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RECOIL POLARIZATION AND STRUCTURE OF THE DEUTERON

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Abstract: Expressions for the recoil deuteron polarization tensors and the differential cross section in elementary particle interactions with the deuteron are obtained as functions of five body form factors of the deuteron taking into account the D-state admixture with an arbitrary relative phase. Considering in particular the elastic scattering of pions and photoproduction of neutral pions, suitable measurements of the observables are suggested to estimate the five form factors. It is also shown that measurements of the differential cross section alone at low values of momentum transfer enable the determination of the D-state probability and the relative phase.

1. INTRODUCTION

The spin polarization of the recoil deuteron is an interesting observable in elastic deuteron elementary particle collisions. A discussion of the polarization tensors representing the deuteron by a pure S-state was considered earlier in the particular cases of neutral pion-photoproduction [1] and pion scattering [2]; this leads us to expect considerable amounts of spin orientation. Some polarization tensors have been studied experimentally in the case of proton scattering recently [3].

The purpose of this paper is to show that precise experimental study of the recoil deuteron polarization provides new possibilities of analysing the structure of the deuteron which could be said to be completely determined if we specify the relative probabilities of the S- and D-states, their relative phase and their respective radial distributions. An alternative way of describing the structure would be to give the three-dimensional Fourier transform of the deuteron wave function which could easily be seen, by the spherical harmonic expansion of the exponential, to be completely specified if essentially a set of five radial integrals, $\langle S|j_0|S\rangle$, $\langle D|j_0|D\rangle$, $\langle S|j_2|D\rangle$, $\langle D|j_2|S\rangle$ and $\langle D|j_2|D\rangle$ involving spherical Bessel functions, j_l are given where the relative weight and phase factors are included. This is a more convenient method of characterization in analysing the results of collision experiments where a certain known amount of momentum is transferred to the nucleus. The objective is thus clearly to determine these five form factors from possible experimental studies of the various observables associated with the deuteron which are, namely, the differential cross section, the three first-rank tensor moments and the five second-rank tensor moments of spin orientation, assuming that the nucleon elementary particle interaction amplitudes are known. We assume the nucleons in the deuteron to be non-relativistic and make use of the impulse approximation to obtain expressions for these observables; no assumptions are made regarding the energies or spins of the elementary particles interacting with the deuteron. Some results for the tensor moments following elastic scattering of electron have been reported [4], but these calculations obviously do not take into account the possibility of a phase difference between the S- and D-states which introduces additional terms. We, however, present in sect. 2 an outline of our calculation and results which are not specialized to any particular reaction. In sects. 3 and 4 we discuss the possible use of these results in elucidating the structure of the deuteron by considering appropriate reactions. In sect. 5 we discuss a possible method of estimating the D-state probability and also the relative phase from measurements of differential cross sections at low values of momentum transfer.

2. CROSS SECTION AND RECOIL DEUTERON POLARIZATION

The amplitude T for an elementary particle interaction on the deuteron may be written in impulse approximation as

$$T = \sum_{j=1,2} \exp(i\boldsymbol{k}\cdot\boldsymbol{r}_j)(i\boldsymbol{\sigma}_j\cdot\boldsymbol{K} + L) \,, \tag{1}$$

where $\boldsymbol{\sigma}_j$ denotes the Pauli spin operator for the jth nucleon whose position co-ordinates are denoted by $\boldsymbol{r}_j$. The momentum transferred to the deuteron during the process is denoted by $\boldsymbol{k}$, while $\boldsymbol{K}$ and L denote the appropriate spin-dependent and spin-independent isoscalar interaction amplitudes with the nucleon.

Taking into account the S- and D-states, the deuteron wave function may be written in the form

$$|\mathrm{d}\rangle = u(r)Y_{00}(\hat{\boldsymbol{r}})\chi_{1m} + w(r)(Y_2(\hat{\boldsymbol{r}})\times\chi_1)^1_m \,, \tag{2}$$

where $Y_{lm}(\hat{\boldsymbol{r}})$ denote normalized spherical harmonics representing the relative motion, χ_{1m} the triplet spin states and $u(r)$ and $w(r)$ the S- and D-state radial wave functions along with the appropriate normalization and phase factors.

The matrix elements of T between the initial and final deuteron states may now be obtained [5] in the form

$$\langle \mathrm{f}|T|\mathrm{i}\rangle = \sum_{\lambda=0}^{2}\sum_{l=0}^{2}\sum_{s=0}^{1}(-1)^{m_\lambda}$$

$$\times\, C(1\lambda 1; m_\mathrm{i}\, m_\lambda\, m_\mathrm{f})(Y_l(\hat{\boldsymbol{k}})\times K_s)^\lambda_{-m_\lambda}\langle \mathrm{f}\|T^\lambda_{ls}\|\mathrm{i}\rangle \,, \tag{3}$$

where $K_0 = L$ and K_1 represents K in spherical tensor notation. The reduced matrix elements $\langle f \| T^{\lambda}_{ls} \| i \rangle$ are given by

$$\langle f \| T^{\lambda}_{ls} \| i \rangle = 2(4\pi)^{\frac{1}{2}} (i)^{l+s} [(2l+1)(s+1)]^{\frac{1}{2}}$$

$$\times [\delta_{s\lambda} \delta_{l0} (-1)^{\lambda} F_0^{uu} + \delta_{l2} \sqrt{\tfrac{3}{5}} (2\lambda+1)^{\frac{1}{2}}$$

$$\times W(1 \lambda 1 s; 1 2)(F_2^{uw} + (-1)^{s+\lambda} F_2^{wu})$$

$$+ (-1)^{\lambda} 3\sqrt{5} (2\lambda+1)^{\frac{1}{2}} C(2 l 2; 0 0)$$

$$\times \begin{Bmatrix} 2 & l & 2 \\ 1 & s & 1 \\ 1 & \lambda & 1 \end{Bmatrix} F_l^{ww}] , \tag{4}$$

where the C, W and $9j$ symbols are respectively the Clebsch-Gordan, Racah and Wigner $9j$ coefficients, and F_l^{gh} denote radial integrals

$$F_l^{gh} = \int_0^{\infty} r^2 \, dr \, g^*(r) \, j_l(\tfrac{1}{2}kr) \, h(r) , \tag{5}$$

involving spherical Bessel functions j_l.

