THEORETICAL PAPERS

AN ALGORITHM TO FIND THE SMALLEST COMMITTEE CONTAINING A GIVEN SET

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Abstract

Billera has proposed an algorithm to find the smallest committee containing a given set T. In this paper a modification, which not only makes the algorithm simpler but also computationally much more efficient, is proposed.

1. Introduction

Conceptually *clutters*, *simple games* and *coherent systems* are equivalent (see [2], [4] and [7]). Let R be a clutter on N and $b[R]$ its blocking clutter. The elements of N are called *players* and *components*, respectively, in the terminologies of simple games and coherent systems. The elements of $R(b[R])$ are referred to as *minimal winning* (*blocking*) *coalitions* in theory of simple games whereas they are called *minimal path* (*cut*) sets in the context of coherent systems.

Birnbaum and Esary introduced the concept of *modular sets* in the context of coherent systems [3]. The same concept has been called committees in the context of simple games by Shapley [10]. Recently there has been some renewed interest in the concept of modular sets or committees ([8] and [9]). Let T be a nonempty subset of N. The problem of finding the *smallest* committee containing T was considered by Billera and he also proposed an algorithm for this purpose [2]. For quite sometime this was the only algorithm available for a general clutter. Recently Mohring and Radermacher have proposed another algorithm [6]. In this paper a modification for Billera's algorithm, which not only makes it simpler but also computationally more efficient, is proposed. In the spirit of Billera [2], the context of clutters is used for this purpose.

2. Clutters and Committees

Most of the introductory material in this section is from [2] and reproduced here for the sake of completeness. By a *family* E on a finite set N, we mean a family of subsets of N. The *support* of E which we denote by $s(E)$ is by definition

$$s(E) = \bigcup_{E \in E} E.$$

If $A \subseteq N$, then $E(A)$ (or some times written as $(E)(A)$) is a family on N defined by

$$E(A) = \{E : E \in E \text{ and } E \cap A \neq \phi\}$$

If $A \subseteq B \subseteq N$, then $E(A) = (E(B))(A)$ and $s(E(A)) \subseteq s(E(B)) \subseteq s(E)$,

A family R on a finite set N is called a *clutter* if $R \neq \phi$, $R \neq \{\phi\}$ and no element of R is properly contained in another element of R. The *blocking clutter* of R (or simply the *blocker* of R) is a clutter, $b[R]$ on N defined by

$b[R] = \{S : S \subseteq N, S \cap P \neq \phi$ for all $P \in R$ and no proper subset of S has this property$\}$.

It is well known (see for example [1, Exercises 3 and 4 on p. 19.]) that $s(b[R]) = s(R)$ and $b[b[R]] = R$. When $s(R) \subset N$, the elements of N-$s(R)$ are called *dummies*. We require the trivial results of Lemma 1 in the sequel.

LEMMA 1. *Let R be a clutter on some finite set* N *and* $\phi \neq J \subseteq s(R)$. *Then* R (J) *is also a clutter on* N *with* $J \subseteq s(R(J)) \subseteq s(R)$. *In particular if* $J = s(R)$ *then* $R(J) = R$.

Throughout the remaining part of this paper, R denotes a clutter on some finite set N. A nonempty subset J of $s(R)$ is called a *committee* of R if and only if

$$R(J) = \{S : S = (P \cap J) \cup (Q - J);\ P, Q \in R(J)\}.$$

For a number of other equivalent characterizations of committees we refer to [4], [8] and [9]. We note that $s(R)$ itself and all singleton subsets of $s(R)$ are committees of R. It is easy to verify that if $s(R(J)) = J$, then J is a committee of R. It is well known (see [4, p. 596]) that the nonempty intersection of two committees of R is again a committee of R.

Let T be any nonempty subset of $s(R)$. We shall denote by C_T the *smallest* committee containing T (that is, the intersection of all committees containing T). An algorithm that is available to find C_T is due to Billera

[2]. We now give a brief description of Billera's algorithm. For each $i \in s(R)$ let $R^i = \{S : S = P - \{i\};\ P \in R(\{i\})\}$. We note that R^i is either a clutter or is equal to $\{\phi\}$. To simplify the description of the algorithm, we define $b[\{\phi\}] = \phi$.

Billera Algorithm

Input : A clutter R on some finite set N and a nonempty subset T of $s(R)$.

Output : C_T the smallest committee containing T.

Step 0 : Put $r = 1$ and $B = T$ and go to Step 1.

Step 1 : Put $T_r = B$ and find $b[R^i]$ for each $i \in T_r$. Let $D = \bigcup_{i \in T_r} b[R^i]$

and $E = \bigcap_{i \in T_r} b[R^i]$. Go to Step 2.

Step 2 : If $(s(D-E)) \cap (s(R) - T_r) = \phi$, then $C_T = T_r$, otherwise put

$r = r+1$ and $B = (s(D-E)) \cup T_r$ and go to Step 1.

For a proof that Billera algorithm terminates in finitely many steps with C_T, we refer to [2]. We shall now propose a modification to this algorithm which not only makes it simpler but also computationally much more efficient. For this purpose, we require the results of Theorems 1 and 2.

THEOREM 1. *A nonempty subset* J *or* $s(R)$ *is a committee of* R *if and only if* $s((b[R(J)])(J)) = J$.

