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r-PARTITE SELF-COMPLEMENTARY GRAPHS—DIAMETERS

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The following results are proved in this paper.

(1) If the diameter of a connected bipartite graph $G(2)$ is larger than six then the diameter of the bipartite complement $\bar{G}(2)$ of $G(2)$ is smaller than five. In particular, the diameter λ of a bipartite self-complementary graph satisfies $3 \le \lambda \le 6$.

(2) If the diameter of a connected r-partite graph $G(r)$, $r \ge 3$, is larger than five then the diameter of the r-partite complement $\bar{G}(r)$ of $G(r)$ is smaller than five. In particular, the diameter λ of an r-partite self-complementary (r-p.s.c.) graph satisfies $2 \le \lambda \le 5$.

(3) If $r \ge 3$ and $G(r)$ is an r-p.s.c. graph with a periodic r-p.c.p. σ, such that any cycle of σ having length >1 intersects at least 3 classes of the r-partition, then the diameter of $G(r)$ is either 2 or 3. As a consequence of the above result it follows that the diameter of a self-complementary graph is either 2 or 3.

1. Introduction

We consider finite, undirected graphs without loops and multiple edges. We follow Harary [4] for the notation and terminology not defined here.

An r-partite graph is said to be *complete r-partite* if each vertex is joined to partitioned into $r \ge 1$ non-empty subsets, also called classes so that no edge has both ends in any one subset. Let $A_1, \dots, A_r$ constitute an r-partition of V with $|A_i| = n_i$, $n_i \ge 1$ for $i = 1, \dots, r$.

An r-partite graph is said to be *complete r-partite* if each vertex is joined to every other vertex that is not in the same class. Such a graph is denoted by $K_{n_1,\dots,n_r}$. Further, if $G(r)$ is uniquely r-colorable for convenience we call $G(r)$ a *uniquely r-partite graph*, where $r = \chi(G)$ or $r = p$, the order of the graph being p.

Bipartition of a connected graph, if it exists, is unique. But, in general, an r-partition of a graph, when it exists, need not be unique. Henceforth, if $G(r)$ is given to be an r-partite graph, we assume that an r-partition of $G(r)$ is prescribed.

The *r-partite complement* $\bar{G}(r)$ of an r-partite graph $G(r)$ is again an r-partite graph with vertex set $V(G(r))$ and satisfying the following conditions:

(i) for $u, v \in A_i$, $1 \le i \le r$: $uv \notin E(\bar{G}(r))$;

(ii) for $u \in A_i$, $v \in A_j$, $1 \le i \ne j \le r$: $uv \in E(\bar{G}(r))$ iff $uv \notin E(G(r))$.

An r-partite graph $G(r)$ with $r \ge 2$ is said to be an *r-partite self-complementary*

(r-p.s.c.) graph if there is an r-partition of $G(r)$ with respect to which $G(r)$ and $\bar{G}(r)$ are isomorphic.

The concepts r-partite complement and r-p.s.c. graphs were first defined and studied in Hebbare [5].

Note that an r-p.s.c. graph may be disconnected.

Remark. The class of classical self-complementary (s.c.) graphs, first studied by Sachs [7] and Ringel [6] is contained in the class of r-p.s.c. graphs, with $r \geq 2$ and $n_1 = \cdots = n_r = 1$. We refer to a survey article by Bhaskara Rao [1] and the references given there for most of the literature on s.c. graphs.

Let $G(r)$ be an r-p.s.c. graph with the vertex set $V = \{1, \ldots, p\}$. Then the isomorphism between $G(r)$ and $\bar{G}(r)$ can be represented as a permutation σ on V. We call σ an *r-partite complementing permutation* (r-p.c.p) for $G(r)$.

Now, let $\sigma = \sigma_1 \cdots \sigma_n$ be the disjoint cycle representation of σ. A cycle σ_i ($i = 1, \ldots, n$) of σ is said to be '*pure*' if $\sigma_i \subseteq A_j$ for some j ($j = 1, \ldots, r$) and '*mixed*' otherwise. Let $\mathscr{C}$, $\mathscr{C}_p$ and $\mathscr{C}_m$ respectively denote the set of all r-p.c.p.'s, the r-p.c.p.'s each of whose cycles is pure, and r-p.c.p.'s each of whose cycles is mixed, of $G(r)$. Also let $I_{\sigma_i} = \{j: \sigma_i$ includes at least one vertex of $A_j, 1 \leq j \leq r\}$, ($i = 1, \ldots, n$).

An r-p.c.p. σ of an r-p.s.c. graph $G(r)$ is said to be periodic if σ maps each A_i into some A_j. It is easily seen that if σ is periodic and $\sigma(A_i) \subseteq A_j$ then equality holds. The class of all periodic r-p.c.p.'s of $G(r)$ is denoted by $\mathscr{C}^*(G(r))$. The following observation is immediate.

Observation 1. Let $G(r)$ be r-p.s.c. and $\sigma \in \mathscr{C}^*(G(r))$. Then $u, v \in A_i$ for some i iff $\sigma(u), \sigma(v) \in A_j$ for some j.