If ρ^i denotes the 3×3 density matrix specifying the initial spin state of the deuterons, the final spin state after interaction may be described by a density matrix ρ^f whose elements are given by

$$\rho^f_{m_f' m_f} = \sum_{m_i} \sum_{m_i'} \langle f' | T | i' \rangle \, \rho^i_{m_i' m_i} \langle f | T | i \rangle^* , \tag{6}$$

or by giving a set of eight parameters

$$\langle \mathcal{T}^{\kappa}_{\mu} \rangle = \mathrm{Tr}\, (\mathcal{T}^{\kappa}_{\mu} \, \rho^f) / \mathrm{Tr}\, \rho^f , \tag{7}$$

for $\kappa = 1, 2$ where $\mathcal{T}^{\kappa}_{\mu}$ denote spherical tensor operators of rank κ in the deuteron spin space [6] and are normalized to satisfy [7]

$$\mathrm{Tr}\, (\mathcal{T}^{\kappa \dagger}_{\mu} \, \mathcal{T}^{\kappa'}_{\mu'}) = 3 \delta_{\kappa\kappa'} \delta_{\mu\mu'} , \tag{8}$$

so that ρ may be represented as

$$\rho = \tfrac{1}{3} \mathrm{Tr}\, \rho \, (1 + \sum_{\kappa} \sum_{\mu} \langle \mathcal{T}^{\kappa}_{\mu} \rangle^* \, \mathcal{T}^{\kappa}_{\mu}) . \tag{9}$$

With these definitions and taking the initial deuterons to be unpolarized, we may calculate

$$\mathrm{Tr}\,\rho^f \langle \mathcal{T}^{\kappa}_{\mu}\rangle = \frac{\sqrt{3}}{4\pi} \sum_{\lambda l s \lambda' l' s'} (-1)^{l+s-\kappa}\, W(\lambda' 1 \kappa 1; 1\lambda) \sum_{LS} [L][S][\lambda][\lambda']$$
$$\times [l][l'] C(l' l L; 0 0) \begin{Bmatrix} l' & s' & \lambda' \\ l & s & \lambda \\ L & S & \kappa \end{Bmatrix} \langle f \| T^{\lambda'}_{l's'} \| i\rangle \langle f \| T^{\lambda}_{ls} \| i\rangle^{*}$$
$$\times \sqrt{\frac{4\pi}{2L+1}} \left(Y_L(\hat{k}) \times (K_{s'} \times K^{*}_{s})^{S} \right)^{\kappa}_{\mu} \,, \tag{10}$$

where $[l] = (2l+1)^{\frac{1}{2}}$. From eq. (10) $\mathrm{Tr}\,\rho^f$ can be calculated by simply substituting $\kappa = 0$. We observe that the dynamics of the interaction is completely incorporated in the spherical tensors $(K_{s'} \times K^{*}_{s})^{S}$ of which there are two different tensors with $S = 0$ (corresponding to the assignments $s = s' = 0$, $s = s' = 1$), three with $S = 1$ ($s = 0$ $s' = 1$, $s = 1$ $s' = 0$, $s = s' = 1$) and one with $S = 2$ ($s = s' = 1$). We also note that while S can take all the three possible values 0, 1, 2, the number L can take only values 0, 2, 4 on account of parity conservation. Thus, in general the expression (10) for $\kappa = 0$, 1 and 2 consists of a sum of 3, 7 and 8 spherical tensors of the appropriate rank as shown in table 1, where they are enumerated with suitable nomenclature. We may therefore write

$$\mathrm{Tr}\,\rho^f = \sum_{i=1}^{3} a_i\, \mathcal{S}_i \,, \tag{11}$$

$$\mathrm{Tr}\,\rho^f \langle \mathcal{T}^{1}_{\mu}\rangle = \sum_{i=1}^{7} b_i \mathcal{V}_i \,, \tag{12}$$

$$\mathrm{Tr}\,\rho^f \langle \mathcal{T}^{2}_{\mu}\rangle = \sum_{i=1}^{8} c_i\, Q_i \,, \tag{13}$$

where the coefficients a_i b_i c_i are determined by eq. (10). The tensors $\mathcal{S}_i$, $\mathcal{V}_i$ and Q_i may be calculated to any desired accuracy once the isoscalar interaction amplitudes with the nucleon appropriate to the given process are known. It may be mentioned that in particular reactions some of the tensors may either vanish or become effectively the same as others. For instance, if we consider elastic scattering of spinless particles on the deuteron, and if the basic interaction is parity conserving, the tensors $\mathcal{S}_3$, $\mathcal{V}_4$, $\mathcal{V}_5$ and Q_7, Q_8 reduce respectively to $\mathcal{S}_2$, $\mathcal{V}_1$, $\mathcal{V}_2$ and Q_1 apart from numerical factors, while the tensors $\mathcal{V}_3$, $\mathcal{V}_6$, $\mathcal{V}_7$ and Q_6 vanish. The expansion coefficients a_i, b_i, c_i explicitly evaluated are given by

$$a_1 = 4(\mathcal{F}_1^2 + \mathcal{F}_3^2) \,, \tag{14}$$

$$a_2 = -\frac{2}{\sqrt{3}}(4\,\mathcal{F}_2^2 + \mathcal{F}_4^2 + 3\,\mathcal{F}_5^2) \,, \tag{15}$$

Table 1
Nomenclature for the spherical tensors

$$(4\pi/2L+1)^{\frac{1}{2}}\,(Y_L(\hat{k})\times(K_{s'}\times K_s^{*})^S)^K_\mu$$

occurring in eq.(10).

