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ON BOUNDED LENGTH SEQUENTIAL CONFIDENCE INTERVALS BASED ON ONE-SAMPLE RANK ORDER STATISTICS[1]

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The problem of obtaining sequential confidence intervals for the median of an unknown symmetric distribution based on a general class of one-sample rank-order statistics is considered. It is shown that the usual one-sample rank-order statistic possesses the martingale or sub-martingale property according as the parent distribution is symmetric about the origin or not. Certain asymptotic almost sure convergence results (with specified order of convergence) for a class of rank-order processes and the empirical distribution are derived, and these are then utilized for the study of the properties of the proposed procedures.

1. Introduction. The problem of finding a bounded length confidence band for the mean of an unknown distribution is studied by Anscombe [1] and by Chow and Robbins [6]. Farrell [8] considers the problem for the p-quantile of a distribution. Sproule [19] has extended the results of Chow and Robbins to the class of Hoeffding's [11] U-statistics, and in the particular cases of the signed-rank and sign statistics (which are both U-statistics), Geertsema [9] considers the problem based on rank estimates of the median.

In the present paper, we consider the problem of providing a sequential confidence interval for the median of a symmetric (but otherwise unknown) distribution based on a general class of one-sample rank order statistics. Of particular interest is the procedure based on the so called one-sample normal scores statistics. This procedure is shown to be asymptotically (i.e., as the prescribed bound on the width of the confidence interval is made to converge to zero) at least as efficient as the Chow–Robbins procedure for a broad class of parent distributions.

In the course of this study, several asymptotic results, having importance of their own, are derived. First, the elegant result of Bahadur [2] on the behavior of the empirical distribution in the neighborhood of a quantile is extended to the entire real line (see Theorem 4.2). Second, the weak convergence results of Sen ([18] Theorem 1) and Jurečková [13] are replaced by almost sure (a.s.) convergence results, under slightly more restrictive conditions on the scores and the underlying distribution (see Theorem 4.3). It is also shown (see Theorem 4.5) that the usual one-sample rank order statistic possesses the martingale or the sub-martingale property according as the parent distribution is symmetric about the origin or not.

Section 2 of the paper deals with the preliminary notions and basic assumptions. The next section describes the proposed procedure and states the main theorem of the paper. Section 4 is concerned with the results stated in the preceding paragraph.

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The proof of the main theorem is supplied in Section 5, and the asymptotic efficiency results are studied in Section 6.

2. Preliminary notions. Let $\{X_1, X_2, \cdots\}$ be a sequence of independent and identically distributed random variables (i.i.d. rv) having an absolutely continuous distribution function (df) $F_\theta(x)$ with location parameter θ (unknown). It is desired to determine (sequentially) a confidence interval $I_N = \{\theta : \hat{\theta}_{L,N} \leqq \theta \leqq \hat{\theta}_{U,N}\}$ of width $\leqq 2d$, where for each positive integer n, $\hat{\theta}_{L,n}$ and $\hat{\theta}_{U,n}$ are two statistics (not depending on d) based on the first n observations, such that $\lim_{n\to\infty} P\{\theta \in I_n\} = 1-\alpha$ (the desired confidence coefficient), while N is the stopping variable defined to be the first integer $n \geqq n_0$ (some positive integer) such that $\hat{\theta}_{U,N} - \hat{\theta}_{L,N} \leqq 2d$.

Our procedure for determining N and $(\hat{\theta}_{L,N}, \hat{\theta}_{U,N})$ rests on the following class of one-sample rank order statistics. Let $c(u) = 0$ or 1 according as $u < 0$ or not. Let then

$$(2.1) \qquad R_{n\alpha} = \textstyle\sum_{\beta=1}^{n} c(|X_\alpha| - |X_\beta|), \quad \alpha = 1, \cdots, n; \qquad \mathbf{X}_n = (X_1, \cdots, X_n).$$

Define

$$(2.2) \qquad T_n = T_n(\mathbf{X}_n) = n^{-1} \textstyle\sum_{\alpha=1}^{n} c(X_\alpha) J_n((n+1)^{-1} R_{n\alpha}),$$

where $\{J_n(u): 0 < u < 1\}$ is generated by a score-function $\{J(u): 0 \leqq u < 1\}$ in either of the following two ways:

(a) $J_n(u) = J(i/(n+1)), (i-1)/n < u \leqq i/n$, for $i = 1, \cdots, n$;

(b) $J_n(u) = EJ(U_{ni}), (i-1)/n < u \leqq i/n, 1 \leqq i \leqq n$, where $U_{n1} < \cdots < U_{nn}$ are the n ordered random variables from the rectangular (0, 1) df. Also, we assume that $J(u) = \Psi^{-1}((1+u)/2), 0 \leqq u < 1$, where $\Psi(x)$ is an absolutely continuous df defined on $(-\infty, \infty)$ satisfying the conditions

$$(2.3) \qquad \text{(a)} \quad \Psi(-x) + \Psi(x) = 1, \text{ i.e., } \Psi'(x) = \psi(x) = \psi(-x), \quad \text{for all real } x,$$

$$(2.4) \qquad \text{(b)} \quad \psi(x)/[1-\Psi(x)] \quad \text{is non-decreasing for all} \quad x \geqq x_0,$$

where $x_0 (\geqq 0)$ is some real number. Thus, the tail of the df $\Psi(x)$ has an increasing failure rate. Note that by definition, $J(0) = 0$ and $J(u)$ is $\uparrow$ in $u: 0 < u < 1$. Also, $\psi(x)/[1-\Psi(x)] = [2J'(2\Psi(x)-1)\{1-\Psi(x)\}]^{-1} = [J'(u)(1-u)]^{-1}, \quad u = 2\Psi(x)-1$. Hence, by (2.4),

$$(2.5) \qquad J'(u) \leqq K(1-u)^{-1}, \qquad 0 < u < 1, \qquad \text{where} \quad 0 < K < \infty,$$

and by integration,

$$(2.6) \qquad J(u) \leqq K[-\log(1-u)], \qquad 0 < u < 1.$$

Finally, by (2.6), there exists a t_0 (where $0 < t_0 < 1/K$), such that

$$(2.7) \quad M(t) = \textstyle\int_0^1 \exp[tJ(u)]\, du \leqq \int_0^1 (1-u)^{-Kt}\, du < \infty, \qquad \text{for all} \quad t \leqq t_0.$$

Note that (2.4) (and hence, (2.5)–(2.7)) hold for the normal, the logistic, double exponential and many other df's. The statistic T_n when Ψ is the *standard normal* df is termed the *normal scores statistic*, and when Ψ is *uniform over* $(-1, 1)$, it is termed the *signed-rank statistic*.

