Table 2. *Comparison of* (4) *with Kimball & Leach's approximations* $\hat{I}$ *and* I^*. I = *exact value of* $I_x(p,q)$; I_N = *approximation* (4)†

q →	2	2	4	2	6	10
p →	2	5	5	12	12	12
x	0·14	0·42	0·29	0·68	0·48	0·37
$10^4 I$	0533	0510	0505	0473	0517	0481
$10^4(\hat{I}-I)$	− 65	+15	−34	+15	+03	− 53
$10^4(I_N-I)$	− 47	−15	−36	−07	−18	− 35
$10^4(I^*-I)$	− 21	−10	−16	−07	−07	− 11
x	0·36	0·64	0·47	0·82	0·61	0·49
$10^4 I$	2955	3006	2999	2920	2923	2989
$10^4(\hat{I}-I)$	− 74	+31	−81	+40	+04	−152
$10^4(I_N-I)$	− 36	−10	−35	−06	−18	− 41
$10^4(I^*-I)$	+121	+20	+36	−02	+08	+ 17
x	0·64	0·82	0·65	0·92	0·73	0·60
$10^4 I$	7045	7044	7064	7206	7011	6914
$10^4(\hat{I}-I)$	− 22	+50	−42	+59	+15	−119
$10^4(I_N-I)$	+ 46	+35	+30	+32	+19	+ 22
$10^4(I^*-I)$	+ 59	+39	+30	+32	+16	+ 16
x	0·86	0·94	0·81	0·97	0·83	0·71
$10^4 I$	9467	9541	9524	9436	9452	9452
$10^4(\hat{I}-I)$	− 35	−23	−19	−16	−02	− 33
$10^4(I_N-I)$	− 15	−25	+03	−21	+04	+ 15
$10^4(I^*-I)$	− 30	+27	−13	−22	−05	− 07

† I would like to thank Dr Kimball for sending full details of his results.

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On certain functions of normal variates which are uncorrelated of a higher order

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It is well known that two linear forms $L_1 = a_1x_1 + \ldots + a_nx_n$ and $L_2 = b_1x_1 + \ldots + b_nx_n$ in independently and normally distributed variates $x_1, \ldots, x_n$ are uncorrelated if, and only if, the relation

$$\mathbf{ab}' = 0 \tag{1a}$$

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is satisfied. Moreover (1a) is equivalent to the stochastic independence of L_1 and L_2. Thus the condition for the stochastic independence of two linear forms can be expressed either in terms of the coefficients as equation (1a) or in terms of the product moment of the forms as

$$E(L_1 L_2) = E(L_1)E(L_2). \tag{1b}$$

A number of authors investigated the independence of two homogeneous forms $Q_1 = \mathbf{xAx}'$ and $Q_2 = \mathbf{xBx}'$ in independently distributed normal variates with zero mean and unit variances. We mention here only Craig (1943), Hotelling (1944) and Matusita (1949). These authors showed that Q_1 and Q_2 are independent if, and only if,

$$\mathbf{AB} = 0. \tag{2a}$$

A different condition, in terms of product moments, was given by Kawada (1950) who proved that the four relations

$$E(Q_1^i Q_2^j) = E(Q_1^i)E(Q_2^j) \quad (i = 1, 2; j = 1, 2) \tag{2b}$$

are equivalent to the condition (2a). Kawada's condition can be formulated more concisely by introducing the following terminology.

Let x and y be two random variables and suppose that the expectation $E(x^r y^s)$ exists. We say that x and y are uncorrelated of order (r, s) if the relations

$$E(x^i y^j) = E(x^i)E(y^j) \quad (i = 1, \ldots, r; j = 1, \ldots, s)$$

hold. In our terminology (2b) means Q_1 and Q_2 are uncorrelated of order (2, 2).

In this note we prove the following theorem

Theorem. Let $x_1, x_2, \ldots, x_n$ be n independent normal variates with zero mean and unit variance. If the inhomogeneous quadratic form $Q = \mathbf{xAx}' + \mathbf{bx}'$ and the linear form $L = \mathbf{cx}'$ are uncorrelated of order (2, 2) then $\mathbf{cA} = 0$ and $\mathbf{cb}' = 0$.

Proof. We note first that $E(x_i^2) = 1$, $E(x_i^4) = 3$, $E(x_i^6) = 15$. Using these values we obtain after some straightforward but tedious computations the following expressions

$$\left.\begin{aligned} E(LQ) - E(L)E(Q) &= \mathbf{cb}', \\ E(L^2Q) - E(L^2)E(Q) &= 2\mathbf{cAc}', \\ E(LQ^2) - E(L)E(Q^2) &= 4\mathbf{cAb}' + 2\mathbf{cb}' \operatorname{tr} \mathbf{A}, \\ E(L^2Q^2) - E(L^2)E(Q^2) &= 8\mathbf{cAA}'\mathbf{c}' + 4\mathbf{cAc}' \operatorname{tr} \mathbf{A} + 2(\mathbf{cb}')^2. \end{aligned}\right\} \tag{3}$$

We write here $\operatorname{tr} \mathbf{A} = \sum_{i=1}^{n} a_{ii}$ for the trace of the matrix $\mathbf{A}$. Since Q and L are uncorrelated of order (2, 2) conditions (3) reduce to

$$\mathbf{cA} = 0, \quad \mathbf{cb}' = 0 \tag{4}$$

so that the theorem is established.

It was shown by Laha (1956) that conditions (4) ensure the independence of L and Q. Therefore the independence of L and Q follows from the assumption that L and Q are uncorrelated of order (2, 2). The converse statement is obvious and we obtain the following corollary to our theorem.

Corollary. Let $x_1, x_2, \ldots, x_n$ be n independent normal variables with zero mean and unit variance. The inhomogeneous quadratic polynomial $Q = \mathbf{xAx}' + \mathbf{bx}'$ and the linear form $L = \mathbf{cx}'$ are independent if, and only if, they are uncorrelated of order (2, 2).

If the polynomial Q is homogeneous, that is if $\mathbf{b} = 0$, then we can conclude from Kawada's result that L and Q are independent if they are uncorrelated of order (4, 2). Our result shows that a weaker assumption is sufficient for the independence of L and Q.

It would be desirable to obtain further results concerning the connexion between the order of uncorrelatedness and the degree of two independent polynomial statistics in independent normal variates.

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