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A NOTE ON FACTORIZATION OF SIMPLE GAMES

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ABSTRACT

The equivalence of coherent systems and simple games has not been generally noticed and exploited. In this note, some known results about factorization of coherent systems are translated into corresponding results for simple games.

1. Introduction and Preliminaries

A *simple game* is an orderd pair (N, v) where N is a non-empty finite set and $v : 2^N \rightarrow \{0, 1\}$ such that (*i*) $v(\phi) = 0$, (*ii*) $v(N) = 1$ and (*iii*) $v(S) \leqslant v(T)$ whenever $S \subseteq T$. Elements of N are called *players* and often identified with the integers $1, 2, \ldots, n$ where $n = |N|$. Elements of the set 2^N are called *coalitions*. A coalition S is called *winning* (*losing*) if $v(S) = 1(0)$. A simple game can also be represented by the pair (N, W) where $W = \{S \in 2^N ; v(S) = 1\}$ is the family of winning coalitions. A coalition S is called *blocking* if $v(N-S) = 0$. A winning (blocking) coalition S is called *minimal* winning (blocking) coalition if there does not exist a winning (blocking) coalition T such that $T \subset S$. A player $i \in N$ is called a *dummy* if $v(S \cup \{i\}) = V(S - \{i\})$ for all coalitions S.

Let $x = (x_1, x_2, \ldots, x_n) \in \{0, 1\}^n$. In *reliability engineering*, a *coherent system* is the orderd pair $[N, \theta]$ where $N = \{1, \ldots, n\}$ is the set of components and $\theta : \{0, 1\}^n \rightarrow \{0, 1\}$ such that (*i*) $\theta(0) = 0$, (*ii*) $\theta(1) = 1$ and (*iii*) $x \leqslant y$ implies $\theta(x) \leqslant \theta(y)$. The function θ is called a structure function.

It is seen that coherent systems of reliability is conceptually equivalent to simple games (See also Butterworth [4]). These two topics have been developed independently of each other. For the same underlying basic concepts, different terminologies have been used. A sample is given below :

Simple Games	*Coherent Systems*
(*i*) Player	Component
(*ii*) Dummy player	Irrelevant component
(*iii*) Winning coalition	Path set

(*iv*) Blocking coalition	Cut set
(*v*) Multilinear extension	Reliability function
(*vi*) Absolute Banzhaf index for player i	Structural importance of component i

The combinatorial and stochastic aspects have been extensively studied in coherent systems while the power indices and their properties have been studied more in simple games. The equivalence of coherent systems and simple games has not been generally noticed and fully exploited. There are a number of results concerning coherent systems which are not known in game theory and vice-versa. For a detailed discussion of coherent systems and simple games we refer to [1], [5] and [3], [6], respectively. In this note we translate a particular set of known results for coherent systems to simple games.

Let $(N_j, \mathbf{W}_j)$, $j = 1, \ldots, m$, be a set of m simple games such that $N_j \cap N_k = \phi$ for $j \neq k$ and also let $(M, \mathbf{V})$ be a simple game with $|M| = m$. The composite game $(N, \mathbf{W})$ is defined by the player set $N = \bigcup N_j$ and

$$\mathbf{W} = \{S \subseteq N: \{j \in M : S \cap N_j \in \mathbf{W}_j\} \in \mathbf{V}\}.$$

Such a procedure is called *composition* of simple games and is extensively studied in game theory. However, the converse problem of decomposing a game $(N, \mathbf{W})$ into sub-games $(N_j, \mathbf{W}_j)$ and a master game $(M, \mathbf{V})$ has not been given much attention. In reliability theory, this problem is known as *Modular Decomposition of Coherent Systems* and has been extensively studied by Birnbaum and Esary [2]. All that we do in this note is state the results of Birnbaum and Esary in game theoretic language. The proofs have been omitted and can be found in [2].

