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COMPLEMENTATION IN THE LATTICE OF BOREL STRUCTURES

BY

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1. Preliminaries. Let $\boldsymbol{A}$ and $\boldsymbol{C}$ be substructures (i.e., sub-σ-algebras) of the Borel σ-algebra $\boldsymbol{B}$ on $I = [0, 1]$. Denote by $\boldsymbol{A} \vee \boldsymbol{C}$ the σ-algebra on I generated by $\boldsymbol{A} \cup \boldsymbol{C}$ and put $\boldsymbol{A} \wedge \boldsymbol{C} = \boldsymbol{A} \cap \boldsymbol{C}$. We say that $\boldsymbol{C}$ is a *complement of* $\boldsymbol{A}$ *relative to* $\boldsymbol{B}$ if $\boldsymbol{A} \vee \boldsymbol{C} = \boldsymbol{B}$ and $\boldsymbol{A} \wedge \boldsymbol{C} = \{\emptyset, I\}$. A relative complement $\boldsymbol{C}$ of $\boldsymbol{A}$ is said to be *minimal* if no proper substructure of $\boldsymbol{C}$ is a complement of $\boldsymbol{A}$ relative to $\boldsymbol{B}$.

In [6], B. V. Rao raised the following question:

What are those countably generated substructures of $\boldsymbol{B}$ on I which have complements relative to $\boldsymbol{B}$? (P 741).

In this note we prove that every countably generated substructure of $\boldsymbol{B}$ has, in fact, a minimal complement relative to $\boldsymbol{B}$ (theorem 2). For this purpose, we need the following results:

(a) If $\boldsymbol{A}$ and $\boldsymbol{C}$ are substructures of $\boldsymbol{B}$ such that $\boldsymbol{A} \vee \boldsymbol{C} = \boldsymbol{B}$, and $\boldsymbol{A} \vee \boldsymbol{C}_1 \neq \boldsymbol{B}$ for any proper substructure $\boldsymbol{C}_1$ of $\boldsymbol{C}$, then $\boldsymbol{A} \wedge \boldsymbol{C} = \{\emptyset, I\}$, and whence $\boldsymbol{C}$ is a minimal complement of $\boldsymbol{A}$ relative to $\boldsymbol{B}$ (see [5], p. 100-101, or [6], theorem 2).

(b) If $\boldsymbol{A}$ is a substructure of $\boldsymbol{B}$, then, for any substructure $\boldsymbol{C}$ of $\boldsymbol{B}$ with $\boldsymbol{A} \vee \boldsymbol{C} = \boldsymbol{B}$, there exists a countably generated substructure $\boldsymbol{C}_1$ of $\boldsymbol{B}$ such that $\boldsymbol{C}_1 \subset \boldsymbol{C}$ and $\boldsymbol{A} \vee \boldsymbol{C}_1 = \boldsymbol{B}$ (see [5], p. 103).

(c) Let X be a Borel subset of a complete separable metric space and let $\boldsymbol{B}_X$ be the Borel σ-algebra on X. If $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are countably generated substructures of $\boldsymbol{B}_X$ which have the same atoms, then $\boldsymbol{A}_1 = \boldsymbol{A}_2$ (see [1], or [5], p. 69).

2. Main results.

THEOREM 1. *Let* $\boldsymbol{A}$ *and* $\boldsymbol{C}$ *be countably generated substructures of* $\boldsymbol{B}$ *on* I. *Then* $\boldsymbol{C}$ *is a minimal complement of* $\boldsymbol{A}$ *relative to* $\boldsymbol{B}$ *if and only if*

(i) *every atom of* $\boldsymbol{C}$ *is a partial selector for* $\boldsymbol{A}$ (*i.e., it is a Borel set containing at most one point from each atom of* $\boldsymbol{A}$), *and*

(ii) *$C_1 \cup C_2$ is not a partial selector for A for any distinct atoms C_1 and C_2 of $\mathbf{C}$.*

To prove this theorem, we need the following

Remark 1. *If $\mathbf{A}$ and $\mathbf{C}$ are countably generated substructures of $\mathbf{B}$ on I, then $\mathbf{A} \vee \mathbf{C} = \mathbf{B}$ if and only if* (i) *holds.*

Proof. As $\mathbf{A} \vee \mathbf{C}$ is countably generated, it separates points if and only if (i) holds. Hence, by (c), (i) is equivalent to $\mathbf{A} \vee \mathbf{C} = \mathbf{B}$.

