# A NEW GENERAL SERIES OF BALANCED INCOMPLETE BLOCK DESIGNS

C. RAMANUJACHARYULU

0. **Summary.** Let $v$ be any integer with $s_1$ the least prime power factor, the other prime power factors being $s_2, \cdots, s_m$. Assume $v$ is odd and consider the cartesian product of the $m$ Galois fields $\mathrm{GF}(s_1), \cdots, \mathrm{GF}(s_m)$ of orders $s_1, \cdots, s_m$ respectively. Let $x_i$ denote the primitive root of $\mathrm{GF}(s_i)$, $i=1, 2, \cdots, m$. Then labelling the points

$$\alpha_{j+1} = (x_1^j, x_2^j, \cdots, x_m^j), \qquad j = 0, 1, \cdots, s_1 - 2;$$

and arbitrarily labelling the remaining points $\alpha$ of the product space, defining addition and multiplication of $\alpha$'s coordinate-wise in their respective fields, we take the initial blocks.

$$(0, \beta_1\alpha_1, \beta_1\alpha_2, \cdots, \beta_1\alpha_{k-1})$$
$$(0, \beta_2\alpha_1, \beta_2\alpha_2, \cdots, \beta_2\alpha_{k-1})$$
$$\vdots$$
$$(0, \beta_{(v-1)/2}\alpha_1, \beta_{(v-1)/2}\alpha_2, \cdots, \beta_{(v-1)/2}\alpha_{k-1})$$

where $k \leqq s_1$ if $m>1$ and $k<s_1$ if $m=1$; $0=(0, 0, \cdots, 0)$ and $\beta_j$'s are such that no two $\beta_j$'s add up to 0 (= the null vector): The theorem proved here is that by adding each of the points $\beta_j, j=0, 1, \cdots, v-1$ of the product space to each of the above initial blocks we get a Balanced Incomplete Block Design with the parameters

$$\left(v, \frac{v\cdot v-1}{2}, \frac{k\cdot v-1}{2}, k, \frac{k\cdot k-1}{2}\right)$$

which is a new series generalising the series given by B. J. Gassner (*Equal difference* BIB *designs*, Proc. Amer. Math. Soc. **16** (1965), 378–380).

1. **Introduction.** Let $G$ be an abelian group of order $v$. A set of $k$ distinct elements of $G$ is called a difference set if the $k\cdot k-1$ differences of the elements of $D$ contain every nonzero element of $G$, $\lambda$ times. These definitions are generalised and in place of a single set $D$, one can take $t$ initial blocks of $k$ elements each where the $t\cdot k\cdot k-1$ differences from the $t$ blocks contain every nonzero element of $G$ the same number of times.

Received by the editors December 24, 1965.

Let $GF(s_i)$ denote a Galois field of order $s_i$, $i=1, 2, \cdots, m$ and $x_i$ be a primitive root in the field. Let $v$ be an odd integer with the following prime power decomposition:

$$v = p_1^{e_1} \cdots p_m^{e_m} = s_1 \cdot s_2 \cdots s_m$$

where $s_i = p_i^{e_i}$; $i=1, 2, \cdots, m$.

Assume that $s_1$ is the least prime power factor of $v$ and let $\beta$ denote a general element of the cartesian product $G$ of the $m$ fields

$$G = GF(s_1) * \cdots * GF(s_m).$$

Let us label some of the $\beta$'s by $\alpha$'s as follows:

$$\alpha_{j+1} = (x_1^j, x_2^j, \cdots, x_m^j), \qquad j = 0, 1, \cdots, s_1 - 2,$$
$$\alpha_0 = (0, 0, \cdots, 0).$$

Let $B$ denote the set of points:

$$B: (\alpha_0, \alpha_1, \cdots, \alpha_{k-1})$$

where $k \leqq s_1$ if $m > 1$ and $k < s_1$ if $m = 1$.