2. MODULAR DECOMPOSITION OF SIMPLE GAMES

Let (N, ν) be a simple game and, for the sake of convenience, we shall assume that there are no dummy players. We consider a subset J of players which is neither empty nor exhaustive, that is $\phi \subset J \subset N$. We call J a *modular* set of (N, ν) if for any $S \subseteq J$

$$\nu(S \cup T) = \nu(J \cup T) \text{ for all } T \subseteq N-J.$$

or

$$\nu(S \cup T) = \nu(T) \text{ for all } T \subseteq N-J.$$

We note that $\nu(J \cup T) = \nu(T)$ for all $T \subseteq N-J$ is *not true* since none of the players are dummies. Define the simple subgame (J, χ) by

$$\chi(S) = \begin{cases} 1 \text{ when } \nu(S \cup T) = \nu(J \cup T) \text{ for all } T \subseteq N-J \\ 0 \text{ when } \nu(S \cup T) = \nu(T) \text{ for all } T \subseteq N-J, \end{cases}$$

for any $S \subset J$. The simple game $(J, \varkappa)$ is called a *module* of (N, v). Define the master game (M, λ) with player set $M = (N-J) \cup \{a\}$; $a \notin N$, by

$$\lambda(\{a\} \cup T) = v(J \cup T)$$

$$\lambda(T) = v(T),$$

for all $T \subseteq N-J$.

Remark 1. If in the master game (M, λ), we replace the player a by the player set J using the simple sub-game $(J, \varkappa)$, the resulting composite game is none other than (N, v).

Example 1. Consider the simple game (N, v) where $N = \{1, 2, 3, 4, 5\}$ and the winning coalitions are $\{1,3,4\}$, $\{1,3,5\}$, $\{2,3,4\}$, $\{2,3,5\}$ $\{1,2,3,4\}$, $\{1,2,3,5\}$, $\{1,3,4,5\}$, $\{2,3,4,5\}$ and $\{1,2,3,4,5\}$.

Minimal winning coalitions : $W_1 = \{1, 3, 4\}$, $W_2 = \{1, 3, 5\}$, $W_3 = \{2, 3, 4\}$, $W_4 = \{2, 3, 5\}$.

Minimal blocking coalitions : $B_1 = \{1, 2\}$, $B_2 = \{3\}$, $B_3 = \{4,5\}$.

Let $J = \{1, 2, 3\}$. We note that

$v(\{1, 3\} \cup T) = v(\{2,3\} \cup T) = v(J \cup T)$ for all $T \subseteq \{4, 5\}$

$v(\{1\} \cup T) = v(\{2\} \cup T) = v(\{3\} \cup T) = v(\{1, 2\} \cup T) = v(T)$,

for all $T \subseteq \{4, 5\}$

Hence J is a modular set of (N, v). The module (J, χ) of (N, v) is defined by

$$\chi(\{1,3\}) = \chi(\{2, 3\}) = \chi(\{1, 2, 3\}) = 1$$

$$\chi(\phi) = \chi(\{1\}) = \chi(\{2\}) = \chi(\{3\}) = \chi(\{1, 2\}) = 0$$

The master game (M, λ) is specified by $M = \{4, 5\} \cup \{a\}$ and

$$\lambda(\phi) = \lambda(\{4\}) = \lambda(\{5\}) = \lambda(\{4, 5\}) = \lambda(\{a\}) = 0$$

$$\lambda(\{a, 4\}) = \lambda(\{a, 5\}) = \lambda(\{a, 4, 5\}) = 1.$$

Theorem 1 gives the relationship between minimal winning (blocking) coalitions of a module with those of the parent simple game.

THEOREM 1. *Let (J, χ) be a module of a simple game (N, v). Further let $W_1, W_2, \ldots, W_r$ $(B_1, B_2, \ldots, B_s)$ be the minimal winning (blocking) coalitions of (N, v). The minimal winning (blocking) coalitions of (J, χ) are those sets $J \cap W_j$, $j = 1, \ldots, r$ $(J \cap B_j, j = 1, \ldots, s)$ which are not empty.*

Example 2. Let (N, v) and (J, χ) be as in Example 1. $J \cap W_1 = J \cap W_2 = \{1, 3\}$ and $J \cap W_3 = J \cap W_4 = \{2, 3\}$. Hence the minimal winning coalitions of (J, χ) are $\{1, 3\}$ and $\{2, 3\}$. Since $J \cap B_1 = \{1, 2\}$ $J \cap B_2 = \{3\}$ and $J \cap B_3 = \phi$, we have $\{1, 2\}$ and $\{3\}$ as the minimal blocking coalitions of (J, χ).

In Theorem 2, a test for modularity of a set is given.