Proof of theorem 1. Let $\mathbf{A}$ and $\mathbf{C}$ satisfy (i) and (ii). We infer from remark 1 that $\mathbf{A} \vee \mathbf{C} = \mathbf{B}$. By (a) and (b), it is enough to prove that, for any countably generated substructure $\mathbf{C}_1$ of $\mathbf{C}$ with $\mathbf{A} \vee \mathbf{C}_1 = \mathbf{B}$, we have $\mathbf{C}_1 = \mathbf{C}$. Suppose C and D are distinct atoms of $\mathbf{C}$. It follows from (ii), and remark 1 applied to $\mathbf{A}$ and $\mathbf{C}_1$, that $C \cup D$ is contained in no atom of $\mathbf{C}_1$. Thus $\mathbf{C}_1$ and $\mathbf{C}$ have the same atoms, so that by (c), $\mathbf{C}_1 = \mathbf{C}$.

To prove the converse, suppose that $\mathbf{C}$ is a minimal complement of $\mathbf{A}$ relative to $\mathbf{B}$. Then, by remark 1, (i) holds. Suppose (ii) does not hold. Let C and D be distinct atoms of $\mathbf{C}$ such that $C \cup D$ is a partial selector for $\mathbf{A}$. Denote by $\mathbf{C}_1$ the σ-algebra on I generated by $\mathbf{C} \cap (I-(C \cup D))$. Then $\mathbf{C}_1$ is a proper substructure of $\mathbf{C}$ which is countably generated. Also, every atom of $\mathbf{C}_1$ is a partial selector for $\mathbf{A}$. Remark 1 now yields $\mathbf{A} \vee \mathbf{C}_1 = \mathbf{B}$, so that $\mathbf{C}$ is not a minimal complement of $\mathbf{A}$ relative to $\mathbf{B}$, a contradiction.

THEOREM 2. *Every countably generated substructure $\mathbf{A}$ of $\mathbf{B}$ on I has a minimal complement relative to $\mathbf{B}$.*

Proof. There are three cases to be considered.

Case 1. $\mathbf{A}$ has a cocountable atom A.

In this case $\mathbf{A}$ has only countably many atoms and all of them, except for A, are countable. Then we can define a countable family $\{G_n: n \geqslant 1\}$ of disjoint Borel sets such that

$$\bigcup_n G_n = I - A$$

and each G_n is a partial selector for $\mathbf{A}$. Let $\{a_n: n \geqslant 1\}$ be a sequence of distinct points in A. Put $H_n = G_n \cup \{a_n\}$. Denote by $\mathbf{C}$ the σ-algebra generated by $\{H_n: n \geqslant 1\}$ and $\mathbf{B} \cap (A - \bigcup_n \{a_n\})$. Clearly, $\mathbf{C}$ is countably generated and the atoms of $\mathbf{C}$ are $\{H_n: n \geqslant 1\}$ and $\{\{x\}: x \in A - \bigcup_n \{a_n\}\}$. By theorem 1, $\mathbf{C}$ is a minimal complement of $\mathbf{A}$ relative to $\mathbf{B}$.

Case 2. All the atoms of $\mathbf{A}$ are countable.

Then there exists a countable family $\{G_n: n \geqslant 1\}$ of disjoint Borel sets such that

$$\bigcup_n G_n = I$$

and each G_n is a non-empty partial selector for $\boldsymbol{A}$. This is a reformulation, with help of the characteristic function of a sequence of sets, of a theorem of Lusin (see [2], p. 335). It is easy to choose the G_n's in such a way that, for distinct G_n and G_m, $G_n \cup G_m$ is not a partial selector for $\boldsymbol{A}$. Denote by $\boldsymbol{C}$ the σ-algebra generated by $\{G_n: n \geqslant 1\}$. The atoms of $\boldsymbol{C}$ are $\{G_n: n \geqslant 1\}$, whence, by theorem 1, $\boldsymbol{C}$ is a minimal complement of $\boldsymbol{A}$ relative to $\boldsymbol{B}$.

Case 3. $\boldsymbol{A}$ has an atom A which is neither countable nor cocountable.

Then A and $I-A$ are uncountable Borel sets. Hence there is a Borel isomorphism $g: A \to I-A$ (see [3], § 37, II). Let $f: I \to I$ be defined by

$$f(x) = \begin{cases} g(x) & \text{if } x \in A, \\ x & \text{if } x \in I-A. \end{cases}$$

Then f is Borel measurable. Put $\boldsymbol{C} = f^{-1}(\boldsymbol{B})$. Clearly, $\boldsymbol{C}$ is countably generated and all the atoms of $\boldsymbol{C}$ are of the form $\{x, g(x)\}$, where $x \in A$. By theorem 1, $\boldsymbol{C}$ is a minimal complement of $\boldsymbol{A}$ relative to $\boldsymbol{B}$.