We denote by $\mathscr{F}_0$ the class of all absolutely continuous df's F symmetric about 0 for which both the density function f and its first derivative f' exist and are bounded for almost all x (a.a. x). Also, let

(2.8) $$\mathscr{F}_0(J) = \{F : F \in \mathscr{F}_0, \text{ and } \lim_{x\to\infty} f(x)J'(F(x)-F(-x)) \text{ is finite}\}.$$

Throughout the paper it will be assumed that $F_\theta(x) = F(x-\theta)$, where $F \in \mathscr{F}_0(J)$. In connection with the finiteness condition in (2.8), we refer to [15] for details.

Introduce the following notations:

(2.9) $$\bar{J}_n = n^{-1}\textstyle\sum_{i=1}^n J_n(i/(n+1)), \qquad A_n^2 = n^{-1}\sum_{i=1}^n J_n^2(i/(n+1));$$

(2.10) $$\mu = \textstyle\int_0^1 J(u)\,du \quad \text{and} \quad A^2 = \int_0^1 J^2(u)\,du.$$

Note that if $\theta = 0$, T_n has a distribution independent of F, and symmetric about $\frac{1}{2}\bar{J}_n$. Hence, there exist two known constants $T_{n,\alpha}^{(1)}$ and $T_{n,\alpha}^{(2)} = \bar{J}_n - T_{n,\alpha}^{(1)}$ and a known α_n (close to the specified α), such that

(2.11) $$P_{\theta=0}\{T_{n,\alpha}^{(2)} \leqq T_n = T_n(\mathbf{X}_n) \leqq T_{n,\alpha}^{(1)}\} = 1-\alpha_n \to 1-\alpha \quad \text{as} \quad n\to\infty.$$

For large n, it is known (cf. Hájek and Šidák [19], page 166) that

(2.12) $$\lim_{n\to\infty} n^{\frac{1}{2}}(T_{n,\alpha}^{(2)} - \tfrac{1}{2}\bar{J}_n) = -\tfrac{1}{2}A\tau_{\alpha/2}, \qquad \lim_{n\to\infty} n^{\frac{1}{2}}(T_{n,\alpha}^{(1)} - \tfrac{1}{2}\bar{J}_n) = \tfrac{1}{2}A\tau_{\alpha/2},$$

where

(2.13) $$\Phi(\tau_{\alpha/2}) = 1-\alpha/2 \quad \text{and} \quad \Phi(x) = (2\pi)^{-\frac{1}{2}}\textstyle\int_{-\infty}^x \exp(-\frac{1}{2}t^2)\,dt.$$

It is also known (cf. [5] Theorem 2 under the existence and a growth condition on $J''(u)$, and [16] Corollary 5.1 without any assumption on $J''(u)$) that under the assumptions made earlier

(2.14) $$n^{-1}\textstyle\sum_{i=1}^n |J_n(i/(n+1)) - J(i/(n+1))| = o(n^{-\frac{1}{2}}),$$

which implies that the use of either of the scores will lead to the same asymptotic results.

3. The procedure for obtaining I_N. Since $J(u)$ is $\uparrow$ in $u: 0 < u < 1$, $T_n(\mathbf{X}_n - a\mathbf{1}_n)$, (where $\mathbf{1}_n = (1, \cdots, 1)$) is $\downarrow$ in a: $-\infty < a < \infty$. Define as in [18]

(3.1) $$\hat{\theta}_{L,n} = \sup\{a : T_n(\mathbf{X}_n - a\mathbf{1}_n) > T_{n,\alpha}^{(1)}\},$$

(3.2) $$\hat{\theta}_{U,n} = \inf\{a : T_n(\mathbf{X}_n - a\mathbf{1}_n) < T_{n,\alpha}^{(2)}\}.$$

It follows from (2.11), (3.1) and (3.2) that $P_\theta\{\hat{\theta}_{L,n} \leqq \theta \leqq \hat{\theta}_{U,n}\} = 1-\alpha_n \to 1-\alpha$, as $n\to\infty$. Our proposed procedure is then framed as in the beginning of Section 2. This is quite similar to the Chow–Robbins procedure, but instead of their confidence interval we use (3.1) and (3.2) which are known to be more robust for outliers or gross errors. The following theorem relates to the asymptotic properties of the proposed procedure.

THEOREM 3.1 *Under the assumptions of Section* 2,

(3.3) $N(=N(d))$ *is a non-increasing function of* d, $N(d)$ *is finite* a.s. *and* $EN(d)<\infty$ *for all* $d>0$; $\lim_{d\to 0} N(d)=\infty$ a.s., *and* $\lim_{d\to 0} EN(d)=\infty$;

$$\lim_{d\to 0} N(d)/\nu(d)=1 \quad \text{a.s.,} \tag{3.4}$$

$$\lim_{d\to 0} P_\theta\{\theta\in I_{N(d)}\}=1-\alpha; \tag{3.5}$$

$$\lim_{d\to 0} E[N(d)]/\nu(d)=1, \tag{3.6}$$

where $\nu(d)=A^2\tau_{\alpha/2}^2/(4d^2B^2(F))$ *and*

$$B(F)=\int_0^\infty (d/dx)J[2F(x)-1]\,dF(x)=\int_0^\infty J'(H(x))f(x)\,dH(x); \tag{3.7}$$
$$H(x)=F(x)-F(-x).$$

The proof of the theorem is postponed to Section 5. In proving this theorem, several other results are needed which are proved in Section 4.

4. Some convergence results on the empirical process and on $\{T_n(\mathbf{X}_n-a\mathbf{1}_n)$: $-\infty<a<\infty\}$. We are primarily interested in the asymptotic (a.s.) linearity of $n^{\frac12}[T_n(\mathbf{X}_n)-T_n(\mathbf{X}_n-a\mathbf{1}_n)]$ in a, when a is "close" to zero. If we define the empirical df's

$$F_n(x)=n^{-1}\textstyle\sum_{i=1}^n c(x-X_i), \quad -\infty<x<\infty, \tag{4.1}$$
$$H_n(x)=F_n(x)-F_n(-x-), \quad x\geqq 0;$$

$$F_{n,a}(x)=F_n(x+a) \quad\text{and}\quad H_{n,a}(x)=F_{n,a}(x)-F_{n,a}(-x-), \tag{4.2}$$

we have from (2.2), (2.14), (4.1) and (4.2) that

$$\begin{aligned} &n^{\frac12}[T_n(\mathbf{X}_n)-T_n(\mathbf{X}_n-a\mathbf{1}_n)]\\ &=n^{\frac12}\{\textstyle\int_0^\infty J_n(nH_n(x)/(n+1))\,dF_n(x)-\int_0^\infty J_n(nH_{n,a}(x)/(n+1))\,dF_{n,a}(x)\}\\ &=n^{\frac12}\textstyle\int_0^\infty [J(nH_n(x)/(n+1))-J(nH_{n,a}(x)/(n+1))]\,dF_{n,a}(x)\\ &\qquad+n^{\frac12}\textstyle\int_0^\infty J(nH_n(x)/(n+1))\,d[F_n(x)-F_{n,a}(x)]+o(1)\\ &=I_{n1}(a)+I_{n2}(a)+o(1). \end{aligned} \tag{4.3}$$