THEOREM 2. *Let $W_1, W_2, \ldots, W_r$ $(B_1, B_2, \ldots, B_s)$ be the minimal winning (blocking) coalitions of a simple game (N, v). A necessary and sufficient condition that a set of players J, $\phi \subset J \subset N$, be a modular set of (N, v) is that $(J \cap W_j) \cup ((N-J) \cap W_k)$ $((J \cap B_j) \cup ((N-J) \cap B_k))$ is a minimal winning (blocking) coalition of (N, v) whenever $J \cap W_j$ and $J \cap W_k$ $(J \cap B_j$ and $J \cap B_k)$ are not empty.*

Example 3. Let (N, v) be the simple game of Example 2 and let $J = \{3, 4, 5\}$. We note that J has non-empty intersection with minimal blocking coalition $B_2 = \{3\}$ and $B_3 = \{4, 5\}$. We further have

$$J \cap B_2 = \{3\}, (N-J) \cap B_2 = \phi,$$

$$J \cap B_3 = \{4, 5\}, (N-J) \cap B_3 = \phi.$$

We note that $(J \cap B_2) \cup ((N-J) \cap B_3)$ and $(J \cap B_3) \cup ((N-J) \cup B_2)$ are minimal blocking coalitions of (N, v). Hence J is a modular set.

The result given in Theorem 3 is known as the *three modules* theorems

THEOREM 3. *Let (N, v) be a simple game with no dummies. Let J_1, J_2, J_3 be non-empty subsets of N such that $J_1 \cup J_2$ and $J_2 \cup J_3$ are modular sets of (N, v). Thon J_1, J_2 and J_3 are modular sets of (N, v). Further, either $J_1 \cup J_2 \cup J_3 = N$ or $J_1 \cup J_2 \cup J_3$ is a modular set (N, v). Finally, the modules (J_1, χ_1), (J_2, χ_2) and (J_3, χ_3) appear in (N, v) either in parallel with one another or in series with one another.*

Example 4. Let (N, v) be as in Example 2 and $J_1 = \{1,2\}$, $J_2 = \{3\}$ and $J_3 = \{4, 5\}$. From Examples 2 and 3, we know that $J_1 \cup J_2$ and $J_2 \cup J_3$ are modular sets of (N, v). Hence by Theorem 3, J_1, J_2 and J_3 are modular sets and $J_1 \cup J_2 \cup J_3 = N$.

$$\chi_1(\{1\}) = \chi_1(\{2\}) = \chi_1(\{1,2\}) = 1, \chi_1(\phi) = 0$$

$$\chi_2(\{3\}) = 1, \chi_2(\phi) = 0$$

$$\chi_3(\{4\}) = \chi_3(\{5\}) = \chi_3(\{4, 5\}) = 1, \chi_3(\phi) = 0.$$

We also note that (J_1, χ_1), (J_2, χ_2) and (J_3, χ_3) appear in series in (N, v).

A set M is a *maximal modular* set J of (N, v) if M is a modular set of

(N, v) and there is no modular set J of (N, v) such that $M \subsetneq J$. Let M_1, $M_2, \ldots, M_k$ be the distinct maximal modular set of (N, v). Let **M** be the class of sets formed by intersecting M_1 or $N-M_1$ with either M_2 or $N-M_2 \ldots$ with either M_k or $N-M_k$. We call the non-empty sets in **M** the *modular factors*. The modular factors form a partition of N into disjoint sets. The modular factor are always modular sets of (N, v). The modular factorization produces a unique decomposition of the simple game (in a qualified sense) into its largest possible disjoint modules. (*See* [2] for an exception to this).

Example 5. let (N, v) be as in Example 1. It is easily verified that $M_1 = \{1,2,3\}$, $M_2 = \{3,4,5\}$ and $M_3 = \{1, 2, 4, 5\}$ are maximal, and

$$M_1 \cap M_2 \cap M_3 = \phi \qquad (N-M_1) \cap M_2 \cap M_3 = \{4,5\}$$

$$M_1 \cap M_2 \cap (N-M_3) = \{3\} \qquad (N-M_1) \cap M_2 \cap (N-M_3) = \phi$$

$$M_1 \cap (N-M_2) \cap M_3 = \{1,2\} \qquad (N-M_1) \cap (N-M_2) \cap M_3 = \phi$$

$$M_1 \cap (N-M_2) \cap (N-M_3) = \phi \qquad (N-M_1) \cap (N-M_2) \cap (N-M_3) = \phi$$

The modular factors are $\{1,2\}$, $\{3\}$ and $\{4,5\}$.

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