Remark 2. As a matter of fact, any substructure $\boldsymbol{A}$ of $\boldsymbol{B}$, which has an atom A being neither countable nor cocountable, has a minimal complement relative to $\boldsymbol{B}$ even if $\boldsymbol{A}$ is not countably generated (see also [6], theorem 3, for a special case). To see this, define $\boldsymbol{C}$ as in case 3 of the proof of theorem 2. Let $\boldsymbol{D}$ be the σ-algebra generated by $\boldsymbol{C} \cup \{A\}$. Then $\boldsymbol{D} \subseteq \boldsymbol{B}$ is countably generated and separates points. Hence, by (c), $\boldsymbol{D} = \boldsymbol{B}$. But $\boldsymbol{D} \subseteq \boldsymbol{A} \vee \boldsymbol{C} \subseteq \boldsymbol{B}$. Hence $\boldsymbol{A} \vee \boldsymbol{C} = \boldsymbol{B}$. To get a contradiction, suppose that there exists a proper substructure $\boldsymbol{C}_1$ of $\boldsymbol{C}$ with $\boldsymbol{A} \vee \boldsymbol{C}_1 = \boldsymbol{B}$. By (b), we can suppose $\boldsymbol{C}_1$ to be countably generated. As $\boldsymbol{C}_1 \subsetneqq \boldsymbol{C}$, there exist atoms $\{x_1, g(x_1)\}$ and $\{x_2, g(x_2)\}$ of $\boldsymbol{C}$, where $x_1, x_2 \in A$ and $x_1 \neq x_2$, such that $\{x_1, x_2, g(x_1), g(x_2)\}$ is contained in an atom of $\boldsymbol{C}_1$. But this implies that $\boldsymbol{A} \vee \boldsymbol{C}_1$ does not separate x_1 and x_2, so that $\boldsymbol{A} \vee \boldsymbol{C}_1 \neq \boldsymbol{B}$. Hence, by (a), $\boldsymbol{C}$ is a minimal complement of $\boldsymbol{A}$ relative to $\boldsymbol{B}$. Thus the converse of theorem 2 is not true. For example, if $\boldsymbol{A}$ is generated by $[0, 1/2)$ and $\{\{x\}: 1/2 \leqslant x \leqslant 1\}$, then $\boldsymbol{A}$ is not countably generated but has a minimal relative complement. We can even construct an $\boldsymbol{A}$ which is not atomic and yet has a minimal complement relative to $\boldsymbol{B}$.

Remark 3. If we wished to prove theorem 2 merely for complements, instead of for minimal complements, the proofs of cases 1 and 2 could be simplified by observing that in these cases there exists a Borel set D such that $D \cap A$ is a singleton for every atom A of $\boldsymbol{A}$. (In case 2, the existence of D also follows from a theorem of Novikoff [4], p. 14.) Then

$$\boldsymbol{C} = \{B \in \boldsymbol{B}: B \supseteq D \text{ or } B \cap D = \emptyset\}$$

is a relative complement of $\boldsymbol{A}$. However, such a D does not exist for all countably generated $\boldsymbol{A} \subseteq \boldsymbol{B}$. To see this, take an analytic set $A \subset I$ which is not Borel. Let $f\colon I \to I$ be Borel measurable and $f(I) = A$. Then $\boldsymbol{A} = f^{-1}(\boldsymbol{B})$ is a countably generated substructure of $\boldsymbol{B}$ for which no such D exists (see [3], § 39, V, theorem 1).

Remark 4. In [6], p. 214, B. V. Rao proved that the countable-cocountable structure on I has no complement relative to $\boldsymbol{B}$. We exhibit another class of structures which have no complements relative to $\boldsymbol{B}$. Let $A \subseteq I$ be any non-Borel set. Write

$$\boldsymbol{B}^A = \{B \in \boldsymbol{B}\colon B \cap A = \emptyset \text{ or } B \supset A\}.$$

To get a contradiction, suppose that $\boldsymbol{B}^A$ has a relative complement $\boldsymbol{C}$. We can suppose $\boldsymbol{C}$ to be countably generated. Then, by (b), there exists a countably generated substructure $\boldsymbol{D}$ of $\boldsymbol{B}^A$ which is a complement of $\boldsymbol{C}$ relative to $\boldsymbol{B}$. As $\boldsymbol{D} \subseteq \boldsymbol{B}^A$, it follows that $\boldsymbol{D}$ has an atom $D \supseteq A$. As D is a Borel set, $D \neq A$. Fix $x \in D - A$. Since $\{x\}$ is an atom of $\boldsymbol{B} = \boldsymbol{D} \vee \boldsymbol{C}$, we have $\{x\} = D \cap C$ for some atom C of $\boldsymbol{C}$. Hence $C \cap A = \emptyset$, so that $C \in \boldsymbol{B}^A$. Thus $\boldsymbol{B}^A \wedge \boldsymbol{C} \neq \{\emptyset, I\}$ which is a contradiction. Therefore, $\boldsymbol{B}^A$ has no complement relative to $\boldsymbol{B}$.

Remark 5. The problem of characterizing the atomic substructures of $\boldsymbol{B}$ which have complements relative to $\boldsymbol{B}$ seems interesting. (**P 899**)

Another interesting question is the following: Does the existence of a relative complement imply the existence of a minimal relative complement? (**P 900**)

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