Thus, the left-hand side (l.h.s.) of (4.3) behaves as a functional of the empirical processes $\{n^{\frac12}[F_n(x)-F(x)]\}$, $\{n^{\frac12}[F_n(x+a)-F_n(x)]\}$ and $\{n^{\frac12}[J(nH_n(x)/(n+1))-J(nH_{n,a}(x)/(n+1))]\}$. We shall study some of the properties of these processes first.

We start with the empirical df F_n. Since F is absolutely continuous, $Y_i=F(X_i)$, $i=1,2,\cdots$, are i.i.d. rv with the rectangular (0, 1) df. Let $G_n(t)=n^{-1}\sum_{i=1}^n c(t-Y_i)$, $0\leqq t\leqq 1$, $n\geqq 1$. Define the empirical process

$$V_n(t)=n^{\frac12}[G_n(t)-t], \qquad 0\leqq t\leqq 1. \tag{4.4}$$

Later on, we shall find it convenient to extend the domain of V_n to $(-\infty, \infty)$ by letting $V_n(t) = 0$ for all t outside the unit interval (0, 1). It is well known (cf. Billingsley [3] pages 103–108) and Hájek and Šidák ([10] page 276) that V_n weakly converges (in the Prohorov sense) to a Brownian Bridge. Using a result of Strassen [20], Brillinger [4] has strengthened this to a.s. convergence for a suitable construction. However, as we shall see later on, for proving (3.6), we need not only the a.s. convergence but also a smooth rate of convergence of the tail probability so as to make its contribution in (3.6) asymptotically negligible. For this reason, we state the following lemma whose proof is contained in [7] page 646.

LEMMA 4.1. *For every finite $s > 0$, there exist positive constants $(c_s^{(1)}, c_s^{(2)})$ and a sample size n_s, such that for all $n \geqq n_s$,*

$$P\{\sup_{0<t<1} |V_n(t)| \geqq c_s^{(1)}(\log n)^{\frac{1}{2}}\} \leqq c_s^{(2)} n^{-s}. \tag{4.5}$$

Let now $g_k(n) = n^{-\frac{1}{2}}(\log n)^k$, $k \geqq 1$. Define

$$K_n(t) = \sup_a \{|V_n(t+a) - V_n(t)| : |a| \leqq g_k(n)\}, \qquad n \geqq 1, \tag{4.6}$$

$$K_n^* = \sup_{0<t<1} K_n(t). \tag{4.7}$$

It is known (cf. [3] Section 13 and [10] page 276) that K_n^* weakly converges to 0. We are interested in the a.s. convergence of K_n^* along with certain asymptotic order for the tail probabilities, so that we can make analogous probability statements for (4.3). This is accomplished here by extending a result of Bahadur ([2] page 578) to the entire real line.

THEOREM 4.2. *For every finite $s > 0$, there exist a $c_s > 0$ and a sample size n_s, such that for all $n \geqq n_s$, $k \geqq 1$,*

$$P\{K_n^* \geqq c_s n^{-\frac{1}{4}}(\log n)^k\} \leqq 4n^{-s}. \tag{4.8}$$

Hence, $K_n^* = O(n^{-\frac{1}{4}}(\log n)^k)$, with probability one, as $n \to \infty$.

PROOF. Let $\xi_{j,n} = j/[n^{\frac{1}{2}}]$, $j = 0, 1, \cdots, [n^{\frac{1}{2}}]$, where $[s]$ stands for the largest integer $\leqq s$. Since for the Y_i, the density function is equal to 1 $(0 < t < 1)$, by the same technique as in Bahadur ([2], namely, his (8), (9), (11), (12) and (13)), it follows that for large n,

$$P\{K_n(\xi_{j,n}) > cn^{-\frac{1}{4}}(\log n)^k\} \leqq 4[\exp(-v_n)], \qquad c > 0, \text{ for } j = 1, \cdots, [n^{\frac{1}{2}}], \tag{4.9}$$

where

$$\begin{aligned} v_n &= -\tfrac{1}{4}\log n + [c^2 n^{\frac{1}{2}}(\log n)^{2k}]/[2\{n^{\frac{1}{2}}(\log n)^k + cn^{\frac{1}{4}}(\log n)^k\}] \\ &= -\tfrac{1}{4}\log n + \tfrac{1}{2}c^2(\log n)^k[1 + cn^{-\frac{1}{4}}]^{-1}. \end{aligned} \tag{4.10}$$

Since $k \geqq 1$, by proper choice of $c = c_s/3$, say, v_n can be made greater than $(s+\frac{1}{2})\cdot \log n$ for any given $s(>0)$, when n is taken large. As the right-hand side (r.h.s.) of (4.10) does not depend on j, by the Bonferroni inequality, we obtain for large n

$$\begin{aligned} &P\{K_n(\xi_{j,n}) > (c_s/3)n^{-\frac{1}{4}}(\log n)^k, \text{ for at least one } j = 1, \cdots, [n^{\frac{1}{2}}]\} \\ &\qquad \leqq 4n^{\frac{1}{2}}[\exp(-v_n)] \leqq 4n^{-s}. \end{aligned} \tag{4.11}$$