Periodic complementing permutations have many interesting properties (for details see [2]). In particular we prove the following

Theorem 1.1. *Let $G(r)$ be an r-p.s.c. graph and let $\sigma \in \mathscr{C}^*$. Then $\sigma^2 \in \operatorname{Aut} G(r)$, where* Aut $G(r)$ *denotes the group of all automorphisms of $G(r)$.*

Proof. Let $u, v \in V$. If u, v belong to some class A_i then $\sigma^2(u), \sigma^2(v)$ belong to $\sigma^2(A_i)$. If u, v belong to different classes of the r-partition then by Observation 1, $\sigma^2(u), \sigma^2(v)$ also belong to different classes of the r-partition and $uv \in E(G)$ iff $\sigma(u)\sigma(v) \in E(\bar{G}(r))$ iff $\sigma(u)\sigma(v) \notin E(G)$ iff $\sigma^2(u)\sigma^2(v) \notin E(\bar{G}(r))$ iff $\sigma^2(u)\sigma^2(v) \in E(G)$. This proves the theorem.

Let $G(r)$ be r-p.s.c. and $\sigma \in \mathscr{C}$. A cycle τ of σ is said to be a *(k, α)-cycle* if there exist k distinct indices $i_1, i_2, \ldots, i_k$ such that τ can be written in the form

$$(u_{11}\, u_{21} \cdots u_{k1}\, u_{12}\, u_{22} \cdots u_{k2} \cdots u_{1\alpha}\, u_{2\alpha} \cdots u_{k\alpha})$$

with $u_{lm} \in A_{i_l}$ ($l = 1, \ldots, k$; $m = 1, \ldots, \alpha$).

For an r-p.s.c. graph $G(r)$ with $\mathscr{C}^*(G(r)) \neq \emptyset$, one can easily prove the following.

Theorem 1.2 (for proof see [2]). *Let $G(r)$ be r-p.s.c. and $\sigma \in \mathscr{C}^*$. Let τ be a cycle of σ such that $|I_\tau| = k$. Then*

(i) *τ is a (k, α)-cycle;*

(ii) *if ψ is any other cycle of σ with $I_\psi \cap I_\tau \neq \emptyset$, then* (a) *$I_\psi = I_\tau$ and* (b) *if τ takes vertices in A_i to A_j then so does ψ.*

For further structural properties of r-p.s.c. graphs and r-p.c.p. we refer to [2, 3, 5].

In this paper, the best possible bounds for diameters of connected r-p.s.c. graphs are given. It is shown that, if the diameter of a bipartite graph $G(2)$ is larger than six then the diameter of the bipartite complement $\bar{G}(2)$ of $G(2)$ is smaller than five. As a consequence, it follows that the diameter λ of a connected bi-p.s.c. graph satisfies $3 \leq \lambda \leq 6$. Further, if the diameter of a connected, r-partite graph $G(r)$ is larger than five then the diameter of the r-partite complement $\bar{G}(r)$ of $G(r)$ is smaller than five. As a consequence, it follows that the diameter λ of a connected r-p.s.c. graph satisfies $2 \leq \lambda \leq 5$. Finally, it is shown that if $r \geq 3$ and $G(r)$ is an r-p.s.c. graph with an r-p.c.p. $\sigma \in \mathscr{C}^*(G(r))$ such that any cycle of σ having length >1 intersects at least 3 classes of the r-partition then the diameter of $G(r)$ is either 2 or 3. As a consequence of the above result it follows that the diameter of an s.c. graph is either 2 or 3.

2. The bipartite case

Let $G(r)$ be a connected r-partite graph with diameter λ. Let $x, t \in V$ be such that $d(x, t) = \lambda$ where $d(u, v)$ is the distance function of $G(r)$, for $u, v \in V$. Then V can be partitioned into $\lambda + 1$ non-empty subsets $B_0, B_1, \ldots, B_\lambda$ such that $B_0 = \{x\}$ and

$$B_\mu = \{u \in V: d(x, u) = \mu\}, \quad (\mu = 1, \ldots, \lambda).$$

Theorem 2.1. *Let $G(2)$ be a connected, bipartite graph. If the diameter of $G(2)$ is larger than six then the diameter of $\bar{G}(2)$ is smaller than five.*

Proof. If $x \in A_1$, then $B_\mu \subseteq A_2$, $B_{\mu+1} \subseteq A_1$ for μ odd, $\mu \geq 1$; and if $x \in A_2$, then $B_\mu \subseteq A_1$, $B_{\mu+1} \subseteq A_2$ for all μ odd, $\mu \geq 1$; Thus either $B_\mu \subseteq A_1$ if and only if $B_{\mu+1} \subseteq A_2$, or $B_\mu \subseteq A_2$ if and only if $B_{\mu+1} \subseteq A_1$ for all μ, $0 \leq \mu \leq \lambda - 1$. That is,

$$(B_\mu \cup B_{\mu+1}) \cap A_i \neq \emptyset, \quad (0 \leq \mu \leq \lambda - 1; i = 1, 2).$$

Let $u, v \in V$. We shall prove that $\bar{d}(u, v) \leq 4$ where $\bar{d}$ denotes the distance function of $\bar{G}(2)$. Then two cases arise according as

(1) $u, v \in B_\mu$, for some μ, $0 \leq \mu \leq \lambda$, and

(2) $u \in B_\mu$ and $v \in B_\eta$ $(0 \leq \mu < \eta \leq \lambda)$.

Case 1. $u, v \in B_\mu$, $0 \leq \mu \leq \lambda$. Without loss of generality let $B_\mu \subseteq A_1$. Then u, $v \in A_1$. Now, if $B_\eta \subseteq A_2$ for some $\eta \neq \mu - 1, \mu, \mu + 1$ then $\bar{d}(u, v) = 2$. Otherwise, $B_\eta \subseteq A_1$. But then $B_\eta = \emptyset$ for $\eta \leq \mu - 3$ and $\eta \geq \mu + 3$. This implies that $\mu - 2 \leq 0$ and $\mu + 2 \geq \lambda$. Hence, $\lambda \leq \mu + 2 \leq 4$, a contradiction.