## 2. Some lemmas on $B$.

2.1. Lemma. *Let $\alpha_c$ and $\alpha_d$ be any two distinct elements of $B$. Then $\alpha_c$ and $\alpha_c - \alpha_d$ have multiplicative inverses defined.*

Proof follows easily since no coordinate of either $\alpha_c$ or $\alpha_c - \alpha_d$ is zero and hence a multiplicative inverse exists for each coordinate in their respective fields.

2.2. Lemma. *If $\alpha_c \in B$, $c \neq 0$, 1 and $m > 1$, then $\alpha_c^{-1} \notin B$.*

For, otherwise if a $d$ exists such that

$$\alpha_c \alpha_d = \alpha_{c+d} = \alpha_0$$

then

$$(2.2) \quad c + d = 0 \ \mathrm{mod}\{(s_1 - 1), (s_2 - 1), (s_3 - 1) \cdots, (s_m - 1)\} \cdots$$

since $c, d \leqq k_1 - 1 \leqq s_1 - 1$ and $c \neq d$, and the fields are all odd; $c+d$ can take at most the value $2(s_1-1)-1$. Thus if $c+d = s_1 - 1$ then $c+d \neq 0 \bmod (s_i - 1)$ $i=2, \cdots, m$. Hence in no case can (2.2) be satisfied.

2.3. Proposition. *A set $T$ of $(v-1)/2$ points $\beta_j$, $j=1, 2, \cdots, (v-1)/2$ can be selected from the product space $G$ such that if $\beta_j \in T$, $-\beta_j \notin T$.*

3. THEOREM. *From the initial blocks*

$$B_1\colon (0,\ \beta_1\alpha_1,\ \beta_1\alpha_2,\ \cdots,\ \beta_1\alpha_{k-1})$$

$$B_2\colon (0,\ \beta_2\alpha_1,\ \beta_2\alpha_2,\ \cdots,\ \beta_2\alpha_{k-1})$$

$$\vdots$$

$$B_{(v-1)/2}\colon (0,\ \beta_{(v-1)/2}\alpha_1,\ \beta_{(v-1)/2}\alpha_2,\ \cdots,\ \beta_{(v-1)/2}\alpha_{k-1})$$

*on adding* $\beta_j$, $i=0, 1, 2, \cdots, v-1$ *to each element of each block a balanced incomplete block design with the following parameters results in*:

$$v = v,$$

$$b = v\cdot\frac{v-1}{2}$$

$$r = k\cdot\frac{v-1}{2},$$

$$k = k,$$

$$\lambda = k\cdot\frac{k-1}{2}\,.$$

PROOF. $\{\beta_j\}$, $j=0, 1, 2, \cdots, v-1$ are the $v$ elements of $G$: the product space of the $m$ fields taken as treatments. First we establish that each initial block contains distinct elements then every two initial blocks are distinct if $m>1$ and finally that every treatment appears $r$ times and every pair of treatments appears $\lambda$ times.

If $B_j$ had contained two identical points then we should have:

$$\beta_j\alpha_c = \beta_j\alpha_d, \qquad c \neq d \in \{1, \cdots, k-1\},$$

i.e.

$$\beta_j(\alpha_c - \alpha_d) = 0.$$

Multiplying by $(\alpha_c-\alpha_d)^{-1}$ we should have $\beta_j=0$ which is not true. Hence $B_j$ contains distinct elements.

Consider

$$B_j = (0,\ \beta_j\alpha_1,\ \cdots,\ \beta_j\alpha_{k-1})$$

and

$$B_i = (0,\ \beta_i\alpha_1,\ \cdots,\ \beta_i\alpha_{k-1})$$

If $\beta_j\alpha_1 \notin B_i$ then $B_j$ and $B_i$ are distinct blocks. If $\beta_j\alpha_1 \in B_i$ then we show that $\beta_i\alpha_1 \notin B_j$. Deny this and let $\beta_j\alpha_1 = \beta_i\alpha_c$, $1 \leq c \leq k-1$. Then $\beta_i\alpha_1 = \beta_j\alpha_c^{-1}\alpha_1^2 = \beta_j\alpha_c^{-1}$ ($\alpha_1^2 = \alpha_1$ for $\alpha_1 = \alpha_1^2 = (1, 1, \cdots, 1)$). But $\alpha_c^{-1} \notin B$ and hence $\beta_i\alpha_1 \notin B_j$.