Now, note that if t and $t+a$ both belong to the same interval $[\xi_{j,n}, \xi_{j+1,n}]$, as $\xi_{j+1,n}-\xi_{j,n} = n^{-\frac{1}{2}} < g_k(n)$, $|V_n(t+a)-V_n(t)| \leqq 2K_n(\xi_{j,n})$. On the other hand, if t and $t+a$ belong to two different intervals, say, $t \in [\xi_{j,n}, \xi_{j+1,n}]$ and $t+a \in [\xi_{r,n}, \xi_{r+1,n}]$, $|a| \leqq g_k(n) \Rightarrow \xi_{r,n}-\xi_{j+1,n} \leqq g_k(n)$ whenever $r \geqq j+1$ (otherwise, interchange j and r). Hence,

$$
\begin{aligned}
(4.12)\qquad |V_n(t+a)-V_n(t)| &\leqq |V_n(t+a)-V_n(\xi_{r,n})| + |V_n(\xi_{r,n})-V_n(\xi_{j+1,n})| \\
&\quad + |V_n(t)-V_n(\xi_{j+1,n})| \\
&\leqq K_n(\xi_{r,n}) + 2K_n(\xi_{j+1,n}) \leqq 3[\max_{1\leqq j \leqq [n^{1/2}]} K_n(\xi_{j,n})],
\end{aligned}
$$

for all $0 \leqq j \leqq r \leqq [n^{\frac{1}{2}}]-1$. Thus,

$$
(4.13)\qquad K_n^* \leqq 3[\max_{1\leqq j \leqq [n^{1/2}]} K_n(\xi_{j,n})],
$$

and, as a result, (4.8) follows from (4.11) and (4.13). □

REMARK. Whenever F is such that $\sup_x f(x) = f_0 < \infty$, $|a| \leqq g_k(n) \Rightarrow |F(x+a)-F(x)| \leqq f_0 g_k(n)$, so that by (4.8), for $n \geqq n_s$ (where $g_k^*(n) = f_0^{-1} g_k(n)$)

$$
\begin{aligned}
(4.14)\qquad & P\{\sup_x \sup_{|a|\leqq g_k^*(n)} n^{\frac{1}{2}} |F_n(x+a)-F_n(x)-F(x+a)+F(x)| > c_s n^{-\frac{1}{4}}(\log n)^k\} \\
&\leqq P\{\sup_{0<t<1} \sup_{|a|\leqq g_k(n)} |V_n(t+a)-V_n(t)| > c_s n^{-\frac{1}{4}}(\log n)^k\} \leqq 4n^{-s}.
\end{aligned}
$$

It may also be remarked that in Theorem 4.2, one can also work with any $g_k(n)$ where $n^{\frac{1}{2}} g_k(n)$ increases (as $n \to \infty$) at a rate not slower than that of $\log n$ but not faster than that of n^λ for $\lambda < \frac{1}{2}$.

Define $H(x) = F(x)-F(-x)$, $x \geqq 0$, and

$$
(4.15)\qquad F_a(x) = F(x+a), \qquad H_a(x) = F_a(x)-F_a(-x) = P[|X-a| \leqq x].
$$

Note that H_a depends on n whenever a depends on n. Since $F \in \mathscr{F}_0$, for large n,

$$
(4.16)\qquad \sup_{|a|\leqq g_k(n)} \sup_x |H_a(x)-H(x)| = O(n^{-1}(\log n)^{2k}).
$$

Hence, upon noting that $|H_{n,a}(x)-H_n(x)-H_a(x)+H(x)| \leqq |F_{n,a}(x)-F_n(x)-F(x+a)+F(x)| + |F_{n,a}(-x-)-F_n(-x-)-F(-x+a)+F(-x)|$, we obtain from (4.16) and Theorem 4.2 that as $n \to \infty$,

$$
\begin{aligned}
(4.17)\qquad & P\{\sup_{|a|\leqq g_k(n)} \sup_x n^{\frac{1}{2}} |H_{n,a}(x)-H_n(x)| \\
&\geqq 2c_s n^{-\frac{1}{4}}(\log n)^k[1+O(n^{-\frac{1}{4}}(\log n)^k)]\} \leqq 4n^{-s}.
\end{aligned}
$$

We shall make use of the above probability inequalities for the study of the asymptotic (a.s.) convergence of our rank order process

$$
(4.18)\qquad W_n(a) = n^{\frac{1}{2}}[T_n(\mathbf{X}_n)-T_n(\mathbf{X}_n-a\mathbf{1}_n)-aB(F)], \qquad -\infty < a < \infty,
$$

where we assume that $\theta = 0$, as otherwise one should replace $\mathbf{X}_n$ by $\mathbf{X}_n-\theta\mathbf{1}_n$.

(4.18) is, in turn, useful for proving the a.s. convergence of $n^{\frac{1}{2}}[\hat{\theta}_{U,n}-\hat{\theta}_{L,n}]$ to $A\tau_{\alpha/2}/B(F)$; the corresponding weak convergence result was proved in [18]. Also, for $n^{-\frac{1}{2}}a$ belonging to a bounded interval, the weak convergence of $W_n(a)$ to 0 follows along the lines of Jurečková [13].

THEOREM 4.3. *Under* (2.3), (2.4) *and* (2.8), *for every* $s(>0)$, *there exist positive constants* $(k_s^{(1)}, k_s^{(2)})$ *and a sample size* n_s, *such that for all* $n \geqq n_s$, $k \geqq 1$,

$$P\{\sup_{|a|\leqq g_k(n)}|W_n(a)| > k_s^{(1)}n^{-\frac{1}{4}}(\log n)^{2k}\} \leqq k_s^{(2)}n^{-s}. \tag{4.19}$$

Hence, $\sup_{|a|\leqq g_k(n)}|W_n(a)|$ *converges to* 0, *with probability* 1, *as* $n\to\infty$.

PROOF. By (4.3) and (4.18), $W_n(a)+n^{\frac{1}{2}}aB(F) = I_{n1}(a)+I_{n2}(a)+o(1)$. We shall consider only the case of $0 < a \leqq g_k(n)$ as the case of negative a follows similarly. First, consider $\sup_{a\leqq g_k(n)}|I_{n1}(a)|$. Define x_n^* by $1-H(x_n^*) = 4c_s n^{-\frac{1}{2}}(\log n)^k$, where c_s and k are defined in (4.8). Then

$$\begin{aligned} I_{n1}(a) &= (\textstyle\int_0^{x_n^*}+\int_{x_n^*}^{\infty})\{J(nH_n(x)/(n+1))-J(nH_{n,a}(x)/(n+1))\}\,dF_n(x+a) \\ &= I_{n11}(a)+I_{n12}(a). \end{aligned} \tag{4.20}$$

Now, proceeding as in (8) and (9) of Bahadur [2] and using our (4.16), we can bound $\sup_{0\leqq a\leqq g_k(n)}|H_{n,a}(x_n^*)-H_a(x_n^*)|$ by $\{\max_{0\leqq j\leqq[n^{1/4}]}|H_{n,a_j}(x_n^*)-H_{a_j}(x_n^*)|\}+O(n^{-\frac{3}{4}}(\log n)^{2k})$, where $a_j = jn^{-\frac{1}{2}}(\log n)/[n^{\frac{1}{4}}]$, $j = 0, 1, \cdots, [n^{\frac{1}{4}}]$. Also, for each j, nH_{n,a_j} involves a sum of i.d.d. bounded rv's on which Theorem 1 of Hoeffding [12] yields a bound essentially the same as in (4.9)-(4.10), and hence, by the Bonferroni inequality along with (4.16), we obtain the following:

