EFFICIENCY BALANCED BLOCK DESIGNS

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ABSTRACT

The efficiency-factor of a treatment contrast in the sense of Jones (1959) is examined from a classical view-point. An upper bound for the efficiency-factor of any treatment contrast is obtained and designs are characterised for which the upper bound of the efficiency-factor is attainable by every contrast. Two concepts of balancing in block designs, namely, Efficiency-balance and Variance-balance are studied and inter-relationships between these concepts are established.

1. INTRODUCTION

Throughout this paper, block designs with v treatments and b blocks are considered. It is assumed that the ith treatment is replicated r_i times, $i = 1,\ldots,v$ and the jth block contains k_j (not necessarily distinct) treatments, $j = 1,\ldots,b$. Let $\underline{r} = (r_1,\ldots,r_v)'$, $\underline{k} = (k_1,\ldots,k_b)'$, $R = \text{diag}(r_1,\ldots,r_v)$, $K = \text{diag}(k_1,\ldots,k_b)$ and N be the $v \times b$ incidence matrix of the design. If $\underline{T}$ denotes the column vector of treatment totals then $\underline{s}'\underline{T}$ is called a contrast of treatment totals if $\underline{s}'\underline{r}=0$, where $\underline{s}$ is a column vector. The 'intra-block component' of $\underline{s}'\underline{T}$ is defined by Jones (1959) as $\underline{s}'\underline{Q}$ where $\underline{Q}$ is the vector of adjusted treatment totals, given by $\underline{Q} = \underline{T}-NK^{-1}\underline{B}$, $\underline{B}$ being the vector of block totals.

Jones (1959) showed that if $\underline{s}$ is a right eigenvector of the matrix $M = R^{-1}NK^{-1}N'$ corresponding to an eigenvalue $\varepsilon(\neq 1)$, then the loss of information on the 'intra-block component' of $\underline{s}'\underline{T}$ is ε so that the efficiency-factor of the 'intra-block component' is $1 - \varepsilon$.

Since $\underline{s}'\underline{Q}$ (the intra-block component of $\underline{s}'\underline{T}$) is a function of observations and not of parameters (treatment effects) the concept of 'loss of information' or 'efficiency-factor' of $\underline{s}'\underline{Q}$ is a little confusing when viewed from the classical definition of loss of information. In the classical sense, the loss of information refers to the loss incurred in estimating a certain contrast of treatment effects through a design, in relation to an orthogonal design.

In section 2 of the paper, an attempt has been made to resolve this anomaly by deriving the result of Jones (1959) using the classical approach. An upper bound for the efficiency-factor of any contrast of treatment effects is also obtained. Designs for which this upper bound is attainable by any treatment contrast are characterised.

A block design for which every contrast has the same loss of information (or, equivalently, same efficiency-factor) may be termed Efficiency-Balanced. The concept of efficiency-balance is different from the one used commonly, according to which design is balanced if every elementary contrast is estimated through the design with the same variance. To avoid confusion, the latter concept is called Variance-Balance (see e.g., Hedayat and Federer, 1974). In Section 3, certain results on efficiency-balanced designs are reported. Interrelationships between efficiency-and variance-balanced designs are also discussed.

Throughout, only connected designs are considered.

2. EFFICIENCY FACTOR OF A CONTRAST AND AN UPPER BOUND

Consider a block design $D(v,b,\underline{r},\underline{k})$. Let us postulate the usual homoscedastic fixed effects model

$$y = m\underline{1} + D_1'\underline{t} + D_2'\underline{b} + \underline{e} \qquad (2.1)$$

where $\underline{y}$ denotes the observations vector, $m,\underline{t},\underline{b}$ denote respectively the general mean, the vector of treatment effects and the vector of block effects, D_1 and D_2 are respectively the treatment x observations and block x observations incidence matrices, $\underline{1}$ is a column vector of all unities and $\underline{e}$ is the vector of residuals with usual assumptions, viz., $E(\underline{e}) = \underline{0}$, $E(\underline{e}\,\underline{e}') = \sigma^2 I$. The block and treatment effects are assumed to be fixed. It may be noted here that Jones (1959) considers a model with random block effects. Suppose $\underline{s}'\underline{T}$ is a contrast of treatment totals. The 'intra-block component' of $\underline{s}'\underline{T}$, viz., $\underline{s}'\underline{Q}$, under (2.1) has expectation

