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Majorization and Singular Values

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A majorization result is proved which involves the singular values of a matrix A and those of $\sum D_k A E_k$, where D_k, E_k are arbitrary matrices of the same order. The result generalizes previous majorization results concerning eigenvalues of hermitian matrices.

If A is a complex square matrix, we write $A \geqslant 0$ to indicate that A is hermitian positive semidefinite. If $A \geqslant 0$ is an $n \times n$ matrix, then $\lambda_1(A) \geqslant \lambda_2(A) \geqslant \cdots \geqslant \lambda_n(A)$ will denote the eigenvalues of A and $\lambda(A)$ will denote the column vector with its i-th entry equal to $\lambda_i(A)$, $i = 1, 2, \ldots, n$.

If A is an $n \times n$ matrix, then $S_1(A) \geqslant S_2(A) \geqslant \cdots \geqslant S_n(A)$ will denote the singular values of A, which by definition are the eigenvalues of the matrix $|A| = (A^*A)^{1/2}$, and we denote by $S(A)$, the column vector of the singular values of A arranged in nonincreasing order.

If $x \in R^n$ is a column vector, then $x_{[1]} \geqslant x_{[2]} \geqslant \cdots \geqslant x_{[n]}$ will denote the components of x in nonincreasing order and x will denote the corresponding column vector.

If $x, y \in R^n$, recall that x is said to be majorized by y, and we write $x \prec y$, if

$$\sum_{i=1}^{k} x_{[i]} \leqslant \sum_{i=1}^{k} y_{[i]}, \qquad k = 1, 2, \ldots, n-1, \tag{i}$$

and

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i. \tag{ii}$$

Also, x is said to be weakly majorized by y, $x \prec_w y$, if in the above definition condition (ii) is changed to $\sum_{i=1}^{n} x_i \leqslant \sum_{i=1}^{n} y_i$.

If A, B are $n \times n$ matrices, $A \circ B$ will denote their Schur (or Hadamard) product, which by definition is $A \circ B = ((a_{ij}b_{ij}))$. The same notation will apply to vectors. If B is an $n \times n$ matrix, diag (B) will denote the column vector $(b_{11}, b_{22}, \ldots, b_{nn})'$.

The following results were obtained in [1]:

(i) If $A \geqslant 0$, $B \geqslant 0$, then $\lambda(A \circ B) \prec_w \lambda(A) \circ \mathrm{diag}(B)$;

(ii) If $A \geqslant 0$ and if $D_1, D_2, \ldots, D_m$ are $n \times n$ matrices, then

$$\lambda\left(\sum D_k A D_k^*\right) \prec_w \lambda(A) \circ \beta$$

where β is any vector which weakly majorizes both $\lambda(\sum D_k D_k^*)$ and $\lambda(\sum D_k^* D_k)$.

When the matrices $D_1, D_2, \ldots, D_m$ are all taken to be diagonal matrices, (ii) reduces to (i).

The purpose of this note is to extend (ii) to a result concerning singular values. We make use of a technique employed by Okubo [4] to obtain majorization results for singular values from the corresponding results for eigenvalues. We state this technique in the form of a lemma.

LEMMA 1 *If* $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix} \geqslant 0$, *where* X_{11}, X_{22} *are square and of the same order, then*

$$S(X_{12}) \prec_w \tfrac{1}{2}\{\lambda(X_{11}) + \lambda(X_{22})\}.$$

Proof Since $X \geqslant 0$, there exists a matrix W such that all its singular values are less than or equal to one and

$$X_{12} = X_{11}^{1/2} W X_{22}^{1/2}.$$

It is well-known ([3], p. 249) that for any $n \times n$ matrices U, V: $S(UV) \prec_w S(U) \circ S(V)$. Hence

$$\begin{aligned} S(X_{12}) &\prec_w S(X_{11}^{1/2}) \circ S(W) \circ S(X_{22}^{1/2}) \\ &\prec_w S(X_{11}^{1/2}) \circ S(X_{22}^{1/2}) \\ &= \lambda(X_{11}^{1/2}) \circ \lambda(X_{22}^{1/2}). \end{aligned}$$

The result follows by an application of the arithmetic mean–geometric mean inequality.

The following is the main result.