LEMMA 4.4. *For every* $s(>0)$, *there exist positive constants* $(c_s^{(1)}, c_s^{(2)})$ *and an* n_s, *such that for all* $n \geqq n_s$,

$$P\{\sup_{0\leqq a\leqq g_k(n)}|H_{n,a}(x_n^*)-H(x_n^*)| > c_s^{(1)}n^{-\frac{3}{4}}(\log n)^k\} \leqq c_s^{(2)}n^{-s}. \tag{4.21}$$

By (4.17), (4.21) and some simple manipulations, we have for $n \geqq n_s$,

$$\begin{aligned} P\{\sup_{0\leqq a\leqq g_k(n)}[1-H_{n,a}(x)] \leqq 3[1-H_n(x)], \text{ for all } 0 \leqq x \leqq x_n^*\} \\ \geqq 1-(4+c_s^{(2)})n^{-s}. \end{aligned} \tag{4.22}$$

A direct use of (4.22) along with (2.5) leads to the following: for all $n \geqq n_s$

$$\begin{aligned} P\left\{ \begin{array}{c} \left|\left(J'\frac{n}{n+1}\{\theta H_n(x)+(1-\theta)H_{n,a}(x)\}\right)\right| \leqq 3K\left[1-\frac{n}{n+1}H_{n,a}(x)\right]^{-1}, \\ \text{for all } 0 \leqq x \leqq x_n^*, 0 \leqq a \leqq g_k(n) \text{ and } 0<\theta<1 \end{array} \right\} \\ \geqq 1-(4+c_s^{(2)})n^{-s}. \end{aligned} \tag{4.23}$$

Thus, by (4.17) and (4.23), for $n \geq n_s$, with a probability $\geq 1-(4+c_s^{(2)})n^{-s}$,

$$\sup_{0<a\leq m(n)}|I_{n11}(a)|$$
$$\leq \sup_{0<a\leq m(n)}\left|\int_0^{x_n^*} n^{\frac{1}{2}}[H_n(x)-H_{n,a}(x)]\cdot J'(n\{\theta_n H_n(x)+(1-\theta_n)H_{n,a}(x)\}/(n+1))\,dF_{n,a}(x)\right| \quad (\text{where } 0<\theta_n<1)$$

(4.24)
$$\leq \{\sup_{0<a\leq m(n)}\sup_x n^{\frac{1}{2}}|H_{n,a}(x)-H_n(x)|\}\cdot\left\{\sup_{0<a\leq m(n)}\left[3K\int_0^{x_n^*}\left\{1-\frac{n}{n+1}H_{n,a}(x)\right\}^{-1}dF_{n,a}(x)\right]\right\}$$
$$\leq \{2n^{-\frac{1}{2}}c_s(\log n)^k[1+O(n^{-\frac{1}{2}}(\log n)^k]\}\cdot\left\{3K\int_0^{\infty}\left[1-\frac{n}{n+1}H_{n,a}(x)\right]^{-1}dH_{n,a}(x)\right\}$$
$$=[O(n^{-\frac{1}{2}}(\log n)^k)][O(\log n)]=O(n^{-\frac{1}{2}}(\log n)^{k+1})],$$

as $\int_0^{\infty}[1-nH_{n,a}/(n+1)]\,dH_{n,a}=n^{-1}\sum_{i=1}^n(1-i/(n+1))^{-1}\leq((n+1)/n)(1+\log n)$. Also,

(4.25)
$$\sup_{0<a\leq m(n)}|I_{n12}(a)|\leq \sup_{0<a\leq m(n)}[n^{\frac{1}{2}}\textstyle\int_{x_n^*}^{\infty}J(nH_n(x)/(n+1))\,dF_n(x+a)] + \sup_{0<a\leq m(n)}[n^{\frac{1}{2}}\textstyle\int_{x_n^*}^{\infty}J(nH_{n,a}(x)/(n+1))\,dF_n(x+a)]$$

As $J(u)$ is ↑ in u and $dF_n \leq dH_n$, by (2.6), for every $a>0$,

$$n^{\frac{1}{2}}\textstyle\int_{x_n^*}^{\infty}J(nH_n(x)/(n+1))\,dF_n(x+a)$$
$$\leq n^{\frac{1}{2}}\textstyle\int_{x_n^*+a}^{\infty}J(nH_n(x)/(n+1))\,dF_n(x)$$

(4.26)
$$\leq n^{\frac{1}{2}}\textstyle\int_{x_n^*}^{\infty}J(nH_n(x)/(n+1))\,dF_n(x)$$
$$\leq Kn^{\frac{1}{2}}\textstyle\int_{x_n^*}^{\infty}\{-\log(1-nH_n(x)/(n+1))\}\,dH_n(x)$$
$$\leq n^{\frac{1}{2}}K[1-H_n(x_n^*)](\log(n+1))$$
$$\leq 8Kc_s n^{-\frac{1}{2}}(\log n)^{k+1}, \qquad \text{with probability } \geq 1-c_s^{(2)}n^{-s},$$
for all $n \geq n_s$,

by Lemma 4.4 and the definition of $H(x_n^*)$. A similar proof holds for the second term on the r.h.s. of (4.25). Hence, for $n \geq n_s$,

(4.27)
$$P\{\sup_{0<a\leq m(n)}|I_{n12}(a)|>K_s^{(1)}n^{-\frac{1}{2}}(\log n)^{k+1}\}\leq K_s^{(2)}n^{-s},$$

where $K_s^{(1)}, K_s^{(2)}$ are positive constants, depending on $s(>0)$. Now, writing $X_{(n)}=\max_{1\leq i\leq n}X_i$, we may rewrite $I_{n2}(a)$ in (4.3) as

(4.28)
$$I_{n2}(a)=0, \qquad \text{if } X_{(n)}\leq 0,$$
$$=I_{n21}(a)+I_{n22}(a), \qquad \text{if } X_{(n)}>0,$$

where

$$I_{n21}(a) = n^{\frac{1}{2}}\int_0^{X_{(n)}} J[H(x)]\, d[F_n(x) - F_n(x+a)] = n^{\frac{1}{2}}\int_0^{X_{(n)}}[F_n(x+a) - F_n(x)]J'(H(x))\, dH(x), \tag{4.29}$$

$$I_{n22}(a) = n^{\frac{1}{2}}\int_0^{X_{(n)}}[J(nH_n(x)/(n+1)) - J(H(x))]\, d[F_n(x) - F_n(x+a)]. \tag{4.30}$$