$$E(\underline{s}'\underline{Q}) = \underline{s}'C\underline{t} \qquad (2.2)$$

where

$$C = R - NK^{-1}N'. \qquad (2.3)$$

If $M = R^{-1}NK^{-1}N'$, we may write
$= RNK^{-1}N'$

$$E(\underline{s}'\underline{Q}) = \underline{s}'(I - M')R\underline{t}. \qquad (2.4)$$

It is easily seen that $\underline{1}$ is a right eigenvector of M corresponding to the simple eigenvalue unity. Suppose $\underline{s}$ is a right eigenvector of M corresponding to an eigenvalue ε $(\neq 1)$, that is

$$M\,\underline{s} = \varepsilon\,\underline{s}. \qquad (2.5)$$

Then, under (2.5),

$$E(\underline{s}'\underline{Q}) = (1-\varepsilon)\underline{s}'R\underline{t}, \qquad (2.6)$$

so that an unbiased estimator of $\underline{s}'R\underline{t}$ in the design D is $\underline{s}'\underline{Q}/(1-\varepsilon)$. The variance of this estimator is given by

$$V(\text{Est. } \underline{s}'R\underline{t})_D = \sigma^2\,\underline{s}'\,R\,\underline{s}/(1-\varepsilon). \qquad (2.7)$$

Consider an orthogonal design, corresponding to the design D. For the orthogonal design, an unbiased estimator of $\underline{s}'R\underline{t}$ is $\underline{s}'\underline{T}$ and the variance of this estimator is

$$V(\text{Est. } \underline{s}'R\underline{t})_0 = \sigma^2\,\underline{s}'\,R\,\underline{s}. \qquad (2.8)$$

The loss of information on $\underline{s}'R\underline{t}$, estimated through D, relative to an orthogonal design is given by

$$L(\underline{s}'R\underline{t}) = 1 - V(\text{Est. } \underline{s}'R\underline{t})_0/V(\text{est. } \underline{s}'R\underline{t})_D$$
$$= \varepsilon. \qquad (2.9)$$

The efficiency-factor of the contrast $\underline{s}'R\underline{t}$ in the design D

is $1-\varepsilon$. It is thus seen that the concept of loss of information on the intra-block component is actually the loss on $\underline{s}'R\underline{t}$. Clearly, $\underline{s}'R\underline{t}$ is a contrast of treatment effects. Note that any treatment contrast is expressible in the form $\underline{s}'R\underline{t}$, for, if $\underline{p}'\underline{t}$ is a contrast $(\underline{p}'\underline{1} = 0)$, then $\underline{p}'\underline{t} = \underline{p}'R^{-1}R\underline{t} = \underline{s}'R\underline{t}$ with $\underline{s} = R^{-1}\underline{p}$.

We now proceed to obtain an upper bound for the efficiency-factor of a contrast of treatment effects. Suppose $\underline{p}'\underline{t}$ is a contrast. The best linear unbiased estimator of $\underline{p}'\underline{t}$ is $\underline{p}'\hat{\underline{t}}$ where $\hat{\underline{t}}$ is a solution of the equations

$$C\underline{t} = \underline{Q}. \tag{2.10}$$

Thus, $\underline{p}'\hat{\underline{t}} = \underline{p}'C^{-}\underline{Q}$ where C^{-} is a generalised inverse of C, i.e., $CC^{-}C = C$. The variance of $\underline{p}'\hat{\underline{t}}$ is $\sigma^2\underline{p}'C^{-}\underline{p}$. The variance of $\underline{p}'\hat{\underline{t}}$ in a corresponding orthogonal design is $\sigma^2\underline{p}'R^{-1}\underline{p}$. Thus, the efficiency-factor of $\underline{p}'\underline{t}$ is given by

$$E = (\underline{p}'R^{-1}\underline{p}) \;/\; (\underline{p}'C^{-}\underline{p}). \tag{2.11}$$

If A is a Gramian matrix, the Cauchy-Schwarz inequality states that for real vectors $\underline{x}$ and $\underline{y}$,

$$(\underline{x}'A\underline{x})(\underline{y}'A\underline{y}) \geq (\underline{x}'A\underline{y})^2. \tag{2.12}$$

The equality in (2.12) is attained if and only if (iff) $\underline{x}$ is proportional to $\underline{y}$.