Case 2. $u \in B_\mu$ and $v \in B_\eta$ $(\mu < \eta)$. Then two subcases arise according as (a) B_μ, B_η are contained in the same class, say A_1 and (b) B_μ, B_η are contained in distinct classes, say $B_\mu \subseteq A_1, B_\eta \subseteq A_2$.

Case 2(*a*). $B_\mu, B_\eta \subseteq A_1$. Then $u, v \in A_1$. If $\mu \geq 3$, then for any $w \in (B_0 \cup B_1) \cap A_2$, $uw, vw \in \bar{E}$ implying that $\bar{d}(u, v) = 2$, where $\bar{E}$ denotes the edge set of $\bar{G}(2)$. Now, let $\mu = 2$. If $\emptyset \neq B_\alpha \subseteq A_2$ for some $\alpha \neq 1, 2, 3, \eta - 1, \eta, \eta + 1$, $0 \leq \alpha \leq \lambda$ then $\bar{d}(u, v) = 2$. Otherwise, $B_\alpha \subseteq A_1$ implying that $\eta \leq 6$ and $\lambda \leq \eta + 2$. Now, $B_2 \subseteq A_1$ implies that $B_\alpha \subseteq A_1$ for all α even, and $B_\alpha \subseteq A_2$ for all α odd. But, $B_\eta \subseteq A_1$ implies that η must be even. Since $\eta \leq 6$ and $\eta > \mu$, it follows that $\eta = 4$ or 6. If $\eta = 4$ then $\lambda \leq \eta + 2 = 6$, a contradiction. Suppose that $\eta = 6$. Let $w \in B_3 \subseteq A_2$ and $y \in B_5 \subseteq A_2$. Since $x \in B_0 \subseteq A_1$, $u \in B_2 \subseteq A_1$ and $v \in B_6 \subseteq A_1$, we get that $uy, yx, xw, wv \in \bar{E}$. Hence $\bar{d}(u, v) \leq 4$. Next let $\mu = 1$. If $B_\alpha \subseteq A_2$ for some $\alpha \neq 0, 1, 2, \eta - 1, \eta, \eta + 1$, $0 \leq \alpha \leq \lambda$, then $\bar{d}(u, v) = 2$. Otherwise $B_\alpha \subseteq A_1$. This implies $\eta \leq 5$ and $\lambda \leq \eta + 2$. Now, $B_1 \subseteq A_1$ implies that $B_\alpha \subseteq A_1$ for all α odd, and $B_\alpha \subseteq A_2$ for α even. But $B_\eta \subseteq A_1$ implies that η is odd. Since $\mu = 1$ we get that $\eta = 3$ or 5. If $\eta = 3$ then $\lambda \leq \eta + 2 \leq 5$, a contradiction. If $\eta = 5$ then for any $w \in B_4$ and $z \in B_7$ $(B_7 \neq \emptyset$, since $\lambda \geq 7)$ we get that $w \in A_2, z \in A_1$ and $uw, wz, zx, xv \in \bar{E}$ and hence $\bar{d}(u, v) \leq 4$. Finally if $\mu = 0$ then $B_\alpha \subseteq A_1$ if α is even and $B_\alpha \subseteq A_2$ if α is odd. Also since $B_\eta \subseteq A_1$, η is even. If $\eta \leq 4$ then let $w \in B_7$ and if $\eta \geq 6$ let $w \in B_3$. Then $uw, vw \in \bar{E}$ and $\bar{d}(u, v) = 2$.

Case 2(*b*). $B_\mu \subseteq A_1$, $B_\eta \subseteq A_2$. Then $u \in A_1$ and $v \in A_2$. If $\eta - \mu \neq 1$, $uv \in \bar{E}$ and $\bar{d}(u, v) = 1$. Otherwise $\eta = \mu + 1$. If $2 \leq \mu \leq \lambda - 3$, then for any $y \in B_{\mu-2} \subseteq A_1$ and $z \in B_{\mu+3} \subseteq A_2$, we have that $uz, zy, yv \in \bar{E}$ and hence $\bar{d}(u, v) \leq 3$. Otherwise, either $\mu \leq 1$ or $\mu \geq \lambda - 2$. If $\mu \leq 1$ then for any $y \in B_{\mu+3} \subseteq A_2$ and $z \in B_{\mu+6} \subseteq A_1$ $(B_{\mu+6} \neq \emptyset$, since $\mu \leq 1$ and $\lambda \geq 7)$, we have that $uy, yz, zv \in \bar{E}$ and hence $\bar{d}(u, v) \leq 3$. Finally, if $\mu \geq \lambda - 2$ for any $y \in B_{\mu-2} \subseteq A_1$ and $z \in B_{\mu-5} \subseteq A_2$ $(B_{\mu-5} \neq \emptyset$, since $\mu - 5 \geq \lambda - 7 \geq 0)$, we have that $uz, zy, yv \in \bar{E}$ and hence $\bar{d}(u, v) \leq 3$.

Thus we have shown that $\bar{d}(u, v) \leq 4$ for any $u, v \in V$ and hence $\bar{\lambda} \leq 4$ follows, where $\bar{\lambda}$ is the diameter of $\bar{G}(2)$.