By (3.6) and (4.29), whenever $X_{(n)} > 0$,

$$\begin{aligned} &\sup_{0\leq a\leq g_k(n)}|I_{n21}(a) - n^{\frac{1}{2}}aB(F)| \\ &\leq \sup_{0\leq a\leq g_k(n)}\{|an^{\frac{1}{2}}\int_{X_{(n)}}^{\infty} f(x)J'(H(x))\, dH(x)| \\ &\quad + n^{\frac{1}{2}}|\int_0^{X_{(n)}}[F_n(x+a) - F_n(x) - af(x)]J'(H(x))\, dH(x)| \\ &\leq \{(\log n)^k\int_{X_{(n)}}^{\infty} f(x)J'(H(x))\, dH(x)\} \\ &\quad + \sup_{0\leq a\leq g_k(n)}\sup_x\{n^{\frac{1}{2}}|F_n(x+a) - F_n(x) - af(x)|\}J(H(X_{(n)})), \end{aligned} \tag{4.31}$$

while if $X_{(n)} \leq 0$,

$$\sup_{0\leq a\leq g_k(n)}|I_{n21}(a) - n^{\frac{1}{2}}aB(F)| = (\log n)^k B(F). \tag{4.32}$$

Since, $P[F(X_{(n)}) \leq 1 - n^{-\frac{1}{2}}] = (1 - n^{-\frac{1}{2}})^n \leq cn^{-s}$, $c > 0$, for all $n \geq n_s$, and (2.8) holds, the first term on the r.h.s. of (4.31) is bounded by $2[\sup_x f(x)J'(H(x))][1 - F(X_{(n)})]$. $(\log n)^k \leq n^{-\frac{1}{2}}(\log n)^k[\sup_x f(x)J'(H(x))]$, with probability $\geq 1 - cn^{-s}$, when $n \geq n_s$. Again, $P[1 - F(X_{(n)}) < n^{-(s+1)}] = 1 - (1 - n^{-s-1})^n < n^{-s}$, and hence, by using (2.6) and Theorem 4.2, for all $F \in \mathscr{F}_0(J)$, the second term on the r.h.s. of (4.31) is bounded above by $c_s n^{-\frac{1}{2}}(\log n)^k[1 + O(n^{-\frac{1}{2}}(\log n)^k)]K\{-\log[1 - (1 - n^{-s-1})]\} = c_s n^{-\frac{1}{2}}(\log n)^{k+1}.\ K(s+1)[1 + O(n^{-\frac{1}{2}}(\log n)^k]$, with probability $\geq 1 - 5n^{-s}$. Further, (4.32) occurs with probability 2^{-n}. Hence, upon noting that 2^{-n} can be made smaller than n^{-s}, $s > 0$, for large n, we obtain that for all $n \geq n_s$,

$$\sup_{0\leq a\leq g_k(n)}|I_{n21}(a) - n^{\frac{1}{2}}aB(F)| \leq K_s^{(3)}n^{-\frac{1}{2}}(\log n)^{k+1}, \tag{4.33}$$

with probability $\geq 1 - K_s^{(4)}n^{-s}$, where $(K_s^{(3)}, K_s^{(4)})$ are positive constants, depending on $s(> 0)$. Finally, we rewrite $I_{n22}(a)$ in (4.30) as

$$n^{-\frac{1}{2}}\sum_{i=1}^n\{J(nH_n(|X_i|)/(n+1)) - J(H(|X_i|))\}\{c(X_i) - c(X_i + a)\}. \tag{4.34}$$

Since $c(X_i) - c(X_i + a) = 0$ unless $-a < X_i < 0$, and $f_0 = \sup_x f(x) < \infty$, on using (4.34), Theorem 4.2, Lemma 4.1, and noting that $J'(u)$ is bounded in the neighborhood of 0, we have

$$\begin{aligned} &\sup_{0\leq a\leq g_k(n)}|I_{n22}(a)| \\ &\leq \{|n^{\frac{1}{2}}[F_n(0) - F_n(-g_k(n))]|\sup_{0\leq a\leq g_k(n)}|J(n/(n+1)H_n(x)) - J(H(x))|\} \\ &\leq [f_0(\log n)^k + c_s^{(1)}n^{-\frac{1}{2}}(\log n)^k][\sup_{0\leq a\leq g_k(n)}J'(H(a))][2c_s n^{-\frac{1}{2}}(\log n)^k] \\ &\leq (\text{const.})(n^{-\frac{1}{2}}(\log n)^{2k}), \quad \text{with probability } \geq 1 - (4 + c_s^{(2)})n^{-s}, \\ &\qquad\qquad\qquad \text{for } n \geq n_s, \end{aligned} \tag{4.35}$$

where $c_s^{(1)}$, $c_s^{(2)}$ are defined in (4.5) and c_s in (4.8). The proof of the theorem then follows from (4.20), (4.24), (4.27), (4.28), (4.33) and (4.35). □

We shall now prove that when F is symmetric about 0 and $J_n(u)$ is specified by (b) of Section 2, then $T_n^* = n(T_n - \mu/2)$, $n \geqq 1$, forms a martingale sequence with respect to a non-decreasing sequence of σ-fields $\mathscr{B}_n$ defined as follows.

Let $\mathbf{c}_n = (c(X_1), \cdots, c(X_n))$ and $\mathbf{R}_n = (R_{n1}, \cdots, R_{nn})$, where the R_{ni} are defined in (2.1). $\mathscr{B}_n$ is the σ-field generated by $(\mathbf{c}_n, \mathbf{R}_n)$. Then, obviously $\mathscr{B}_n$ is ↑ in n, and we prove the following theorem to be used in proving the "uniform continuity in probability" (to be explained in Section 5) of the estimates $\hat{\theta}_{L,n}$ and $\hat{\theta}_{U,n}$.