L, S \ K	0	1	2
0,0	$\mathcal{S}_1, \mathcal{S}_2$		
0,1		$\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3$	
0,2			Q_1
2,0			Q_2, Q_3
2,1		$\mathcal{V}_4, \mathcal{V}_5, \mathcal{V}_6$	Q_4, Q_5, Q_6
2,2	$\mathcal{S}_3$	$\mathcal{V}_7$	Q_7
4,2			Q_8

Of the tensors with $S = 0$ the first in the row refers to the assignment $s = s' = 0$ and the second to $s = s' = 1$. In the case of $S = 1$ the first and second refer, respectively, to the sum of the two tensors with $s = 0\ s' = 1$ and $s = 1\ s' = 0$ and i times the difference of the second of the two from the first; the third one refers to $s = s' = 1$.

$$a_3 = \sqrt{\tfrac{10}{3}}\,(-4\sqrt{2}\,\mathcal{T}_2\mathcal{T}_4 + \mathcal{T}_4^2 - 3\mathcal{T}_5^2)\,, \tag{16}$$

$$b_1 = -\sqrt{6}\,\mathcal{T}_3\mathcal{T}_5\,, \tag{17}$$

$$b_2 = \sqrt{\tfrac{2}{3}}\,(4\mathcal{T}_1\mathcal{T}_2 + \mathcal{T}_3\mathcal{T}_4)\,, \tag{18}$$

$$b_3 = \frac{1}{2\sqrt{3}}\,(8\mathcal{T}_2^2 - \mathcal{T}_4^2 + 3\mathcal{T}_5^2)\,, \tag{19}$$

$$b_4 = -\sqrt{15}\,\mathcal{T}_3\mathcal{T}_5\,, \tag{20}$$

$$b_5 = \sqrt{\tfrac{5}{3}}\,(2\sqrt{2}\,\mathcal{T}_1\mathcal{T}_4 + 2\sqrt{2}\,\mathcal{T}_2\mathcal{T}_3 - \mathcal{T}_3\mathcal{T}_4)\,, \tag{21}$$

$$b_6 = \frac{-\sqrt{5}}{\sqrt{6}}\,(2\sqrt{2}\,\mathcal{T}_2\mathcal{T}_4 + \mathcal{T}_4^2 + 3\mathcal{T}_5^2)\,, \tag{22}$$

$$b_7 = -2\sqrt{5}\,i\,(\mathcal{T}_2 - \frac{1}{\sqrt{2}}\,\mathcal{T}_4)\,\mathcal{T}_5\,, \tag{23}$$

$$c_1 = \frac{1}{10\sqrt{3}}\,(40\mathcal{T}_2^2 + \mathcal{T}_4^2 + 21\mathcal{T}_5^2)\,, \tag{24}$$

$$c_2 = -2\sqrt{2}\,\mathcal{T}_3(2\sqrt{2}\,\mathcal{T}_1 + \mathcal{T}_3)\,, \tag{25}$$

$$c_3 = \frac{1}{\sqrt{6}}(4\sqrt{2}\,\mathcal{F}_2\mathcal{F}_4 - \mathcal{F}_4^2 + 3\mathcal{F}_5^2)\,, \tag{26}$$

$$c_4 = -\sqrt{3}\,(2\sqrt{2}\,\mathcal{F}_1 + \mathcal{F}_3)\,\mathcal{F}_5\,, \tag{27}$$

$$c_5 = \sqrt{3}\,\mathcal{F}_3(2\sqrt{2}\,\mathcal{F}_2 + \mathcal{F}_4)\,, \tag{28}$$

$$c_6 = \sqrt{6}\,i(\sqrt{2}\,\mathcal{F}_2 - \mathcal{F}_4)\,\mathcal{F}_5\,, \tag{29}$$

$$c_7 = \frac{1}{\sqrt{42}}\,(14\sqrt{2}\,\mathcal{F}_2\mathcal{F}_4 + \mathcal{F}_4^2 + 15\mathcal{F}_5^2)\,, \tag{30}$$

$$c_8 = \frac{3}{5}\sqrt{\frac{4}{7}}\,(\mathcal{F}_4^2 + \mathcal{F}_5^2)\,, \tag{31}$$

where

$$\mathcal{F}_1 = F_0^{uu} + F_0^{ww}\,, \tag{32}$$

$$\mathcal{F}_2 = F_0^{uu} - \tfrac{1}{2}F_0^{ww}\,, \tag{33}$$

$$\mathcal{F}_3 = F_2^{uw} + F_2^{wu} - \frac{1}{\sqrt{2}}\,F_2^{ww}\,, \tag{34}$$

$$\mathcal{F}_4 = F_2^{uw} + F_2^{wu} + \sqrt{2}F_2^{ww}\,, \tag{35}$$

$$\mathcal{F}_5 = -i\,(F_2^{uw} - F_2^{wu})\,. \tag{36}$$

3. PARTICULAR PROCESSES AND POSSIBILITY OF DEUTERON STRUCTURE DETERMINATION

An important distinction between the tensors and coefficients considered in sect. 2 relates to their dependence on the kinematical and other variables governing an experiment. While the tensors depend in detail on the energy and angular variables as well as on the initial and final spin states of the particles interacting with the deuteron, the expansion coefficients are purely functions of the square of the momentum transferred to the target.

We shall therefore examine the possibility of determining some of these coefficients (using the technique of performing several experiments changing the energy and other variables but keeping the square of the momentum transfer fixed) by considering some of the well-known processes and hence discuss the possible determination of the five basic form factors F_0^{uu}, F_0^{ww}, F_2^{uw}, F_2^{wu} and F_2^{ww}. It is clear that a knowledge of these form factors would immediately allow determination of the D-state probability P_D and the relative phase δ between the S- and the D-states through

$$P_D = \underset{k\to 0}{Lt} \frac{F_0^{ww}}{F_0^{uu} + F_0^{ww}} , \tag{37}$$

$$i \text{ tg } \delta = \frac{F_2^{uw} - F_2^{wu}}{F_2^{uw} + F_2^{wu}} . \tag{38}$$

