$. Otherwise we have to find an nnd solution for $\mathbf{V}$ satisfying Eqs.(4.22) and (4.24). The conditions of the Lemma 4.1 are satisfied and we have the representation as in (4.21).

Note that for sufficiency, in addition to the representation (4.21), $\mathbf{N'Y}$ should actually be zero with probability 1.

Finally, we consider the problem of representing $\mathbf{V}$ such that the equations

$$\mathbf{X'C} = \mathbf{X'}, \qquad \mathbf{Z'V_0C} = 0, \qquad \mathbf{Z'VC} = 0 \tag{4.25}$$

have a solution. It has been shown in Rao [8], that for (4.25) to have a solution it is necessary and sufficient that $(\mathbf{VZ} : \mathbf{V_0Z})$ and $\mathbf{X}$ are disjoint.

THEOREM 4.6. *If (4.25) admits a solution, then* $\mathbf{V}$ *is of the form*

$$\mathbf{V} = \mathbf{X}\Lambda_1\mathbf{X}' + (\mathbf{S} : \mathbf{V_0Z})\,\Lambda_2(\mathbf{S} : \mathbf{V_0Z})' \tag{4.26}$$

where $\mathbf{S}$ *is any matrix such that* $(\mathbf{S} : \mathbf{V_0Z})$ *and* $\mathbf{X}$ *are disjoint, and* Λ_1, Λ_2 *are nnd matrices.*

The result (4.26) follows by applying Lemma 4.1. Mitra and Moore [4] give a different representation.

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