THEOREM 4.5. *If* $J_n(i) = EJ(U_{ni})$, $i = 1, \cdots, n$, *and if* $F(x)+F(-x) = 1$, *for all real* x, $\{T_n^*, \mathscr{B}_n\}$ *forms a martingale sequence.*

PROOF. By definition in (2.2),

$$(4.36)\qquad \begin{aligned} E(T_{n+1} \mid \mathscr{B}_n) &= (n+1)^{-1} \textstyle\sum_{i=1}^{n+1} E\{c(X_i)J_{n+1}((n+2)^{-1}R_{n+1\,i}) \mid \mathscr{B}_n\} \\ &= (n+1)^{-1}[\textstyle\sum_{i=1}^{n} c(X_i)E\{J_{n+1}((n+2)^{-1}R_{n+1\,i}) \mid \mathscr{B}_n\} \\ &\qquad + E\{c(X_{n+1})J_{n+1}((n+2)^{-1}R_{n+1\,n+1}) \mid \mathscr{B}_n\}]. \end{aligned}$$

Under the hypothesis of the theorem (cf. [10] page 40), $c(X_{n+1})$ is independent of $R_{n+1\,n+1}$ and of $\mathscr{B}_n$. Also, given $(\mathbf{c}_n, \mathbf{R}_n)$, $R_{n+1\,n+1}$ can assume all the $n+1$ values $1, \cdots, n+1$ with the common probability $1/(n+1)$. Hence,

$$(4.37)\qquad \begin{aligned} &E\{c(X_{n+1})J_{n+1}((n+2)^{-1}R_{n+1\,n+1}) \mid \mathscr{B}_n\} \\ &\qquad = [2(n+1)]^{-1} \textstyle\sum_{j=1}^{n+1} J_{n+1}(j/(n+2)) = \frac{1}{2}\bar{J}_{n+1}\,, \end{aligned}$$

where $\bar{J}_{n+1}$ is defined in (2.9). It is easy to see that

$$(4.38)\qquad \bar{J}_n = n^{-1} \textstyle\sum_{i=1}^{n} J_n(i/(n+1)) = \int_0^1 J(u)\,du = \mu, \quad \text{for all} \quad n \geqq 1.$$

Also, given $\mathbf{R}_n$, $R_{n+1\,i}$ can either assume the value R_{ni} or $R_{ni}+1$ with respective conditional probability $(n+1-R_{ni})/(n+1)$ and $R_{ni}/(n+1)$, for $i = 1, \cdots, n$. Hence,

$$(4.39)\qquad \begin{aligned} &E\{J_{n+1}((n+2)^{-1}R_{n+1\,i}) \mid \mathscr{B}_n\} \\ &\quad = \{(n+1-R_{ni})/(n+1)\}J_{n+1}(R_{ni}/(n+2)) \\ &\qquad + \{R_{ni}/(n+1)\}J_{n+1}((R_{ni}+1)/(n+2)) \\ &\quad = J_n(R_{ni}/(n+1)), \qquad\qquad 1 \leqq i \leqq n, \end{aligned}$$

after using the fact that by definition of $J_{n+1}(i/(n+2))$, $1 \leqq i \leqq n+1$,

$$(4.40)\qquad \begin{aligned} &\{(n+1-i)/(n+1)\}J_{n+1}(i/(n+2)) + \{i/(n+1)\}J_{n+1}((i+1)/(n+2)) \\ &\quad = J_n(i/(n+1)), \qquad\qquad 1 \leqq i \leqq n. \end{aligned}$$

Hence, from (4.37) through (4.39), we obtain that

$$(4.41)\qquad \begin{aligned} E(T_{n+1} \mid \mathscr{B}_n) &= (n+1)^{-1}\{\textstyle\sum_{i=1}^{n} c(X_i)J_n(R_{ni}/(n+1)) + \frac{1}{2}\mu\} \\ &= (n+1)^{-1}\{nT_n + \frac{1}{2}\mu\}, \qquad\qquad n \geqq 1. \end{aligned}$$

This implies that for $T_n^* = n(T_n - \frac{1}{2}\mu)$,

$$E(T_{n+1}^* \mid \mathscr{B}_n) = T_n^* \quad \text{for all} \quad n \geqq 1. \quad \square \tag{4.42}$$

REMARK. The theorem may not hold when J_n is defined by (a) of Section 2. However, if $J(u)$ is convex, then it can be shown by the same technique that $\{nT_n\}$ forms a sub-martingale sequence with respect to $\mathscr{B}_n$ even when J_n is defined by (a) of Section 2. Also, if F is not symmetric about 0, the martingale property does not hold. However, as $J_n > 0$, it follows by the same technique that $\{nT_n\}$ forms a sub-martingale sequence with respect to $\mathscr{B}_n$ when J_n is defined by (b) of Section 2.

5. Proof of Theorem 3.1. We do it in several steps. First, let us prove the following lemmas.

LEMMA 5.1. *For every $s(>0)$, there exist positive constants $(c_s^{(1)}, c_s^{(2)})$ and a sample size n_s, such that for all $n \geqq n_s$,*

$$P\{n^{\frac{1}{2}}(\hat{\theta}_{L,n} - \theta) + \tau_{\alpha/2} A/2B(F) < -c_s^{(1)}(\log n)^2\} \leqq c_s^{(2)} n^{-s}, \tag{5.1}$$

$$P\{n^{\frac{1}{2}}(\hat{\theta}_{U,n} - \theta) - \tau_{\alpha/2} A/2B(F) > c_s^{(1)}(\log n)^2\} \leqq c_s^{(2)} n^{-s}. \tag{5.2}$$

PROOF. We only prove (5.1) as (5.2) follows by analogy. Now,

$$\begin{aligned} &P\{n^{\frac{1}{2}}(\hat{\theta}_{L,n} - \theta) + \tau_{\alpha/2} A/2B(F) < -c_s^{(1)}(\log n)^2\} \\ &\quad = P\{\hat{\theta}_{L,n} < \theta - n^{-\frac{1}{2}}\tau_{\alpha/2} A/2B(F) - n^{-\frac{1}{2}} c_s^{(1)}(\log n)^2\} \\ &\quad = P_{\theta=0}\{T_n(\mathbf{X}_n + \{n^{-\frac{1}{2}}[\tau_{\alpha/2} A/2B(F) + c_s^{(1)}(\log n)^2]\}\mathbf{1}_n) \leqq T_{n,\alpha}^{(1)}\}, \end{aligned} \tag{5.3}$$

where $\quad T_{n,\alpha}^{(1)} = \frac{1}{2}\mu + \frac{1}{2}(An^{-\frac{1}{2}}\tau_{\alpha/2}) + o(n^{-\frac{1}{2}})$.