3.1. *Elastic scattering of pions*

The isoscalar amplitudes of pion-nucleon scattering have the form

$$L = g(E, \cos\theta) , \tag{39}$$

$$K = (\hat{q}_2 \times \hat{q}_1)\, h(E, \cos\theta) , \tag{40}$$

where $\hat{q}_1$ and $\hat{q}_2$ denote unit vectors parallel to the incident and scattered pions and g and h the scalar functions of energy and scattering angle. Since K is perpendicular to the scattering plane, $\mathcal{G}_3$ reduces to $\mathcal{G}_2$, and the differential cross section is proportional to

$$a_1 LL^* - \frac{1}{\sqrt{3}} (a_2 + \frac{1}{\sqrt{10}} a_3) K\cdot K^* . \tag{41}$$

Among the first-rank tensors only $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_4$ and $\mathcal{V}_5$ are non-zero and

$$\mathcal{V}_1 = -\sqrt{6}\,\mathcal{V}_4 = L^*K + LK^* , \tag{42}$$

$$\mathcal{V}_2 = -\sqrt{6}\,\mathcal{V}_5 = i(L^*K - LK^*) , \tag{43}$$

so that the "polarization" is perpendicular to the scattering plane. If we choose this direction to be the z-axis we have

$$\text{Tr}\, \rho^f \langle \mathcal{T}_0^1 \rangle = (b_1 + \frac{1}{\sqrt{10}} b_4)(L^*K + LK^*)_z + (b_2 + \frac{1}{\sqrt{10}} b_5)\, i(L^*K - LK^*)_z , \tag{44}$$

$$\text{Tr}\, \rho^f \langle \mathcal{T}_{\pm 1}^1 \rangle = 0 . \tag{45}$$

Of the second-rank tensors, Q_6 vanishes and Q_1, Q_7, Q_8 are equivalent so that we have

$$\text{Tr}\, \rho^f \langle \mathcal{T}_0^2 \rangle = (c_1 + \frac{1}{\sqrt{14}} c_7 + \frac{3}{4\sqrt{14}} c_8)(K_1 \times K_1^*)_0^2$$

$$- \tfrac{1}{2} c_2\, LL^* + \frac{1}{2\sqrt{3}} c_3\, K\cdot K^* , \tag{46}$$

$$\text{Tr}\, \rho^f \langle \mathcal{T}_{\pm 1}^2 \rangle = 0 , \tag{47}$$

$$\mathrm{Tr}\,\rho^{\mathrm{f}}\langle\mathcal{T}^2_{\pm 2}\rangle = \Big[\frac{\sqrt{3}}{2\sqrt{2}}c_2 LL^* - \frac{1}{2\sqrt{2}}c_3\,\boldsymbol{K}\cdot\boldsymbol{K}^* \pm \tfrac{1}{2}c_4(L^*\boldsymbol{K}+L\boldsymbol{K}^*)_z$$
$$\pm \tfrac{1}{2}c_5\, i(L^*\boldsymbol{K}-L\boldsymbol{K}^*)_z + (\tfrac{1}{2}\sqrt{\tfrac{3}{7}}\,c_7 - \tfrac{5}{8}\tfrac{1}{\sqrt{21}}\,c_8)(\boldsymbol{K}_1\times\boldsymbol{K}^*_1)^2_0\Big]\,\mathrm{e}^{\pm 2i\theta_k}\,, \tag{48}$$

where θ_k denotes the angle made by the unit vector $\hat{k}$ in the plane of scattering. In the lab frame, θ_k denotes actually the angle of the recoil deuteron.

3.2. *Photoproduction of neutral pions using unpolarized photons*

The isoscalar amplitudes of photoproduction of neutral pions on the nucleon are somewhat more complicated [8] due to the presence of an extra vector $\boldsymbol{\varepsilon}$ the photon polarization characterising the process. However, if the incident photons are unpolarized and an average over the independent polarizations are taken, we would find that in addition to $\frac{1}{2}\sum_{\varepsilon} LL^*$ and $\frac{1}{2}\sum_{\varepsilon} \boldsymbol{K}\cdot\boldsymbol{K}^*$ the scalar $\frac{1}{2}\sum_{\varepsilon} |\hat{k}\cdot\boldsymbol{K}|^2$ also survives, while the vectors

$$\tfrac{1}{2}\sum_{\varepsilon}(L^*\boldsymbol{K}+L\boldsymbol{K}^*)\,,\quad \tfrac{1}{2}\sum_{\varepsilon} i(L^*\boldsymbol{K}-L\boldsymbol{K}^*)\quad\text{and}\quad \tfrac{1}{2}\sum_{\varepsilon} i\,\boldsymbol{K}\times\boldsymbol{K}^*$$

are in general non-zero and perpendicular to the reaction plane, which could easily be seen from invariance considerations. This would imply that $\mathcal{V}_7$ vanishes and $\mathcal{V}_4$, $\mathcal{V}_5$, $\mathcal{V}_6$ are respectively equivalent to $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_3$. However all the five components of the second-rank tensor

$$\tfrac{1}{2}\sum_{\varepsilon}(\boldsymbol{K}_1\times\boldsymbol{K}^*_1)^2_{\mu}$$

survive in general. Thus we have the differential cross section proportional to

$$\tfrac{1}{2}\sum_{\varepsilon}\Big[a_1 LL^* - \frac{1}{\sqrt{3}}\big(a_2 + \frac{1}{\sqrt{10}}a_3\big)\boldsymbol{K}\cdot\boldsymbol{K}^* + \frac{\sqrt{3}}{\sqrt{10}}|\hat{k}\cdot\boldsymbol{K}|^2\Big]\,, \tag{49}$$

and choosing the normal to the plane of the reaction to be the z-axis, we have

$$\mathrm{Tr}\,\rho^{\mathrm{f}}\langle\mathcal{T}^1_0\rangle = \tfrac{1}{2}\sum_{\varepsilon}\Big[\big(b_1 + \frac{1}{\sqrt{10}}b_4\big)(L^*\boldsymbol{K}+L\boldsymbol{K}^*)_z + \big(b_2+\frac{1}{\sqrt{10}}b_5\big)$$
$$\times\, i(L^*\boldsymbol{K} - L\boldsymbol{K}^*)_z + \big(\frac{1}{\sqrt{2}}b_3 + \frac{1}{2\sqrt{5}}b_6\big)\, i(\boldsymbol{K}\times\boldsymbol{K}^*)_z\Big]\,, \tag{50}$$

