THE ISOMORPHISM BETWEEN GRAPHS AND THEIR ADJOINT GRAPHS

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(received May 25, 1964)

1. Introduction. A graph G is defined as a set $X = \{x_1, \ldots, x_n\}$ of elements x_i called vertices, and a collection Γ of (not necessarily distinct) unordered pairs of distinct vertices, called edges. An edge (x_i, x_j) is said to be incident to x_i and x_j which are its end-vertices.

DEFINITION 1. The adjoint (or the interchange graph) I(G) of a given graph $G = (X, \Gamma)$ is defined as follows. The edges of G form the vertices of I(G), and two vertices in I(G) are joined by zero, one or two edges according as the corresponding edges in G have zero, one or two common end-vertices respectively.

For example, in Fig. 1 we see the graphs G_1, G_2 and G_3 and their adjoints $I(G_1)$, $I(G_2)$ and $I(G_3)$. The edges have been called e_1, e_2, e_3.

DEFINITION 2. $I^n(G)$ is defined recursively by

$$I^n(G) = I[I^{n-1}(G)], \quad n \geq 2 .$$

DEFINITION 3. Two graphs G and G' are isomorphic if there exists a one-one correspondence between their vertices such that if $x_i, x_j \in G$ correspond to vertices $x_{i'}, x_{j'} \in G'$

Canad. Math. Bull. vol. 8, no. 1, February 1965

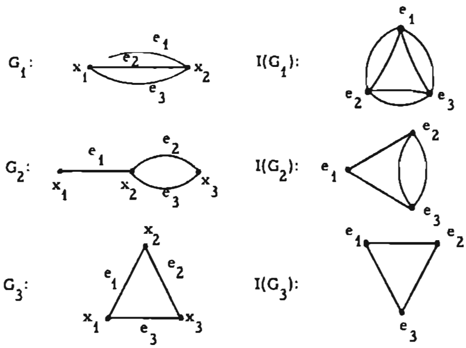

Fig. 1

respectively, then the edge (x_i, x_j) exists in G if and only if the edge $(x_{i'}, x_{j'})$ exists in G'.

DEFINITION 4. The degree of a vertex x_i is the number of edges incident to it.

The problem dealt with in this paper is that of determining graphs which are isomorphic to their adjoints; and in general, of graphs G which are isomorphic to $I^n(G)$ for some n. The latter is the generalisation of a problem suggested in Ore [1].

The solution of this problem occurs as Theorem 2 in section 3. The theorem 1 is a general result applicable to any graph. The proofs of these theorems also appear in section 3. In section 2 are given certain obvious results which are useful in simplifying the proof of the main theorem.

2. Preliminary remarks. First we define the connected components of a graph. A graph is said to be connected if for any pair of vertices x_i, x_j there exists a sequence $u_1, \ldots, u_k$ of edges of the graph such that (1) u_1 is incident to x_i and u_k

to x_j, and (2) u_{i-1} is incident to one end-vertex of u_i, and u_{i+1} to the other, for $2 \leq i \leq k-1$. In other words, between every pair of vertices there exists a chain of edges. Any given graph can be partitioned into components, called the connected components of the graph, such that each component is a connected graph and there are no edges joining vertices belonging to different components.

Considering a graph G, we see that the edges in a connected component of G form the vertices of a connected component of I(G), and vice-versa.

From the definition of an adjoint graph, we can easily verify the following lemmas.

LEMMA 1. Let the graph G consist of n edges in a chain $(n \geq 1)$, as shown in Fig. 2(a), then the adjoint I(G) consists of $n-1$ edges in a chain, as in Fig. 2(b). Conversely, if I(G) consists of $n-1$ edges in a chain, then the relevant connected component of G consists of n edges in a chain.

(a) (b)

Fig. 2

LEMMA 2. In the graph G let there be a vertex x_1 of degree 1 (called a pendant vertex) such that starting from x_1 there is a chain of n edges $(n \geq 1)$ before the first vertex of a degree exceeding 2 is encountered, as in Fig. 3(a). Then the corresponding portion in I(G) has a similar configuration with $n-1$ edges, as in Fig. 3(b). Conversely, if I(G) has the form shown in Fig. 3(b), then the relevant connected component of G has the form shown in Fig. 3(a).