Corollary 2.2. *If $G(2)$ is a connected bipartite graph with diameter larger than six then the bipartite complement $\bar{G}(2)$ is connected.*

Corollary 2.3. *If $G(2)$ is a connected bi-p.s.c. graph with diameter λ, then* $3 \leq \lambda \leq 6$.

Remark 1. Connected bi-p.s.c. graphs with diameter λ exist for all λ, $3 \leq \lambda \leq 6$. The graphs in Fig. 1 illustrate this fact.

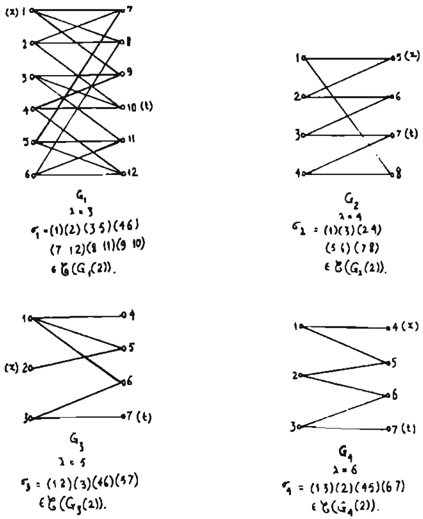

Fig. 1.

Remark 2. Let $G(r)$ be an r-p.s.c. graph with diameter λ. Construct a new graph $G^*(r)$ from $G(r)$ replacing each vertex $u \in V$ by a set U of n distinct vertices and whenever $uw \in E$ define a complete bipartite graph $K_{n,n}$ between U and W in $G^*(r)$. Then $G^*(r)$ is again an r-p.s.c. graph with diameter λ. Further if $G(r)$ is a (p, q)-graph then $G^*(r)$ is a (pn, n^2q) graph. Using the above construction and the graphs given in Fig. 1, one can construct an infinite class of bi-p.s.c. graphs with diameter λ for all λ, $3 \leq \lambda \leq 6$.

3. The r-partite case, $r \geq 3$

Theorem 3.1. *Let $G(r)$ be a connected r-partite graph with $r \geq 3$. If the diameter of $G(r)$ is larger than five then $\bar{G}(r)$ must have diameter smaller than five.*

Proof. For any $u, v \in V$, we shall prove that $\bar{d}(u, v) \leq 4$. We prove this in four cases. Let

$$S_\mu = B_\mu \cap \Big(\bigcup_{k\neq i} A_k\Big),$$

$$S^*_\mu = B_\mu \cap \Big(\bigcup_{k\neq i,j} A_k\Big),$$

for some fixed i, j, $1 \le i < j \le r$ and $0 \le \mu \le \lambda$.

Case 1. $u, v \in A_i \cap B_\mu$ $(1 \le i \le r,\ 1 \le \mu \le \lambda)$. If $S_\alpha \neq \emptyset$ for some $\alpha \neq \mu-1, \mu, \mu+1$; $0 \le \alpha \le \lambda$ then for any $w \in S_\alpha$, $uw, wv \in \bar{E}$ and hence $\bar{d}(u, v) \le 2$. Otherwise, $S_\alpha = \emptyset$ and hence $B_\alpha \subseteq A_i$. But then $B_\alpha = \emptyset$ whenever $\alpha \le \mu - 3$ and $\alpha \ge \mu + 3$. That is, $\mu - 2 \le 0$ and $\mu + 2 \ge \lambda$ which imply that $\lambda \le 4$, a contradiction.

Case 2. $u, v \in B_\mu$; $u \in A_i$, $v \in A_j (0 \le \mu \le \lambda;\ 1 \le i < j \le r)$. If $S^*_\alpha \neq \emptyset$ for some $\alpha \neq \mu-1, \mu, \mu+1$; $0 \le \alpha \le \lambda$ then for any $w \in S^*_\alpha$ we have that $uw, wv \in \bar{E}$ and hence $\bar{d}(u, v) \le 2$. Otherwise, $S^*_\alpha = \emptyset$ implying that $B_\alpha \subseteq A_i \cup A_j$. Then $(B_\alpha \cup B_{\alpha+1}) \cap A_i \neq \emptyset$ and $(B_\alpha \cup B_{\alpha+1}) \cap A_j \neq \emptyset$. Now, if $\mu \le 1$, let $w \in B_3$. Without loss of generality let $w \in A_j$. Let $z \in (B_5 \cup B_6) \cap A_i$. Then $uz, zw, wv \in \bar{E}$ and hence $\bar{d}(u, v) \le 3$. If $\mu = 2$ without loss of generality let $x \in A_i$ then for any $z \in (B_4 \cup B_5) \cap A_j$, $uz, zx, xv \in \bar{E}$ and hence $\bar{d}(u, v) \le 3$. If $3 \le \mu \le \lambda - 3$, then for $w \in (B_{\mu-3} \cup B_{\mu-2}) \cap A_i$ and $z \in (B_{\mu+2} \cup B_{\mu+3}) \cap A_j$ we have that $uz, zw, wv \in \bar{E}$ and hence $\bar{d}(u, v) \le 3$. If $\mu = \lambda - 2$, let $z \in B_\lambda$ and without loss of generality let $z \in A_j$ then for any $w \in (B_{\lambda-5} \cup B_{\lambda-4}) \cap A_i$ we have that $uz, zw, wv \in \bar{E}$ and hence $\bar{d}(u, v) \le 3$. If $\mu = \lambda - 1$ let $z \in B_{\lambda-3}$ and without loss of generality let $z \in A_j$, then for any $w \in (B_{\lambda-6} \cup B_{\lambda-5}) \cap A_i$, $uz, zw, wv \in \bar{E}$ and hence $\bar{d}(u, v) \le 3$. Finally, if $\mu = \lambda$, then for $z \in (B_{\lambda-3} \cup B_{\lambda-2}) \cap A_j$ and $w \in (B_{\lambda-6} \cup B_{\lambda-5}) \cap A_i$ we have that $uz, zw, wv \in \bar{E}$ and hence $\bar{d}(u, v) \le 3$.