Proof. We note from Lemma 1 that $J \subseteq s(R(J)) = s(b[R(J)])$. Let J be a committee of R. If $s(R(J)) = J$, it follows Lemma 1 that $(b[R(J)](J) = b[R(J)]$, that is $s((b[R(J)])(J)) = J$. Now consider the case when $J \subset s(R(J))$. Let D and E be the families defined by

$$D = \{S : S = P \cap J\ ;\ P \in R(J)\}.$$

$$E = \{S : S = P - J\ ;\ P \in R(J)\}.$$

The hypothesis that J is a committee implies

$$R(J) = \{S : S = P \cup Q;\ P \in D,\ Q \in E\}.$$

Therefore we note that D is a clutter and $s(D) = J$. Further the additional hypothesis that $J \subset s(R(J))$ implies that E is also a clutter with

$s(E) = s(R(J))—J$. It is easy to verify that $b[R(J)] = b[D] \cup b[E]$. We therefore have $(b[R(J)])\,(J) = b[D]$. The required assertion is then immediate.

Suppose now $s(b[R(J)])\,(J)) = J$. If $s(b[R(J)]) = J$, then $s(R(J)) = J$ and J is trivially a committee. Consider now the case when $J \subset s(b[R(J)])$. The hypothesis that $s((b[R(J)])\,(J)) = J$ implies $P \subseteq J$ or $P \subseteq s(R(J))—J$ for all $P \in b[R(J)]$. Further the assumption that $J \subset s(b[R(J)])$ implies the existence of at least one $P \in b[R(J)]$ such that $P \subseteq s(R(J))—J$. Consider the families F and G defined by

$$F = \{S : S \in b[R(J)];\ S \subseteq J\},$$

$$G = \{S : S \in b[R(J)];\ S \subseteq s(R(J))—J\}.$$

We note that F and G are clutters and $b[R(J)] = F \cup G$. It follows that $s(F) = J$ and $s(G) = s(R(J))—J$. It is easy to verify that

$$\begin{aligned} R(J) = b[b[R(J)]) &= \{S : S = P \cup Q;\ P \in b[F],\ Q \in b[G]\} \\ &= \{S : S = (P \cap J) \cup (Q—J),\ P,\ Q \in R(J)\}. \end{aligned}$$

Therefore J is a committee of R.

THEOREM 2. *Let* J *and* K *be nonempty subsets of* $s(R)$ *such that* $J \supseteq K$. *If* J *is a committee of* R, *then* $s((b[R(K)])\,(K)) \subseteq J$.

Proof. Let J be a committee of R and further let D and E be the clutters defined in the proof of Theorem 1. Recall that $s(D) = J$, $s(E) = s(R(J))—J$ and

$$R(J) = \{S : S = P \cup Q;\ P \in D,\ Q \in E\},$$

It follows that

$$\begin{aligned} R(K) = (R(J))\,(K) &= \{S : S \cap K \neq \phi,\ S \in R(J)\} \\ &= \{S : S = P \cup Q;\ P \cap K \neq \phi,\ P \in D,\ Q \in E\} \\ &= \{S : S = P \cup Q,\ P \in D\,(K),\ Q \in E\}. \end{aligned}$$

From Lemma 1, we note that $R(K)$ and $D(K)$ are also clutters. It is easy to see that $b[R(K)] = b[D(K)] \cup b[E]$. Therefore it follows that $(b[R(K)])\,(K) = b[D(K)]$. Since $s(b[D(K)]) = s(D(K)) \subseteq s(D) = J$, the required result follows.

We are now in a position to state the *modified version* of Billera's algorithm.

Modified Billera Algorithm

Input: A clutter R on some finite set N and a nonempty subset T of $s(R)$.

Output: C_T the smallest committee containing T.

Step 0: Put $r = 1$ and $B = T$ and go to Step 1.

Step 1: Put $T_r = B$ and find $b[R(T_r)]$ and go to step 2.

Step 2: If $s((b[R(T_r)])(T_r)) = T_r$, then $C_T = T_r$. Otherwise put $r = r+1$ and $B = s((b[R(T_r)])(T_r))$ and go to Step 1.

The convergence properties of the above algorithm are established in Theorem 3.

THEOREM 3. *Let* T *be any nonempty subset* $s(R)$. *The modified Billera algorithm terminates in finitely many step with* C_T.

Proof. Suppose the algorithm terminates at $r = k$. We note that $T = T_1 \subset T_2 \dots \subset T_k$. It follows that $k \leq | s(R) - T |$. By Theorem 1 we conclude that T_k is a committee of R. Let J be any committee containing T. By assumption we note that $J \supseteq T_1$. Using Theorem 2 we have $J \supseteq T_2$. Repeated application of Theorem 2 implies $J \supseteq T_k$. Since T_k is a committee, it follows that $C_T = T_k$.

The main computional effort required in the implementation of Billera's algorithm as well as its modified version consists in finding the blocker of a clutter. This problem is equivalent to that of finding the *prime covers* in a *set covering problem* and this is known to be NP *hard*. In the Billera Algorithm, at Step 1 we have to find $b[R^i]$ for each $i \in T_r$, where as in the Modified Billera Algorithm we need find only $b[R(T_r)]$. Consequently the modified algorithm is computationally superior.

It should be noted that polynomial algorithms are available for special types of clutters like the circuits of a matroid [5]. It has also been shown in [6] that the general algorithm of Mohring and Radermacher is polynomial only for certain types of clutters. However these results seem not extendable to the general case.

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