Thus the $b$ blocks are distinct, we will show that each treatment appears $r$ times. Let $\beta$ be any point in $G$. Consider the $v$ blocks generated by $B_j$ for some fixed $j$, $j = 1, 2, \cdots, (v-1)/2$. Let

$$\beta - \beta_j\alpha_c = \beta_c \quad \text{for } c = 0, 1, \cdots, k-1$$

then $\beta$ appears in the $k$ blocks $\{B_j + \beta_c\}$, $c = 0, 1, \cdots, k-1$. Hence as $j = 1, 2, \cdots, (v-1)/2$ we observe that every treatment appears in $r = k-(v-1)/2$ blocks.

Now we proceed to determine $\lambda$. Consider any two points $\beta_1\alpha_c \neq \alpha_d\beta_1 \in B_1$. Let $\beta_1(\alpha_c - \alpha_d) = \beta_1\alpha$. Then in the initial blocks $B_j$ the corresponding difference is $\beta_j\alpha$. The differences $\{\beta_j\alpha, -\beta_j\alpha\}$ $j = 1, 2, \cdots, (v-1)/2$ are all distinct. For if $\beta_j\alpha = \beta_{j'}\alpha$ then multiplying by $\alpha^{-1}$ we should have $j = j'$ or if $\beta_j\alpha = -\beta_{j'}\alpha$ then again $(\beta_j + \beta_{j'}) = 0$ which is not true by choice. Thus in the initial block the differences between $c$ and $d$ elements produce all the nonzero elements of $G$ exactly once. Given two points $\beta_i$ and $\beta_j$ let $\beta_i - \beta_j = \beta$ say. In the initial block choose any two distinct points $\alpha_c$ and $\alpha_d$ then there exists a unique $\beta_l$, $l = 1, 2, \cdots, (v-1)/2$ such that

$$\beta_l\alpha_c - \beta_l\alpha_d = \beta,$$

since $\pm(\alpha_c - \alpha_d)\beta_t$, $t = 1, 2, \cdots, (v-1)/2$ gives all nonzero $\beta$'s exactly once. In the set of $v$ blocks generated by $B_l$ then, $\beta_i$ and $\beta_j$ occur together in exactly one block. Since we have $C_{k,2}$ pairs of $(\alpha_c, \alpha_d)$ every pair of treatments appears in $k \cdot k - 1/2$ blocks.

An example of $v = 9$, $b = 36$, $r = 16$, $k = 4$, $\lambda = 6$, constructed using the 4 initial blocks

$$(0, 1, -1, x);\ (0, x, -x, -1);\ (0, x+1, -x-1, x-1);$$
$$(0, x-1, -x+1, -x-1);$$

in the field GF($3^2$) with the irreducible function $x^2 + 1 = 0$:

| | | | | | |
|---|---|---|---|---|---|
| (1 2 3 4) | (2 3 1 6) | (3 1 2 8) | (1 4 5 3) | (2 6 9 1) | (3 8 7 2) |
| (1 6 7 8) | (2 8 4 5) | (3 4 9 6) | (1 8 9 7) | (2 4 5 7) | (3 6 5 9) |
| (4 6 8 5) | (5 9 7 1) | (6 8 4 9) | (4 5 1 8) | (5 1 4 7) | (6 9 2 4) |
| (4 9 3 7) | (5 2 8 3) | (6 7 1 5) | (4 7 2 3) | (5 3 6 8) | (6 5 1 3) |
| (7 5 9 3) | (8 4 6 7) | (9 7 5 2) | (7 3 8 9) | (8 7 3 6) | (9 2 6 5) |
| (7 1 6 2) | (8 5 2 9) | (9 3 4 1) | (7 2 4 6) | (8 9 1 2) | (9 1 8 4) |

## References

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3. B. J. Gassner, *Equal difference* BIB *designs*, Proc. Amer. Math. Soc. 16 (1965), 378–380.

Indian Statistical Institute, Calcutta