$$\mathrm{Tr}\,\rho^{\mathrm{f}}\langle\mathcal{T}^1_{\pm 1}\rangle = 0\,, \tag{51}$$

$$\mathrm{Tr}\,\rho^{\mathrm{f}}\langle\mathcal{T}^2_0\rangle = \tfrac{1}{2}\sum_{\varepsilon}\Big[\big(c_1 + \frac{1}{\sqrt{14}}c_7 + \frac{3}{4\sqrt{14}}c_8\big)(\boldsymbol{K}_1\times\boldsymbol{K}^*_1)^2_0 - \tfrac{1}{2}c_2 LL^*$$
$$+\frac{1}{2\sqrt{3}}c_3\,\boldsymbol{K}\cdot\boldsymbol{K}^* + \Big(\tfrac{1}{2}\frac{\sqrt{3}}{\sqrt{7}}c_7 - \frac{5}{8\sqrt{21}}c_8\Big)$$
$$\times\big\{(\boldsymbol{K}_1\times\boldsymbol{K}^*_1)^2_{+2}\,\mathrm{e}^{-2i\theta_k} + (\boldsymbol{K}_1\times\boldsymbol{K}^*_1)^2_{-2}\,\mathrm{e}^{2i\theta_k}\big\}\Big]\,, \tag{52}$$

$$\mathrm{Tr}\,\rho^f\langle\mathcal{T}^2_{\pm1}\rangle = \tfrac{1}{2}\sum_{\mathcal{E}}\Big[\Big(c_1 + \frac{1}{2\sqrt{14}}c_7 - \frac{1}{2\sqrt{14}}c_8\Big)(K_1\times K_1^*)^2_{\pm1} + \Big(\frac{3}{2\sqrt{14}}c_7 + \frac{5}{6\sqrt{14}}c_8\Big)(K_1\times K_1^*)^2_{\mp1}e^{\pm2i\theta_k}\Big]\,, \tag{53}$$

$$\begin{aligned}\mathrm{Tr}\,\rho^f\langle\mathcal{T}^2_{\pm2}\rangle = \tfrac{1}{2}\sum_{\mathcal{E}}\Big[\Big(c_1 - \frac{1}{\sqrt{14}}c_7 + \frac{1}{8\sqrt{14}}c_8\Big)(K_1\times K_1^*)^2_{\pm2} &+ \Big\{\frac{\sqrt{3}}{2\sqrt{2}}c_2LL^* - \frac{1}{2\sqrt{2}}c_3\,K\cdot K^* \pm \tfrac{1}{2}c_4(L^*K + LK^*)_z \\ &\pm \tfrac{1}{2}c_5\,i(L^*K - LK^*)_z \pm \frac{1}{2\sqrt{2}}c_6\,i(K\times K^*)_z \\ &+ \tfrac{1}{2}\frac{\sqrt{3}}{\sqrt{7}}\big(c_7 - \tfrac{1}{12}c_8\big)(K_1\times K_1^*)^2_0\Big\}e^{\pm2i\theta_k} \\ &+ \frac{5\sqrt{7}}{24\sqrt{2}}c_8(K_1\times K_1^*)^2_{\mp2}e^{\pm4i\theta_k}\Big]\,.\end{aligned} \tag{54}$$

It may be noted that in the above equations the quantities LL^*, $K\cdot K^*$, (L^*K+LK^*), $i(L^*K-LK^*)$, $iK\times K^*$ and $(K_1\times K_1^*)^2_0$ are real, while $(K_1\times K_1^*)^2_{\pm1}$ and $(K_1\times K_1^*)^2_{\pm2}$ are in general complex. It is convenient to define real quantities [1]

$$Q_{\alpha\beta} = \tfrac{1}{2}\big[\tfrac{3}{2}(K_\alpha K_\beta^* + K_\alpha^* K_\beta) - K\cdot K^*\,\delta_{\alpha\beta}\big]\,; \qquad \alpha,\beta = x, y, z, \tag{55}$$

in terms of which

$$(K_1\times K_1^*)^2_0 = \sqrt{2}\,Q_{zz}\,, \tag{56}$$

$$(K_1\times K_1^*)^2_{\pm1} = \mp\frac{2}{\sqrt{3}}(Q_{xz} \pm iQ_{yz})\,, \tag{57}$$

$$(K_1\times K_1^*)^2_{\pm2} = \frac{1}{\sqrt{3}}(Q_{xx} - Q_{yy} \pm 2iQ_{xy})\,. \tag{58}$$

3.3. *Elastic scattering of electrons, nucleons and other processes*

In the case of elastic scattering of electrons or nucleons, we would expect from invariance considerations that the results will follow the same pattern as in the previous case, if we sum and average over the spins. If we assume however that the electron scattering proceeds through one-photon exchange, considerable reduction occurs. But the nucleon-photon vertex introduces the electromagnetic form factors of the nucleon and one would in fact expect to determine these quantities precisely from an improved understanding of the structure of the deuteron. This is the case also with possible muon scattering or electro-pion production experiments.

4. DISCUSSION

Amongst the various observables considered above, the differential cross sections are by far the easiest to measure. Further their knowledge is necessary for a meaningful analysis of data on the various tensor moments $\langle \mathcal{T}^{\kappa}_{\mu} \rangle$ unless the product of a tensor moment with the differential cross section is measured directly in experiments. This is because of the appearance of the quantity Tr ρ^f in eqs. (41) and (44)-(54), which we have now to solve to determine the necessary coefficients a, b and c to enable evaluation of the basic form factors. At the outset we notice that since each of these coefficients is of the form of a homogeneous quadratic expression in the form factors, the solutions would not determine an overall sign for the form factors, although we know that F_0^{uu} and F_0^{ww} are positive at the origin. Further since the differential cross section provides at best only the three coefficients a_1, a_2 and a_3, it is not possible generally to determine the form factors from data on differential cross sections alone. We have therefore in addition to make use of data on the spin tensor moments. We shall now outline a suitable set of measurements for the purpose.