Now since $\underline{p}'\underline{t}$ is estimable, $\underline{p}'C^{-}C = \underline{p}'$. Since C is Gramian, choosing $\underline{x} = \underline{p}$, $A = C$ and $\underline{y} = (C^{-})'\,\underline{p}$ in (2.12) and simplifying, we get

$$E \leq (\underline{p}'R^{-1}\underline{p})(\underline{p}'C\,\underline{p}) \;/\; (\underline{p}'\underline{p})^2. \tag{2.13}$$

This gives an upper bound for the efficiency-factor of $\underline{p}'\underline{t}$. The equality in (2.13) is attainable iff

$$\underline{p} = \alpha \ (C^{-})'\underline{p} \qquad (2.14)$$

where α is a scalar. This implies

$$C\ \underline{p} = \alpha\ C\ (C^{-})'\ \underline{p}$$

or

$$C\ \underline{p} = \alpha\ \underline{p}. \qquad (2.15)$$

Thus any contrast $\underline{p}'\underline{t}$ is most efficiently estimated through a design D iff $\underline{p}$ is an eigenvector of the C-matrix of the design D. Under (2.15), the efficiency-factor of $\underline{p}'\underline{t}$ is given by

$$E = (\alpha\ \underline{p}'R^{-1}\underline{p})\ /\ (\underline{p}'\underline{p}). \qquad (2.16)$$

Consider now a variance-balanced design. It is known (Rao, 1958), that a connected design is variance-balanced iff its C-matrix is given by

$$C = \theta\ (I - v^{-1}\ \underline{1}\underline{1}'), \qquad (2.17)$$

where θ is a positive scalar. For variance-balanced designs, $C\ \underline{p} = \theta\underline{p}$ for all $\underline{p}$ such that $\underline{p}'\underline{1} = 0$. Thus, for variance-balanced designs, any contrast of treatment effects has the maximum efficiency-factor.

3. EFFICIENCY - AND VARIANCE - BALANCED DESIGNS

In this section, results on efficiency-and variance-balanced designs are discussed.

It has been established by Calinski (1971) and Puri and Nigam (1975) that a sufficient condition for a design to be efficiency-balanced is that its M-matrix is given by

$$M = \varepsilon\, I + (1-\varepsilon)\underline{1}\ \underline{r}' / n, \qquad (3.1)$$

where n is the total number of observations in the design. That (3.1) is necessary as well for a design to be efficiency-balanced was shown by Williams (1975). In what follows, an alternative proof of the result is given.

Theorem 3.1 A necessary and sufficient condition for a design to be efficiency-balanced is that (3.1) holds.

Proof (Necessity) Let the design be efficiency-balanced, i.e., $M\,\underline{s} = \varepsilon\,\underline{s}$ for all $\underline{s}$ such that $\underline{s}'\underline{r} = 0$. This implies that

$$(M-\varepsilon\, I)\underline{s} = \underline{0}$$

for all $\underline{s}$ such that $\underline{s}'\underline{r} = 0$. Thus, $\underline{r}'$ belongs to the row space of $M-\varepsilon\, I$. Since $M-\varepsilon\, I$ is of rank unity, it follows that there exists a (column) vector $\underline{\beta}$ such that

$$M-\varepsilon\, I = \underline{\beta}\,\underline{r}'$$
$$\Rightarrow (M-\varepsilon\, I)\ \underline{1} = \underline{\beta}\,\underline{r}'\ \underline{1} = n\ \underline{\beta}.$$

Further, $M\,\underline{1} = \underline{1}$ and thus $\underline{\beta} = (1-\varepsilon)\underline{1}/n$, which proves the necessity. Sufficiency is obvious.

Q.E.D.

As mentioned earlier, the concepts of efficiency-and variance-balance are in general different. The following result gives the relationship between these two notions of balance.

