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A NOTE ON THE POSSIBILITY OF DECENTRALIZATION IN A MODEL OF ALLOCATION OF RESOURCES*†

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The decomposition algorithm for the solution of large-scale linear programmes has been interpreted by Baumol and Fabian (*Management Science*, September, 1964) as a procedure for decentralized decision by the multi-division company or the multi-sector economy. While commenting on the possibility of decentralization in this context, they maintain that an economy's optimum point will not *in general* yield a sectoral optimum. The present note demonstrates that this contention is not valid.

Dantzig and Wolfe [2], [3] have developed a computational technique, called the decomposition algorithm, for solving linear programmes of certain specific types. Baumol and Fabian [1] have given an economic interpretation of this algorithm along with some remarks on its implication for planning with decentralization. The present note maintains that one of these remarks is erroneous.

We assume that the economy consists of K firms. The kth firm has at its disposal a number (n_k) of basic linear productive activities. Each activity requires only two types of inputs: (a) plant capacity elements which are rigidly attached to any given firm, and (b) resources whose supplies are given for the economy as a whole. The allocation of these resources among the competing firms for given gross revenues from the different activities is the main problem to be solved by a Central Executive Authority (denoted by C. E. A. in short).

Let R^n denote the n-dimensional real vector space, $M(m, n)$, the sets of all $m \times n$ order matrices with real coefficients and $\in$, the relation "belongs to."

Let the requirements of the resources and of the plant capacity elements by the different activities of the kth firm be represented by the semipositive matrices[1] $A_k \in M(m_0, n_k)$ and $B_k \in M(m_k, n_k)$ respectively. The rows of the matrices correspond to the different input elements, while the columns correspond to the activities. The row vector $c_k \in R^{n_k}$ and the column vector $x_k \in R^{n_k}$ denote the gross revenues and the levels of operation of those activities respectively. The positive column vector $b_k \in R^{m_k}$ represents the supplies of the plant capacity elements of the kth firm while the positive column vector $s \in R^{m_0}$ represents those of the resources for the economy as a whole.

Further let the set $X_k = \{x_k \mid B_k x_k \leqq b_k ; x_k \geqq 0\}$. Obviously this set is nonempty, convex and compact. Let x_{kr} ($r = 1, 2, \cdots, R_k$) be the column vector representing the rth extreme point of the set X_k. Let the matrix $E_k \in M(n_k, R_k)$ be formed by these extreme vectors. Finally let $\bar{c}_k = c_k E_k$ and $\bar{A}_k = A_k E_k$.

The central problem of allocation that the C. E. A. has to solve is then formally equivalent to the following one.

(P1) Find x_k ($k = 1, 2, \cdots, K$) which would maximise $\sum_k c_k x_k$ subject to $\sum_k A_k x_k \leqq s$; $B_k x_k \leqq b_k$, $x_k \geqq 0$ for all k.

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[1] A semipositive matrix has each of its columns semipositive. A semipositive column vector has all its elements nonnegative and at least one of its elements strictly positive.

The decomposition algorithm solves this linear programme by an iterative method at two levels—at the C. E. A. level and at the firm level. At the iteration t of the algorithm, the C. E. A. imputes a provisional rental ρ_k^t to the kth firm and announces its readiness to sell the resources remaining at its disposal to the firms at some provisional prices represented by the row vector $p^t \in R^{m_0}$. Taking these prices as data each firm solves the following allocation problem. (We write the problem for the kth firm.)

$(P2)_k^t$ Find x_k which would maximise $(c_k - p^t A_k)x_k$ subject to $B_k x_k \leqq b_k$; $x_k \geqq 0$.

The kth firm then reports[2] to the C. E. A. its demand for resources a_{kr}^t and promises to earn the revenue (net of resource cost) δ_k^t, where $a_{kr}^t = A_k x_{kr}^t$, $\delta_k^t = (c_k - p^t A_k)x_{kr}^t$, x_{kr}^t being the extreme point of X_k solving $(P2)_k^t$. At the beginning of the $(t+1)$th iteration the C. E. A. compares δ_k^t with ρ_k^t for each k. Of those demand vectors it 'accepts' one from some firm k_0 provided $\delta_{k_0}^t > \rho_{k_0}^t$. Let t_k denote the number of demand vectors 'accepted' in this sense by the C. E. A. from the kth firm upto the iteration t and let those 'accepted' vectors correspond to the first t_k columns of E_k, where[3] $1 \leqq t_k \leqq R_k$. (Obviously $(t+1)_k = t_k + 1$ for $k = k_0$ and $(t+1)_k = t_k$ for $k \neq k_0$.) At the $(t+1)$th iteration, the C. E. A. frames the following programme.