We observe that if we consider pion scattering at 180°, either of the tensors $\langle \mathcal{T}^2_0 \rangle$ or $\langle \mathcal{T}^2_{\pm 2} \rangle$ are given by just one term proportional to c_2/a_1 and consequently their measurements would lead to a direct determination of c_2. Observing that c_3 is directly proportional to a_3, measurement of the quantity Tr $\rho^f \langle \mathcal{T}^2_0 \rangle$ in the same process in general at other angles provides a linear equation in $(c_1 + (1/\sqrt{14})c_7 + (3/4\sqrt{14})c_8)$ and c_2. Therefore from measurements at different angles corresponding to the same k, these two unknowns could be estimated. A knowledge of c_2 together with a_1 enables us to solve for

$$\mathcal{F}_1 = \tfrac{1}{6}(\alpha_1 + 2\alpha_2) , \tag{59a}$$

$$\mathcal{F}_3 = \frac{1}{3\sqrt{2}} (\alpha_1 - \alpha_2) , \tag{60a}$$

or

$$\mathcal{F}_1 = \tfrac{1}{6}(\alpha_1 - 2\alpha_2) , \tag{59b}$$

$$\mathcal{F}_3 = \frac{1}{3\sqrt{2}} (\alpha_1 + \alpha_2) , \tag{60b}$$

apart from an overall sign, where

$$\alpha_1 = (a_1 - \sqrt{2}\, c_2)^{\frac{1}{2}} , \tag{61}$$

$$\alpha_2 = (a_1 + \frac{1}{\sqrt{2}} c_2)^{\frac{1}{2}} . \tag{62}$$

The solutions (59) and (60) may conveniently be obtained, for example, geometrically as the intersections of a pair of straight lines

$$x + \sqrt{2}\, y = \pm \tfrac{1}{2} \alpha_1 , \tag{63}$$

or

$$x - \frac{1}{\sqrt{2}} y = \pm \tfrac{1}{2} \alpha_2 \,, \tag{64}$$

with the circle

$$x^2 + y^2 = \tfrac{1}{4} a_1 \,. \tag{65}$$

The alternate solutions (59a), (60a), (59b) and (60b) would collapse into one if α_1 or α_2 is zero and for solutions of physical interest to exist, we obtain the condition

$$-\sqrt{2} a_1 \leq c_2 \leq \frac{1}{\sqrt{2}} a_1 \,. \tag{66}$$

We further have from eq. (14)

$$a_1 \geq 0 \,. \tag{67}$$

We next observe that measurement of the quantity $\mathrm{Tr}\,\rho^f(\langle \mathcal{T}^2_{+2}\rangle + \langle \mathcal{T}^2_{-2}\rangle)$ at $\theta_k = 45^\circ$ and of $i\mathrm{Tr}\,\rho^f(\langle \mathcal{T}^2_{+2}\rangle - \langle \mathcal{T}^2_{-2}\rangle)$ at $\theta_k = 0^\circ$ in the case of photoproduction of neutral pions leads us to a pair of linear equations in $(c_1 + (\sqrt{2}/\sqrt{7})c_7 + (1/8\sqrt{14})c_8)$ and c_8 which could therefore be estimated normally. It is readily seen that from a knowledge of a_2, a_3 and c_8, we can solve for

$$\mathcal{F}_2 = \frac{1}{2\sqrt{2}} \left(\frac{\beta_2}{\beta_1} - \beta_1 \right) , \tag{68}$$

$$\mathcal{F}_4 = \frac{1}{2} \left(\frac{\beta_2}{\beta_1} + \beta_1 \right) , \tag{69}$$

apart from an overall sign, and

$$\mathcal{F}_5^2 = \frac{5\sqrt{7}}{9\sqrt{6}} c_8 - \frac{1}{4} \left(\frac{\beta_2}{\beta_1} + \beta_1 \right)^2 , \tag{70}$$

where

$$\beta_1 = \left(\frac{\sqrt{3}}{2\sqrt{10}} a_3 - \frac{\sqrt{3}}{4} a_2 \right)^{\frac{1}{2}} , \tag{71}$$

$$\beta_2 = \frac{\sqrt{3}}{4} a_2 + \frac{5\sqrt{7}}{6\sqrt{6}} c_8 \,. \tag{72}$$

Solutions (68) and (69) may also be obtained geometrically as before as the intersections of a pair of straight lines

$$\sqrt{2}\, x - y = \pm \beta_1 \,, \tag{73}$$

or

$$\sqrt{2}\, x + y = \pm \frac{\beta_2}{\beta_1} \,, \tag{74}$$

with the hyperbola

$$-2x^2 + y^2 = \beta_2 \,, \tag{75}$$

which can conveniently be made into a rectangular hyperbola by choosing $x' = \sqrt{2}x$; $y' = y$, when the straight lines make angles 45° and 135°. It may be mentioned that since the straight lines are parallel to the asymptotes, we obtain only two finite points of intersection which go into one another under $(x, y) \to (-x, -y)$ and the condition for real solutions to exist is