Case 3. $u, v \in A_i$; $u \in B_\mu$, $v \in B_\eta$ $(1 \le i \le r,\ 0 \le \mu < \eta \le \lambda)$. If $S_\alpha \neq \emptyset$, for some $\alpha \neq \mu-1, \mu, \mu+1, \eta-1, \eta, \eta+1$ and $0 \le \alpha \le \lambda$, then for $w \in S_\alpha$, $uw, wv \in \bar{E}$ and hence $\bar{d}(u, v) \le 2$. Otherwise, $S_\alpha = \emptyset$ and hence $B_\alpha \subseteq A_i$. Hence, $\mu \le 2$, $\eta + 2 \ge \lambda$ and $\mu + 2 \le \eta \le \mu + 4$ (if $\eta = \mu + 1$ then $\lambda \le \eta + 2 = \mu + 3 \le 5$, a contradiction. At this stage, we consider three subcases of $\mu = 0, 1, 2$.

Case 3(*a*). $\mu = 0$. If $\eta \le 3$ then $\lambda \le \eta + 2 \le 5$, a contradiction. If $\eta = 4$ then $B_\mu = \{u\}$. Further, $u \in A_i$ implies that $S_1 \neq \emptyset$. Again $B_6 \subseteq A_i$ implies that $S_5 \neq \emptyset$ (otherwise, $\lambda \le 5$ a contradiction). If possible, let $w \in B_1 \cap A_k$ and $y \in B_5 \cap A_l$, $k \neq l \neq i$. Then $uy, yw, wv \in \bar{E}$ and hence $\bar{d}(u, v) \le 3$. Otherwise, for some $k \neq i$, $B_1 \subseteq A_k$, $B_5 \subseteq A_k \cup A_i$. Let $w \in B_1 \cap A_k$. Since $r \ge 3$, and $B_2 \subseteq A_i$, $B_\alpha \cap A_l \neq \emptyset$ for some $\alpha = 3, 4$. Let $z \in B_\alpha \cap A_l$. Then $uz, zw, wv \in \bar{E}$ and hence $\bar{d}(u, v) \le 3$.

Case 3(*b*). $\mu = 1$. If $\eta = 3$ then since $\eta + 2 \ge \lambda$, $\lambda \le 5$, a contradiction. Hence, let $\eta \ge 4$. Since $u \in B_1 \cap A_i$, $x \in A_k$, for some $k \neq i$. Now, $S_\eta \cup S_{\eta+1} \neq \emptyset$ for otherwise we have $S_\eta = \emptyset$ and $S_{\eta+1} = \emptyset$. This implies that $B_\eta, B_{\eta+1} \subseteq A_i$ and hence $B_{\eta+1} = \emptyset$. So $\lambda = \eta \le \mu + 4 = 5$, a contradiction. Hence, $S_\eta \cup S_{\eta+1} \neq \emptyset$. If possible, let $w \in (S_\eta \cup S_{\eta+1}) \cap A_l$, $l \neq k \neq i$. Then $uw, wx, xv \in \bar{E}$ and hence $\bar{d}(u, v) \le 3$. Otherwise, let $w \in S_\eta \cup S_{\eta+1} \subseteq A_k$. Since $r \ge 3$, $B_\alpha \cap A_l \neq \emptyset$ for some $\alpha = 1, 2, \eta - 1$ $l \neq k \neq i$. Let $z \in B_\alpha \cap A_l$. If $z \in B_1 \cup B_2$ then $uw, wz, zv \in \bar{E}$ and hence $\bar{d}(u, v) \le 3$. If $z \in B_{\eta-1}$ then $uz, zx, xv \in \bar{E}$ and hence $\bar{d}(u, v) \le 3$.

Case 3(*c*). $\mu=2$. Let $\eta=4$. Then $B_0, B_6 \subseteq A_i$ implying $S_1 \neq \emptyset$, $S_5 \neq \emptyset$. If possible, let $w \in S_1$ and $y \in S_5$ such that $w \in A_k$, $y \in A_l$ where $k \neq l \neq i$, then $uy, yw, wv \in \bar{E}$ and $\bar{d}(u,v) \leq 3$. Otherwise, $B_1, B_5 \subseteq A_k \cup A_i$, $k \neq i$. Then, since $r \geq 3$, $B_\alpha \cap A_l \neq \emptyset$, for some $\alpha = 2, 3, 4$, follows. Let $z \in B_\alpha \cap A_l$. If $z \in B_2$ then $uy, yz, zv \in \bar{E}$ and hence $\bar{d}(u,v) \leq 3$. if $z \in B_3$, then $uy, yz, zw, wv \in \bar{E}$ and hence $\bar{d}(u,v) \leq 4$. If $z \in B_4$, then $uz, zw, wv \in \bar{E}$ and hence $\bar{d}(u,v) \leq 3$.