Theorem 2 *Let A; D_k, E_k, $k = 1, 2, \ldots, m$ be $n \times n$ matrices. Then*

$$S(\sum D_k A E_k) \prec_w S(A) \circ \beta,$$

where β is any vector which weakly majorizes all the vectors: $\lambda(\sum D_k D_k^*)$, $\lambda(\sum D_k^* D_k)$, $\lambda(\sum E_k E_k^*)$, $\lambda(\sum E_k^* E_k)$.

Proof As noted by Okubo [4],

$$\begin{bmatrix} |A^*| & A \\ A^* & |A| \end{bmatrix} \geq 0. \tag{1}$$

Pre- and post-multiply (1) by

$$\begin{bmatrix} D_k & 0 \\ 0 & E_k^* \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} D_k^* & 0 \\ 0 & E_k \end{bmatrix},$$

respectively, and sum over $k = 1, 2, \ldots, m$; to get

$$\begin{bmatrix} \sum D_k |A^*| D_k^* & \sum D_k A E_k \\ \sum E_k^* A^* D_k^* & \sum E_k^* |A| E_k \end{bmatrix} \geq 0. \tag{2}$$

By Lemma 1,

$$S(\sum D_k A E_k) \prec_w \tfrac{1}{2}\{\lambda(\sum D_k |A^*| D_k^*) + \lambda(\sum E_k^* |A| E_k)\}.$$

Now the theorem follows from result (ii).

We note a few consequences of the theorem.

Corollary 3 *If A, B are $n \times n$ matrices, then $S(A \circ B) \prec_w S(A) \circ \beta$, where β is any vector which weakly majorizes* diag$|B|$ *and* diag$|B^*|$.

Proof We can write $B = U \Delta V$, where U, V are unitary and Δ is the diagonal matrix which carries the singular values of B along its diagonal. Let D_k be the diagonal matrix whose diagonal entries are exactly the corresponding entries of the k-th column of $U\Delta^{1/2}$ and let E_k be the diagonal matrix whose diagonal entries are exactly the corresponding entries of the k-th row of $\Delta^{1/2} V$, $k = 1, 2, \ldots, n$. Then $A \circ B = \sum D_k A E_k$ and it can be verified that the result follows by an application of Theorem 2.

The following result has been obtained by Horn and Johnson [2] and by Okubo [4].

Corollary 4 *If A, B are $n \times n$ matrices, then*

$$S(A \circ B) \prec_w S(A) \circ S(B).$$

Proof By a well-known result of Schur the eigenvalues of a hermitian matrix majorize its diagonal elements. Thus $S(B)$ majorizes both diag$|B|$ and diag$|B^*|$, so the result follows from Corollary 3.

COROLLARY 5 *If A, B are $n \times n$ matrices where $B \geqslant 0$ and $b_{ii} = 1, i = 1, 2, \ldots, n$; then*

$$S(A \circ B) \prec_w S(A).$$

We remark that according to Corollary 5, if A, B are $n \times n$ matrices where $B \geqslant 0$ and $b_{ii} = 1$, $i = 1, 2, \ldots, n$; and if $\|\cdot\|$ is any unitarily invariant norm (see, for example, [3], p. 264), then

$$\|A \circ B\| \leqslant \|A\|.$$

We conclude by indicating yet another application of Lemma 1. It is well-known ([3], p. 243) that if A, B are $n \times n$ matrices, then

$$S(A + B) \prec_w S(A) + S(B). \tag{3}$$

It is possible to assert a bit more:

LEMMA 6 *If A, B are $n \times n$ matrices, then*

$$S(A + B) \prec_w \tfrac{1}{2}\{\lambda(|A| + |B|) + \lambda(|A^*| + |B^*|)\}$$
$$\prec_w S(A) + S(B).$$

Proof We have

$$\begin{bmatrix} |A^*| + |B^*| & A + B \\ A^* + B^* & |A| + |B| \end{bmatrix} \geqslant 0.$$

The first majorization of the lemma now follows by Lemma 1. To get the second assertion, note that $\lambda(|A| + |B|)$ and $\lambda(|A^*| + |B^*|)$ are both majorized by $S(A) + S(B)$ by an application of (3) to hermitian matrices.

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