$$a_3 \geq \frac{\sqrt{5}}{\sqrt{2}} a_2 \,. \tag{76}$$

From eq. (15) we also have

$$a_2 \leq 0 \,. \tag{77}$$

The choice of observables considered above is however, not unique since several combinations of c_1, c_7 and c_8 can be determined from experiments and each of these coefficients as well as a_2 and a_3 are functions of $\mathcal{F}_2$, $\mathcal{F}_4$ and $\mathcal{F}_5$ only. For instance, we have already mentioned that $(c_1 + (1/\sqrt{14})\, c_7 + (3/4\sqrt{14})\, c_8)$ can be estimated from measurements of $\mathrm{Tr}\, \rho^f \langle \mathcal{T}^2_0 \rangle$ in pion scattering. One might also solve for $[c_1 + (1/2\sqrt{14}) \times (c_7 - c_8)]$ and $(9c_7 + 5c_8)$ from suitable measurements of $\mathrm{Tr}\, \rho^f \langle \mathcal{T}^2_{\pm 1} \rangle$ in the photoproduction process. The tensor moment, $\mathrm{Tr}\, \rho^f \langle \mathcal{T}^2_0 \rangle$ on the same process may even be utilised to solve for $(c_1 + (1/\sqrt{14})\, c_7 + (3/4\sqrt{14})\, c_8)$ and $(24 c_7 - 5 c_8)$ assuming c_2 and c_3 or to determine the latter combination of the c-values assuming the former and c_2 and c_3 which are obtainable from other experiments. Thus several other sets of non-redundant equations might also be formed and the determination of $\mathcal{F}_2$ and $\mathcal{F}_4$ together with their relative sign and of $|\mathcal{F}_5|$ can be carried out choosing any other convenient set of observables as well. They may be also expected to lead to additional conditions similar to (66), (67), (76) and (77) which may be considered as checks on the model underlying the analysis.

Finally we observe that measurement of $\mathrm{Tr}\, \rho^f \langle \mathcal{T}^1_0 \rangle$ in the case of pion scattering leads to a linear equation in $(b_1 + (1/\sqrt{10})\, b_4)$ and $(b_2 + (1/\sqrt{10})\, b_5)$. These two quantities may therefore be determined by taking measurements at different angles and energies corresponding to the same value of k. Assuming knowledge of $\mathcal{F}_2$ and $\mathcal{F}_4$, we readily see that $(b_2 + (1/\sqrt{10})\, b_5)$ leads to a linear relation between $\mathcal{F}_1$ and $\mathcal{F}_3$. This equation could be utilised to choose between the solutions (a) and (b) of eqs. (59) and (60) with the correct relative sign with respect to $\mathcal{F}_2$ and $\mathcal{F}_4$. The sign of $(b_1 + (1/\sqrt{10})\, b_4)$ clearly determines the relative sign of $\mathcal{F}_5$ with respect to the other four. Alternatively one could also consider $\mathrm{Tr}\, \rho^f \langle \mathcal{T}^1_0 \rangle$ in the case of photoproduction but this leads to solving for three unknown quantities $(b_1 + (1/\sqrt{10})\, b_4)$, $(b_2 + (1/\sqrt{10})\, b_5)$ and $(b_3 + (1/\sqrt{20})\, b_6)$, where the third unknown might be expected to be dominant. One may also consider the coefficients c_4 and c_5 for the purpose but their determination from measurements of $\mathrm{Tr}\, \rho^f \langle \mathcal{T}^2_{\pm 2} \rangle$ might necessitate assumption of c_2, c_3, c_7 and c_8.

The determination of the five factors $\mathcal{F}$ with their relative signs is equivalent to determining the five basic form factors F apart from a common sign; they are given by

$$F_0^{uu} = \tfrac{1}{2}(\mathcal{F}_1 + 2\,\mathcal{F}_2) \,, \tag{78}$$

$$|F_2^{uw}|\cos\delta = \tfrac{1}{4}(2\mathcal{F}_3 + \mathcal{F}_4)\ , \tag{79}$$

$$|F_2^{uw}|\sin\delta = \tfrac{1}{4}\ \mathcal{F}_5\ , \tag{80}$$

$$F_0^{ww} = \tfrac{2}{3}(\mathcal{F}_1 - \mathcal{F}_2)\ , \tag{81}$$

$$F_2^{ww} = \tfrac{1}{3}\sqrt{2}\,(\mathcal{F}_4 - \mathcal{F}_3)\ . \tag{82}$$

On the basis of existing estimates of the D-state admixture and the generally held belief that the S- and D-state radial shapes are more or less alike, one would expect $\mathcal{F}_1$ and $\mathcal{F}_2$ and likewise $\mathcal{F}_3$ and $\mathcal{F}_4$ to have the same order of magnitude and sign and consequently one might expect to determine the form factors F_0^{uu}, $|F_2^{uw}|\cos\delta$ and $|F_2^{uw}|\sin\delta$ as appropriate experimental data on recoil deuteron polarization become available, while the determination of F_0^{ww} and F_2^{ww}, which are given by the respective differences between $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$, $\mathcal{F}_4$ might have to await increased accuracy of experimental observation and theoretical analysis. But it is interesting to note that a knowledge of the first three form factors as functions of k would allow us, in addition to determining δ using (38) at any value of k, to calculate

$$|u(r)|^2 = \frac{1}{2\pi}\int_0^\infty F_0^{uu}(k)\, j_0(\tfrac{1}{2}kr)\, k^2\, \mathrm{d}k\ , \tag{83}$$

$$u(r)w(r) = \frac{1}{2\pi}\, \mathrm{e}^{i\delta}\int_0^\infty |F_2^{uw}(k)|\, j_2(\tfrac{1}{2}kr)\, k^2\, \mathrm{d}k\ , \tag{84}$$

using the self-inverse Hankel transform. It might be added that making use of the finite radius of the deuteron one could also express the integrals on the right hand side of eqs. (83) and (84) as sums [9] over conveniently chosen discrete sets of values of k. We shall see in sect. 5 that the finite radius of the deuteron enables us also to estimate F_0^{ww} at low values of k from measurements of differential cross section alone so that one readily obtains P_D using eq. (37).