Suppose now that $\eta \geq 5$. If $S_{\eta+1} \neq \emptyset$ let $y \in S_{\eta+1}$ and if possible let $w \in S_1$ ($S_1 \neq \emptyset$, since $B_0 \subseteq A_i$) such that $w \in A_k$, $y \in A_l$, $k \neq l \neq i$. Then $uy, yw, wv \in \bar{E}$ and hence $\bar{d}(u,v) \leq 3$. Otherwise, $B_1 \subseteq A_k \cup A_i$ and $B_{\eta+1} \subseteq A_k \cup A_i$. Let $w \in B_1 \cap A_k$ and $y \in B_{\eta+1} \cap A_k$, $k \neq i$. Since $r \geq 3$, $B_\alpha \cap A_l \neq \emptyset$ for some l, $l \neq k \neq i$, $1 \leq l \leq r$ and for some $\alpha = 2, 3, \eta-1, \eta$. Let $z \in B_\alpha \cap A_l$. If $z \in B_2$ then $uy, yz, zv \in \bar{E}$ and hence $\bar{d}(u,v) \leq 3$. If $z \in B_3$ or $B_{\eta-1}$ then $uy, yz, zw, wv \in \bar{E}$ and hence $\bar{d}(u,v) \leq 4$. If $z \in B_\eta$, then $uz, zw, wv \in \bar{E}$ and hence $\bar{d}(u,v) \leq 3$.

If $S_{\eta+1} = \emptyset$, then $B_{\eta+1} \subseteq A_i$. Suppose $S_\eta \neq \emptyset$. Then, let $y \in S_\eta$ and if possible let $w \in S_1$ such that $w \in A_k$, $y \in A_l$, $k \neq l \neq i$. Then $uy, yw, wv \in \bar{E}$ and hence $\bar{d}(u,v) \leq 3$. Otherwise, $B_1 \subseteq A_k \cup A_i$ and $B_\eta \subseteq A_k \cup A_i$, for some $k \neq i$. Let $w \in B_1 \cap A_k$ and $y \in B_\eta \cap A_k$, $k \neq i$. Since $r \geq 3$, $B_\alpha \cap A_l \neq \emptyset$ for some l, $1 \leq l \leq r$, $l \neq k \neq i$ and $\alpha = 2, 3, \eta-1$. Let $z \in B_\alpha \cap A_l$. If $z \in B_2$ or B_3 then $uy, yz, zv \in \bar{E}$ and hence $\bar{d}(u,v) \leq 3$. If $z \in B_{\eta-1}$ then $uz, zw, wv \in \bar{E}$ and hence $\bar{d}(u,v) \leq 3$.

If $S_\eta = \emptyset$, then $B_\eta \subseteq A_i$. But $B_{\eta+1} \subseteq A_i$ implies that $B_{\eta+1} = \emptyset$. Hence, $6 \leq \lambda \leq \eta \leq \mu+4 = 6$ and so $\eta = 6$. If possible let $w \in S_1$ and $y \in S_5$ such that $w \in A_k$ and $y \in A_l$, $k \neq l \neq i$. Then $\bar{d}(u,v) \leq 3$. Otherwise, $B_1, B_5 \subseteq A_k \cup A_i$ for some $k \neq i$. Let $w \in B_1 \cap A_k$, $y \in B_5 \cap A_k$, $k \neq i$. But, since $r \geq 3$, $B_\alpha \cap A_l \neq \emptyset$ for some α ($\alpha = 2, 3$), $k \neq l \neq i$. Let $z \in B_\alpha \cap A_l$, then $uy, yz, zv \in \bar{E}$ and hence $\bar{d}(u,v) \leq 3$.

Case 4. $u \in A_i \cap B_\mu$, $v \in A_j \cap B_\eta$ ($1 \leq i < j \leq r$; $0 \leq \mu < \eta \leq \lambda$). If $\eta - \mu \geq 2$ then $uv \in \bar{E}$. So let $\eta = \mu+1$. If $S_\alpha^* \neq \emptyset$ for some $\alpha \neq \mu-1, \mu, \mu+1, \mu+2$ let $w \in S_\alpha^*$. Then $uw, wv \in \bar{E}$ and hence $\bar{d}(u,v) \leq 2$. Otherwise, $B_\alpha \subseteq A_i \cup A_j$. Then five subcases arise according as $\mu = 0, 1, 2, 3$ and $\mu \geq 4$.

Case 4(*a*). $\mu = 0$. Then $u \in B_0$, $v \in B_1$ and $B_\alpha \subseteq A_i \cap A_j$ for all $\alpha \geq 3$. If $B_3 \cap A_i \neq \emptyset$, let $w \in B_3 \cap A_i$. Note that, $(B_5 \cup B_6) \cap A_j \neq \emptyset$. Otherwise, $B_5 \cap A_j = \emptyset$ and $B_6 \cap A_j = \emptyset$ imply that $B_5 \subseteq A_i$, $B_6 \subseteq A_i$ and hence $B_6 = \emptyset$. But then $\lambda \leq 5$, a contradiction. Let $z \in (B_5 \cup B_6) \cap A_j$. Then $uz, zw, wv \in \bar{E}$ and hence $\bar{d}(u,v) \leq 3$. Finally, if $B_3 \cap A_i = \emptyset$ it follows that $B_3 \subseteq A_j$. Let $w \in B_3$. Then, for $z \in (B_5 \cup B_6) \cap A_i \neq \emptyset$ we have that $uw, wz, zv \in \bar{E}$ and hence $\bar{d}(u,v) \leq 3$.