Theorem 3.2 If a design has any two of the following three properties

(i) Efficiency-balance
(ii) Variance-balance
(iii) Equal replication

then it has the third.

Proof (a) (i) and (iii) → (ii).

Let $r_i = r$ for all i since the design is equireplicate. Then, the C-matrix of the design is given by

$$C = r\ (I-M). \tag{3.2}$$

Also, since the design is efficiency-balanced,

$$M = \varepsilon\ I + (1-\varepsilon)\ r\ \underline{1}\ \underline{1}'\ /\ n$$

so that (3.2) boils down to

$$C = r\ (1-\varepsilon)\ (I-v^{-1}\underline{1}\ \underline{1}'). \tag{3.3}$$

It follows that the design is variance-balanced, and (a) is proved.

(b) (ii) and (iii) → (i).

Since the design is variance-balanced, its C-matrix must be of the form

$$C = \alpha\ (I-v^{-1}\underline{1}\ \underline{1}')$$

where α is the unique non-zero eigenvalue of C. Now, since the design is also equireplicate,

$$\begin{aligned} M &= I - C\ /\ r \\ &= (1-\alpha\ /\ r)\ I + \underline{1}\ \underline{1}'\ /\ vr \\ &= \varepsilon\ I + (1-\varepsilon) r\ \underline{1}\ \underline{1}'\ /\ n \end{aligned}$$

where $\varepsilon = 1 - \alpha\ /\ r$. Thus (b) is proved.

Note that since the eigenvalues of C for a variance-balanced design are zero and α , the eigenvalues of the matrix $P = NK^{-1}N'$ are r and $r-\alpha$ for an equireplicate negative.

(c) (i) and (ii) ⇒ (iii).

Since the design is both efficiency-and variance-balanced its M - and C - matrices are given by

$$M = \varepsilon I + (1-\varepsilon) \underline{1}\, \underline{r}' / n$$
$$\text{and} \quad C = \alpha (I - v^{-1} \underline{1}\, \underline{1}').$$

which holds iff $r_i = r$ for all i. Thus, (i) and (ii) imply (iii).

Q.E.D.

A design is called proper if $k_j = k$ for all j and binary if the incidence matrix N is a zero-one matrix. The following result characterises proper, binary efficiency-balanced designs.

Theorem 3.3 In the class of proper, binary designs, the balanced incomplete block (BIB) design is the only efficiency-balanced design, if it exists.

Proof Since the design is efficiency-balanced,

$$M = \varepsilon I + (1-\varepsilon) \underline{1}\, \underline{r}' / n$$
$$\Rightarrow \quad P = NK^{-1}N' = \varepsilon R + (1-\varepsilon) \underline{r}\, \underline{r}' / n.$$

If the design is proper, $K^{-1} = I / k$ and thus

$$P = NN'/k = \varepsilon R + (1-\varepsilon) \underline{r}\, \underline{r}' / n. \quad (3.4)$$

Since the design is binary, (3.4) implies that

$$r_i = (1/k-\varepsilon) n/(1-\varepsilon).$$

Thus any proper, binary efficiency-balanced design is equireplicate and hence (by theorem 3.2) variance-balanced. The only binary, proper, equireplicate variance-balanced design is the BIB design (if it exists).

Q.E.D.

Finally we prove

Theorem 3.4 The inequality $b \geq v$ holds for all non-orthogonal efficiency-balanced designs.

Proof It is well known that a necessary and sufficient condition for a design to be orthogonal is that $N = \underline{r}\,\underline{k}'/n$. Thus, for orthogonal designs, $M = \underline{1}\,\underline{r}'/n$ implying that orthogonal designs are efficiency-balanced with $\varepsilon = 0$. For orthogonal designs M is singular. If the design is non-orthogonal and efficiency-balanced, M is non-singular, for, M has one eigenvalue unity and rest $v-1$ eigenvalues equal to ε, $\varepsilon > 0$. Thus

$$\begin{aligned} v &= \text{Rank}(M) = \text{Rank}(R^{-1}NK^{-1}N') = \text{Rank}(NK^{-1}N') \\ &= \text{Rank}(N) \leq b. \end{aligned}$$

Q.E.D.

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