5. AN APPROXIMATION

Observing that at sufficiently low values of the momentum transfer k such that $\frac{1}{2}kR$ is small, where R denotes the radius of the deuteron, we can approximate [10]

$$j_0(\tfrac{1}{2}kr) = 1,\qquad j_2(\tfrac{1}{2}kr) = \tfrac{1}{15}(\tfrac{1}{2}kr)^2\ , \tag{85}$$

we see that the coefficients of the LL^* and $K \cdot K^*$ terms in the differential cross section may be written respectively as

$$A = 4f_1^2 + 4(\tfrac{1}{60}k^2)^2 f_3^2 \, , \tag{86}$$

$$B = \tfrac{1}{3}[2\sqrt{2}\, f_2 + \tfrac{1}{60}k^2 f_4]^2 + 3(\tfrac{1}{60}k^2)^2 f_5^2 \, , \tag{87}$$

where f_1, f_2 and $\frac{1}{60}k^2$ times f_3, f_4, f_5 denote the respective $\mathcal{F}$ at small values of k and are given by

$$f_1 = P_S + P_D = 1 \, , \tag{88}$$

$$f_2 = P_S - \tfrac{1}{2}P_D = 1 - \tfrac{3}{2}P_D \, , \tag{89}$$

$$f_3 = \int_0^R r^4 \, dr \, (u^*w + w^*u - \frac{1}{\sqrt{2}} w^*w) \, , \tag{90}$$

$$f_4 = \int_0^R r^4 \, dr \, (u^*w + w^*u + \sqrt{2}\, w^*w) \, , \tag{91}$$

$$f_5 = -i \int_0^R r^4 \, dr \, (u^*w - w^*u) \, . \tag{92}$$

It may be observed that the factor f_3 is directly proportional to the deuteron quadrupole moment [11]

$$Q = \frac{1}{10\sqrt{2}} f_3 \, , \tag{93}$$

so that essentially no unknown structure factor is involved in the spin-independent part. The spin-dependent part however contains three unknown quantities f_2, f_4 and f_5 if we retain all the terms, f_2 and f_4 only if we neglect terms proportional to $(\frac{1}{60}k^2)^2$ and f_2 only if we neglect terms proportional to $\frac{1}{60}k^2$ and higher powers. Thus f_2 may be determined directly or more accurately by fitting B to a polynomial expansion in $(\frac{1}{60}k^2)$. Hence we have

$$P_D = \tfrac{2}{3}(1 - f_2) \, . \tag{94}$$

It is interesting to observe that the polynomial expansion of

$$B = B_0 + B_1 \tfrac{1}{60}k^2 + B_2 (\tfrac{1}{60}k^2)^2 \, , \tag{95}$$

allows us to calculate also

$$f_4 = \tfrac{1}{2}\sqrt{3}\, B_0^{-\frac{1}{2}} B_1 \, , \tag{96}$$

$$f_5^2 = \tfrac{1}{3}\left(B_2 - \tfrac{1}{4}\frac{B_1}{B_0}\right) , \tag{97}$$

so that one might also estimate

$$\mathrm{tg}^2\,\delta = \frac{(2f_3 + f_4)^2}{9f_5^2}\,, \tag{98}$$

where the value of f_3 could be obtained either from eq. (93) or from the spin-independent part of the differential cross section.

In processes where the $|\hat{k}\cdot K|^2$ term is non-vanishing, it might be advantageous to make use of the coefficient of the above quantity, which may be written as

$$C = (\tfrac{1}{60}k^2)^2 f_4^2 - 3(\tfrac{1}{60}k^2)^2 f_5^2 - 4\sqrt{2}\,(\tfrac{1}{60}k^2) f_2 f_4\,, \tag{99}$$

for determining $\mathrm{tg}^2\delta$, since once can conveniently express (99) as

$$(\tfrac{1}{60}k^2)^{-1}C = C_0 + (\tfrac{1}{60}k^2)C_1\,, \tag{100}$$

which provides a straight line fit.

It may also be noted that although the determination of the spin-dependent cross section would generally require at least two measurements at a fixed momentum transfer in processes like pion scattering and at least three measurements in processes like neutral pion-photoproduction where an additional term involving the spin-dependent amplitudes also contributes, the number of measurements could be reduced by one if the value of the factor f_3 could be assumed from data on the quadrupole moment, for which a recent estimate [12] gives 2.82×10^{-27} cm^2. Thus single measurements of the differential cross section for the scattering of pions at low values of momentum transfer would be sufficient for the purpose. But, to our knowledge, appropriate data on the process are not presently available.

In summary, the impulse approximation analysis of the deuteron elementary particle collisions where the deuteron remains bound in the final state has shown that the differential cross section as well as the various spin tensor moments of the recoil deuteron are functions of the five form factors F_0^{uu}, F_0^{ww}, F_2^{uw}, F_2^{wu} and F_2^{ww} characterizing the structure of the deuteron and that the observables contain not only the combination $(F_2^{uw}+F_2^{wu})$ but also $(F_2^{uw}-F_2^{wu})$ which makes possible the determination of the relative phase δ between the S- and D-states. Considering (i) the elastic scattering of pions and (ii) the photoproduction of neutral pions, it is found that measurements at any chosen value of momentum transfer of the following observables in the lab frame would enable one to solve for the five form factors:

(i) $\quad \mathrm{Tr}\,\rho^{\mathrm{f}}\langle Q_{zz}\rangle$ or $\mathrm{Tr}\,\rho^{\mathrm{f}}\langle Q_{xx}\rangle$

in process (i) at $\Theta = 0$, where Θ denotes the angle made by the recoil deuterons with the beam and Q the functions of deuteron spin J defined as in (55),

(ii) $\quad \mathrm{Tr}\,\rho^{\mathrm{f}}(\langle Q_{xx}\rangle - \langle Q_{yy}\rangle)$ at $\Theta = 45^\circ$,

and $\mathrm{Tr}\,\rho^{\mathrm{f}}\langle Q_{xy}\rangle$ at $\Theta = 0^\circ$ in process (ii),

(iii) $\quad \mathrm{Tr}\,\rho^{\mathrm{f}}\langle J_z\rangle$

at at least two different angles in process (i),

(iv) The differential cross section at at least three different angles either in process (ii) or together in processes (i) and (ii), where at least one measurement should belong to the process (ii).

In (i), (ii) and (iii) the beam direction is chosen as the x axis and the normal to the reaction plane as the z-axis. Alternative choices for the above measurements also exist.

It is also found that measurements of the differential cross section as a function of the momentum transfer at low values of the variable in either of the processes would independently lead to estimates of the D-state probability and also the relative phase as solution of an equation where $\mathrm{tg}^2\,\delta$ is given. Corresponding to each value of momentum transfer, at least two measurements are required in process (ii) while one measurement in process (i) is sufficient for the purpose.

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