Case 4(*b*). $\mu = 1$. Then $B_\alpha \subseteq A_i \cup A_j$ for all $\alpha \geq 4$ and hence let $w \in (B_5 \cup B_6) \cap A_i \neq \emptyset$ and $z \in (B_5 \cup B_6) \cap A_j \neq \emptyset$. Since $r \geq 3$, $B_\alpha \cap A_k \neq \emptyset$ for some α, $0 \leq \alpha \leq 3$, $k \neq i \neq j$. Let $y \in B_\alpha \cap A_k$. Then $uz, zy, yw, wv \in \bar{E}$ and hence $\bar{d}(u,v) \leq 4$.

Case 4(*c*). $\mu = 2$. Then $B_0, B_\alpha \subseteq A_i \cup A_j$ for $\alpha \geq 5$. Now, let $w \in (B_5 \cup B_6) \cap A_i \neq \emptyset$ and $Z \in (B_5 \cup B_6) \cap A_j \neq \emptyset$. Now if $x \in A_i$ then $uz, zx, xv \in \bar{E}$ and hence $\bar{d}(u,v) \leq 3$. If $x \in A_j$ then $ux, xw, wv \in \bar{E}$ and hence $\bar{d}(u,v) \leq 3$.

Case 4(*d*). $\mu = 3$. Then $B_0, B_1, B_\alpha \subseteq A_i \cup A_j$ for $\alpha \geq 6$. Let $w \in (B_0 \cup B_1) \cap A_i \neq \emptyset$, $z \in (B_0 \cup B_1) \cap A_j \neq \emptyset$ and $y \in B_6$. If $y \in A_i$ then $uz, zy, yv \in \bar{E}$ and hence $\bar{d}(u,v) \leq 3$. If $y \in A_j$ then $uy, yw, wv \in \bar{E}$ and hence $\bar{d}(u,v) \leq 3$.

Case 4(*e*). $\mu \geq 4$. Then $B_\alpha \subseteq A_i \cup A_j$ for $\alpha \neq \mu-1, \mu, \mu+1, \mu+2$. Let $w \in (B_0 \cup B_1) \cap A_i \neq \emptyset$, and $z \in (B_0 \cup B_1) \cap A_j \neq \emptyset$. Since $r \geq 3$, $B_\alpha \cap A_k \neq \emptyset$ for some $\alpha = \mu-1, \mu, \mu+1, \mu+2$; $k \neq i \neq j$. Let $y \in B_\alpha \cap A_k$. Then $uz, zy, yw, wv \in \bar{E}$ and hence $\bar{d}(u, v) \leq 4$.

Thus we have established that $\bar{d}(u, v) \leq 4$ for all $u, v \in V(\bar{G}(r))$ and hence the diameter $\bar{\lambda}$ of $\bar{G}(r)$ satisfies $\bar{\lambda} \leq 4$.

Corollary 3.2. *If $G(r)$ is a connected, r-partite graph with diameter larger than five then the r-partite complement $\bar{G}(r)$ of $G(r)$ is connected.*

Corollary 3.3. *Let $G(r)$ be a connected r-p.s.c. graph with diameter λ. Then* $2 \leq \lambda \leq 5$.

Remark 3. r-p.s.c. graphs, with $r \geq 3$, with diameter λ exist for all λ, $2 \leq \lambda \leq 5$. This is illustrated in Fig. 2. One can construct infinite families of r-p.s.c. graphs for each λ, $2 \leq \lambda \leq 5$, by using the construction described in Remark 2.

4. r-P.s.c. graphs $G(r)$ with $\mathscr{C}^*(G(r)) \neq \emptyset$

In this section we generalise a well-known result of Ringel [6] and Sachs [7] in the following

Theorem 4.1. *Let $r \geq 3$ and $G(r)$ be r-p.s.c. If there exists $\sigma \in \mathscr{C}^*(G(r))$ such that any cycle of σ having length >1 intersects at least three classes of the r-partition, then the diameter of $G(r)$ is either 2 or 3.*

Proof. Let $\sigma \in \mathscr{C}^*(G(r))$ be such that any cycle of σ having length >1, intersects at least three classes. By Theorem 1.1, $\sigma^2 \in \operatorname{Aut} G(r)$. Let $u, v \in V(G(r))$. We first prove the following claims.

Claim 1. *If $\sigma(u) \neq u$, then $d(u, \sigma(u)) \leq 2$.*

Suppose $\sigma(u) \neq u$. Then by hypothesis and Theorem 1.2, the cycle of σ containing u is a (k, α)-cycle for some $k \geq 3$ and some $\alpha \geq 1$. Thus $u, \sigma(u), \sigma^2(u)$ all belong to different classes. Now if $u\sigma(u) \in E$ we are done. Otherwise $u\sigma(u) \in \bar{E}$ and so $\sigma^{-1}(u)u \in E$. Since $\sigma^2 \in \operatorname{Aut} G(r)$, it follows that $\sigma(u)\sigma^2(u) \in E$. Now, either $\sigma^{-1}(u)\sigma(u) \in E$ or $u\sigma^2(u) \notin \bar{E}$ and hence $u\sigma^2(u) \in E$. Thus either $u\sigma^{-1}(u)\sigma(u)$ or $u\sigma^2(u)\sigma(u)$ is a 2-path in $G(r)$. This proves the claim.