(a) (b)

Fig. 3

3. The Main Results.

THEOREM 1. Suppose G is a finite graph without loops (there may be multiple edges); let $x_1, \ldots, x_n$ be its vertices and let d_i be the degree of the vertex x_i, $1 \leq i \leq n$. Then the number of edges in the adjoint $I(G)$ is

$$\sum_{i=1}^{n} \frac{d_i(d_i-1)}{2} .$$

Proof: From the construction of adjoints, we see that if there are d_i edges at the vertex x_i of G, then each of the vertices of $I(G)$ corresponding to these edges will be joined by edges to each of the others if $d_i \geq 2$, and there will be no edges in virtue of edges at x_i if $d_i \leq 1$. In other words, the number of edges in $I(G)$ contributed by edges (of G) at x_i is $\frac{d_i(d_i-1)}{2}$ if $d_i \geq 2$, and 0 if $d_i \leq 1$. The total number of edges in $I(G)$ is, therefore,

$$\Sigma \frac{d_i(d_i-1)}{2} ,$$

where the summation is over all i such that $d_i \geq 2$,

$$= \sum_{i=1}^{n} \frac{d_i(d_i-1)}{2} .$$

Also if (x_i, x_j) is an edge in G then the vertex (x_i, x_j) of $I(G)$ will be joined by edges to (d_i-1) vertices in virtue of the edges at x_i in G, and to (d_j-1) vertices in virtue of the edges at x_j in G. Thus the degree of this vertex in $I(G)$ is $(d_i-1) + (d_j-1)$ $= d_i + d_j - 2$.

THEOREM 2. For a finite graph G without loops, the following statements are equivalent.

a) the degree of each vertex of G is 2,

b) G is isomorphic to $I^k(G)$ for all $k \geq 1$,

c) G is isomorphic to $I^k(G)$ for some k, $(k \geq 1)$.

As a corollary it follows that G is isomorphic to I(G) if and only if the degree of each vertex of G is 2.

Proof: We shall prove the following implications

$$(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a) ,$$

Let x_i, $1 \leq i \leq n$ be the vertices of G and d_i, $1 \leq i \leq n$, their corresponding degrees. Since (b) ⇒ (c) obviously, we shall only prove

$$(a) \Rightarrow (b) \text{ and } (c) \Rightarrow (a) .$$

1. (a) ⇒ (b). For if each $d_i = 2$, then every connected component must be of the following form, as in Fig. 4, (called an elementary cycle), where the vertices are $x_1, x_2, \ldots, x_{\ell-1}, x_\ell$. (For different components, the value of ℓ may be different) and the edges are $(x_1, x_2), \ldots, (x_i, x_{i+1}), \ldots, (x_\ell, x_1)$.

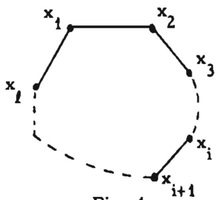

Fig. 4

One readily verifies that the adjoint of such a component is isomorphic to itself. Thus each connected component of G is isomorphic to the corresponding component of I(G), i.e., G is isomorphic to I(G). By induction, we see that G is isomorphic to $I^k(G)$ for every k.

2. (c) ⇒ (a). We first show that G cannot contain vertices of degree zero or one.

If possible, let $d_i = 0$ for some i, i.e., the corresponding vertex x_i is an isolated vertex. Since G and $I^k(G)$ are isomorphic, $I^k(G)$ also contains an isolated vertex. Now applying lemma 1 of section 2 repeatedly, we see that the connected component of G which gave rise to this isolated point of $I^k(G)$ must be a chain of k edges (an isolated point is a chain of zero edges), as in Fig. 5(a).