$(P3)^{t+1}$ Find $\eta_{kr} \geqq 0$ $(k = 1, 2, \cdots, K; r = 1, 2, \cdots, (t+1)_k)$ which would maximise $\sum_{k=1}^{K} \sum_{r=1}^{(t+1)_k} \eta_{kr}\gamma_{kr}$ subject to $\sum_{k=1}^{K} \sum_{r=1}^{(t+1)_k} \eta_{kr}a_{kr} \leqq s$, $\sum_{r=1}^{(t+1)_k} \eta_{kr} = 1$ for all k, where γ_{kr}'s are the elements of $\bar{c}_k$.

The dual to this problem is important.

$(D3)^{t+1}$ Find $p \geqq 0$, ρ_k $(k = 1, 2, \cdots, K)$ which would minimise $ps + \sum_k \rho_k$ subject to $pa_{kr} + \rho_k \geqq \gamma_{kr}$ for all k and for all $r \leqq (t+1)_k$.

Let η_{kr}^{t+1}'s solve $(P3)^{t+1}$ and p^{t+1} and ρ_k^{t+1}'s solve $(D3)^{t+1}$. The optimal solution of $(D3)^{t+1}$ provides the instrumental values of the prices and the rentals with which the C. E. A. would work. The firm problem $(P2)_k^{t+1}$ $(k = 1, 2, \cdots, K)$ would be based on these values of prices and rentals. Successive iterations would follow in the same way.

We now define the column vector $y_k^t \equiv \{\eta_{kr}^t\} \in R_k$ where η_{kr}^t is the optimal value of η_{kr} given by $(P3)^t$ for all k and $r = 1, 2, \cdots, t_k$, and η_{kr}^t is zero for all k and for all r satisfying the inequality $t_k < r \leqq R_k$. We also define $\bar{x}_k^t \equiv E_k y_k^t$ and $\bar{x}^t \equiv |\prod_{k=1}^{K} \bar{x}_k^t|$ ($\prod$ denotes the operation of Cartesian multiplication).

We would now get the following results whose proofs follow immediately from the theory of linear programming. (For proofs see [5].)

LEMMA 1. *For any t and any k, $\rho_k^t = (\bar{c}_k - p^t \bar{A}_k)y_k^t \leqq \delta_k^t$.*

LEMMA 2. *$\bar{x}^t$ solves (P1) if and only if $\delta_k^t = \rho_k^t$ for all k.*

Lemma 2 provides the optimality criterion. While giving an economic meaning of this algorithm, Baumol and Fabian commented that the decentralization in decision

[2] At this point the statements in [1] are confusing. At some places Baumol and Fabian remark that in the decomposition process the central management requires no knowledge about the nature of the internal technological arrangements of the divisions (p. 2 and footnote 13 of p. 10). At some other places they state that a division should submit its optimal solution of activity levels to the central management and the corporate accounting office would compute its resource use (p. 10) and form the executive programme from that information. But in order to compute the resource use from the activity levels, the corporate accounting office would require knowledge about the internal technologies of the divisions. All these would be unnecessary if we interpret the algorithm as follows: firms report to the C. E. A. their demand for resources and their revenues (net of resource cost).

[3] Let the first column of each E_k be the null vector, an extreme point of X_k. To begin with, the C. E. A. assumes that no firm produces anything. Accordingly, at the first iteration $t_k = 1$ for all k. (See [1, p. 25].)

permitted by decomposition may break down owing to the following difficulty: even if $\bar{x}^t$ solves (P1), for some k, the corresponding $\bar{x}_k^t$ may turn out to be an interior point of the set X_k and thereby represent a suboptimal point of the firm problem $(P2)_k^t$. In that case it has been argued[4] that what is optimal for the economy would not necessarily be optimal for all the firms. The fallacy of this argument is immediate from the following theorem.

THEOREM. *If $\bar{x}^t$ solves (P1), then $\bar{x}_k^t$ solves $(P2)_k^t$ for all k.*

PROOF. Since $\bar{x}^t$ solves (P1), by Lemma 2, $\rho_k^t = \delta_k^t$ for all k. Again we should note that $\bar{x}_k^t \in X_k$ for all k. Further we have, for all k, $(c_k - p^t A_k)\bar{x}_k^t = (\bar{c}_k - p^t \bar{A}_k)y_k^t = \rho_k^t$ (by Lemma 1) $= \delta_k^t \geqq (c_k - p^t A_k)x_k$ for all $x_k \in X_k$ (by the definition of δ_k^t). Thus for all k, $\bar{x}_k^t$ solves $(P2)_k^t$. Hence, the theorem is proved.

It is now obvious that the overall optimum solution $\bar{x}^t$ would always yield an optimum solution for each firm problem.[5] Thus $\bar{x}_k^t$ cannot be an interior point of X_k.

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[4] See [1, p. 17].

[5] Gale discusses a similar problem of allocation of resources in [4]. He proves this theorem in his Theorem 3.5 (p. 91–93).