Claim 2. *If $\sigma(u) \neq u$ and $\sigma(v) \neq v$, then either $\sigma(u)$, v belong to different classes or u, $\sigma(v)$ belong to different classes.*

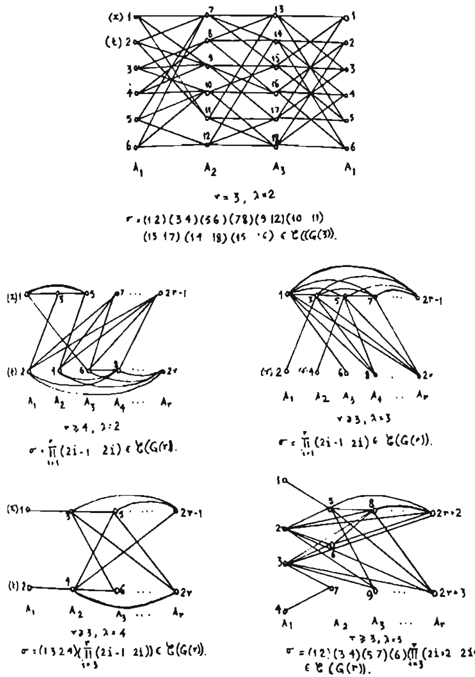

Fig. 2.

If the claim is false, there exist A_i and A_j such that $\sigma(u), v \in A_i, \sigma(v), u \in A_j$. Since $\sigma \in \mathscr{C}^*(G(r))$, it follows that $\sigma(A_i) = A_j$ and $\sigma(A_j) = A_i$. Also by hypothesis and since $\sigma(u) \neq u$, we have $i \neq j$. But then σ has a $(2, \alpha)$-cycle, contradicting the hypothesis. This proves the claim.

We will now prove that for any $u, v \in V(G(r))$, $d(u, v) \leqslant 3$. We consider the following three cases:

Case 1. $\sigma(u) \neq u, \sigma(v) \neq v$. By Claim 2, we assume without loss of generality that $\sigma(u)$, v belong to different classes. Now if $\sigma(u), v \in E$ then by Claim 1, $d(u, v) \leq 3$. Otherwise $\sigma(u)v \in \bar{E}$ and so $u\sigma^{-1}(v) \in E$. By Claim 1, $d(\sigma^{-1}(v), v) \leq 2$ and so $d(u, v) \leq 3$.

Case 2. σ sends exactly one of u, v to itself. Without loss of generality assume that $\sigma(u) \neq u$, $\sigma(v) = v$. If $uv \in E$ we are done. Otherwise $uv \notin E$, hence $\sigma(u)\sigma(v) \notin \bar{E}$, i.e. $\sigma(u)v \notin \bar{E}$. Now if $\sigma(u), v \in A_i$ for some i, then since $\sigma(v) = v$, it follows that $\sigma(A_i) = A_i$. Since $\sigma(u) \in A_i$, it also follows that $u \in A_i$. But $u \neq \sigma(u)$ and so if τ is the cycle of σ containing u then τ has length >1 and τ takes vertices from A_i only, contradicting the hypothesis. Hence $\sigma(u)$, v belong to different classes. Since $\sigma(u)v \notin \bar{E}$, it follows that $\sigma(u)v \in E$. Now by Claim 1, we have $d(u, v) \leq 3$.

Case 3. $\sigma(u) = u$, $\sigma(v) = v$. If u, v belong to different classes then $uv \in E$ iff $\sigma(u)\sigma(v) \in \bar{E}$ iff $uv \notin E$, a contradiction. So $u, v \in A_i$ for some i. Choose and fix a w in some A_j, $j \neq i$. By a similar argument to that above it follows that $\sigma(w) \neq w$. Now by hypothesis and Theorem 1.2 we have that the cycle containing w is a (k, α)-cycle for some $k \geq 3$. Thus, $w, \sigma(w), \sigma^2(w)$ belong to different classes. Also since $\sigma(A_i) = A_i$, we have $w, \sigma(w), \sigma^2(w) \notin A_i$. Now if uw, vw are edges of $G(r)$ we are done. Otherwise without loss of generality we assume that $uw \notin E$. Then $u\sigma(w) \notin \bar{E}$ and so $u\sigma(w) \in E$. Now if $v\sigma(w) \in E$, we are done. Otherwise $v\sigma(w) \notin E$ and so $v\sigma(w) \in \bar{E}$. Since $\sigma^{-1}(v) = v$, it follows that $vw \in E$. Also, since $\sigma^2 \in \operatorname{Aut} G(r)$, we have $v\sigma^2(w) \in E$. Now if $w\sigma(w) \in E$ then $u, \sigma(w), w, v$ is a 3-path in $G(r)$; otherwise $\sigma(w)\sigma^2(w) \in E$, and so $u, \sigma(w), \sigma^2(w), v$ is a 3-path in $G(r)$. In either case, $d(u, v) \leq 3$.

This completes the proof of Theorem 4.1.

As a consequence of the above theorem, we have the following

Corollary 4.2 (Ringel [6], Sachs [7]). *Every s.c. graph G with more than one vertex has diameter* 2 *or* 3.

Finally we remark that the method adopted in proving the theorems of Sections 2 and 3 is due to S. Bhaskara Rao who proved Corollary 4.2 using this method. Originally, Sachs [7] and Ringel [6] proved Corollary 4.2 using complementing permutations.

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