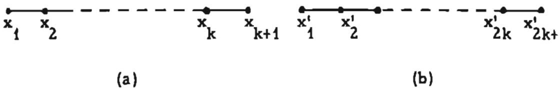

Fig. 5

But G and $I^k(G)$ are isomorphic, so $I^k(G)$ contains such a connected component (a chain of k edges). The corresponding component of G (which reduces to this component in $I^k(G)$) must be, again by repeated applications of lemma 1, a chain of 2k edges, as in Fig. 5(b). Proceeding thus, we see that in G there occur connected components which are chains of k, 2k, 3k, ... edges respectively. This contradicts the finiteness of G.

Now let $d_i = 1$ for some i, i.e., the corresponding vertex x_i is a pendant vertex. Consider the chain (of ℓ edges, say) from x_i to the first vertex of degree exceeding 2 (this chain may be of length 1), or of degree 1. If the latter applies, we can use the above argument. So we can assume that a configuration, as in Fig. 6(a) exists in G, and hence in $I^k(G)$.

By applying k times the lemma 2 of section 2, we see that G must contain the configuration of Fig. 6(b), where there are $\ell + k$ edges from the pendant vertex to the first vertex of degree > 2. Thus, as in the previous case, we can

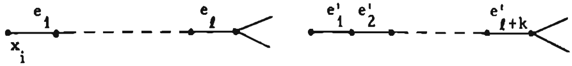

Fig. 6

show that such configurations with ℓ, $\ell+k$, $\ell+2k, \ldots$ edges exist in G, as connected components, which is absurd since G is finite.

Hence we must have $d_i \geq 2$ for all i, in G.

Using theorem 1, we see that if G contains n vertices and m edges, and the degree of each vertex is at least 2, and if I(G) contains m_1 edges, then

i) $m_1 = \sum_1^n \frac{d_i(d_i-1)}{2} \geq n$,

ii) the degree of each vertex of I(G) is also at least 2. The equality $m_1 = n$ holds if and only if each $d_i = 2$.

Let now n_o, m_o be respectively the number of vertices and edges of G, and let n_r, m_r be the corresponding quantities for $I^r(G)$. Then it follows that (since the degree of each vertex in $I^r(G)$ is at least 2 for all $r \geq 0$)

(1) $$m_{r+1} \geq n_r \qquad \text{for } r \geq 0.$$

It is of course true that $n_{r+1} = m_r$, since $I^{r+1}(G) = I[I^r(G)]$, for $r \geq 0$. Now since G and $I^k(G)$ are isomorphic, they have, in particular, the same number of vertices and edges, respectively, i.e., $n_o = n_k$

and $m_o = m_k$.

If k is even, say $k = 2r$, then using the result (1), we obtain

$$m_k = m_{2r} \geq n_{2r-1} = m_{2r-2} \geq \cdots \geq n_1 = m_o$$

and equality holds if and only if each vertex is of degree 2 at all stages. But since $m_o = m_k$, we have each $d_i = 2$ for G.

If k is odd, say $2r+1$, then using the result (1), we have

$$m_k = m_{2r+1} \geq n_{2r} = m_{2r-1} \geq \cdots \geq n_2 = m_1 \geq n_o$$

and

$$n_k = m_{k-1} = m_{2r} \geq m_o ,$$

whence $n_o = n_k \geq m_o = m_k \geq n_o$. Thus equality holds and hence each $d_i = 2$.

<u>Special case</u>. If we are given that G and I(G) are isomorphic, we can simplify the last stage of the proof considerably. Because if each $d_i \geq 2$, then the condition of equality of the number of edges in G and I(G) gives

$$\sum_{i=1}^{n} \frac{d_i(d_i - 1)}{2} = m_1 = m_o = \frac{1}{2} \sum_{1}^{n} d_i$$

$$\text{i.e.,} \quad \sum_{i=1}^{n} d_i(d_i - 2) = 0 ,$$

whence it follows that $d_i = 2$ for all i.

Remark. We can put the condition that each $d_i = 2$, in the alternative form that the graph consists of disjoint elementary cycles.

REFERENCE

1. O. Ore, Theory of Graphs, American Mathematical Society Colloquium Publications, Vol. XXXVIII, 1962, Section 1.5, problem 5.

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