Let us define $\tilde{T}_n = T_n(\mathbf{X}_n + n^{-\frac{1}{2}}(A\tau_{\alpha/2}/2B(F))\mathbf{1}_n)$ (where $\theta = 0$ without any loss of generality), and let $\hat{T}_n = T_n(\mathbf{X}_n + \{n^{-\frac{1}{2}}(\tau_{\alpha/2}A/2B(F) + c_s^{(1)}(\log n)^2)\}\mathbf{1}_n]$. Then, by Theorem 4.3, with probability $\geqq 1 - k_s^{(2)}n^{-s}$, $|n^{\frac{1}{2}}(\hat{T}_n - \tilde{T}_n) - B(F)c_s^{(1)}(\log n)^2| \leqq k_s^{(1)}n^{-\frac{1}{4}}(\log n)^4$. Hence, it suffices to prove that for large n, $P\{n^{\frac{1}{2}}|\tilde{T}_n - T_{n,\alpha}^{(1)}| \geqq B(F)c_s^*(\log n)^2\} \leqq c_s^{**}n^{-s}$, where c_s^* $(< c_s^{(1)})$ and c_s^{**} are positive constants. We now write $a = n^{-\frac{1}{2}}A\tau_{\alpha/2}/2B(F)$, and define $H_a(x)$, $H_{n,a}(x)$ and $F_{n,a}(x)$ as in Section 4. Then $\tilde{T}_n = \int_0^\infty J(nH_{n,a}(x)/(n+1))\, dF_{n,a}(x)$, and it can be shown by some standard computations, as in Section 4, that $T_{n,\alpha}^{(1)} = \int_0^\infty J(H_a(x))\, dF_a(x) + o(n^{-\frac{1}{2}})$. Hence, it is enough to show that for every $s > 0$, there exists an n_s such that for $n \geqq n_s$,

$$\begin{aligned} &P\{n^{\frac{1}{2}}|\textstyle\int_0^\infty J(nH_{n,a}(x)/(n+1))\, dF_{n,a}(x) - \int_0^\infty J(H_a(x))\, dF_a(x)| > c_s^*(\log n)^2\} \\ &\qquad \leqq c_s^{**}n^{-s}. \end{aligned} \tag{5.4}$$

The l.h.s. of the inequality within the parentheses in (5.4) can be written as

$$\begin{aligned} &n^{\frac{1}{2}}\textstyle\int_0^\infty J(nH_{n,a}(x)/(n+1))\, d[F_{n,a}(x) - F_a(x)] \\ &\qquad + n^{\frac{1}{2}}\textstyle\int_0^\infty \{J(nH_{n,a}(x)/(n+1)) - J(H_a(x))\}\, dF_a(x) = I_1 + I_2 \quad \text{(say)}. \end{aligned} \tag{5.5}$$

Then, by Lemma 4.1 and (2.5), we have

$$n^{-\frac{1}{2}}|I_1| = (n/(n+1))\left|\int_0^\infty [F_{n,a}(x)-F_a(x)]J'(nH_{n,a}(x)/(n+1))\,dH_{n,a}(x)\right|$$
$$\leq (n+1)^{-1}\sum_{i=1}^n |F_n(X_i+a)-F(X_i+a)|\,|J'((n+1)^{-1}R_{ni}(a))|$$
$$\leq [\sup_{1\leq i\leq n}|F_n(X_i+a)-F(X_i+a)|][(n+1)^{-1}\sum_{i=1}^n |J'(i/(n+1))|] \tag{5.6}$$
$$\leq [c_s^{(1)}n^{-\frac{1}{2}}(\log n)^{\frac{1}{2}}][K(n+1)^{-1}\sum_{i=1}^n (n+1)/(n+1-i)]$$
$$\leq Kc_s^{(1)}n^{-\frac{1}{2}}(\log n)^{\frac{1}{2}}[1+(\log n)], \quad \text{with probability } \geq 1-c_s^{(2)}n^{-s},$$

where $R_{ni}(a) = \sum_{j=1}^n c(|X_i-a|-|X_j-a|)$. Again, essentially repeating the steps as in (4.20) through (4.27), it follows that for every $s(>0)$, there exist two positive constants $d_s^{(1)}$, $d_s^{(2)}$ and an n_s, such that for all $n \geq n_s$,

$$P\{|I_2| \geq d_s^{(1)}n^{-\frac{1}{4}}(\log n)^{\frac{1}{2}}\} \leq d_s^{(2)}n^{-s}. \tag{5.7}$$

This completes the proof of the lemma.

LEMMA 5.2. *For every $s(>0)$, there exist positive constants (c_{s1}^*, c_{s2}^*) and an n_s, such that for all $n \geq n_s$,*

$$P\{|B(F)n^{\frac{1}{2}}(\hat\theta_{U,n}-\hat\theta_{L,n})/A\tau_{\alpha/2}-1| > c_{s1}^*\, n^{-\frac{1}{4}}(\log n)^3\} \leq c_{s2}^*\, n^{-s}. \tag{5.8}$$

PROOF. By virtue of Lemma 5.1, we have with probability $\geq 1-2c_s^{(2)}n^{-s}$,

$$\theta - n^{-\frac{1}{2}}\{(A\tau_{\alpha/2}/2B(F))+c_s^{(1)}(\log n)^2\} \leq \hat\theta_{L,n} \leq \hat\theta_{U,n} \tag{5.9}$$
$$\leq \theta + n^{-\frac{1}{2}}\{(A\tau_{\alpha/2}/2B(F))+c_s^{(1)}(\log n)^2\}.$$

Hence, the proof directly follows from Theorem 4.3. □

LEMMA 5.3. *$\{\hat\theta_{L,n}\}$ and $\{\hat\theta_{U,n}\}$ are uniformly continuous in probability with respect to $\{n^{-\frac{1}{2}}\}$, i.e., for every positive ε and η, there exists a $\delta(>0)$, such that as $n\to\infty$,*

$$P\{\sup_{|n'-n|<\delta n}|n^{\frac{1}{2}}(\hat\theta_{L,n'}-\hat\theta_{L,n})| > \eta\} < \varepsilon; \tag{5.10}$$

$$P\{\sup_{|n'-n|<\delta n}|n^{\frac{1}{2}}(\hat\theta_{U,n'}-\hat\theta_{U,n})| > \eta\} < \varepsilon. \tag{5.11}$$

PROOF. Write $n^{\frac{1}{2}}(\hat\theta_{L,n'}-\hat\theta_{L,n}) = (n/n')^{\frac{1}{2}}(n')^{\frac{1}{2}}(\hat\theta_{L,n'}-\theta)-n^{\frac{1}{2}}(\hat\theta_{L,n}-\theta)$. By Lemma 5.1, Theorem 4.3 and (2.12), we have with probability $\geq 1-c_{s2}^*(n^{-s}+n'^{-s})$ (for large n),

$$n^{\frac{1}{2}}(\hat\theta_{L,n'}-\hat\theta_{L,n}) = \tfrac{1}{2}(A\tau_{\alpha/2}/B(F))[(n/n')^{\frac{1}{2}}-1]$$
$$+[n^{\frac{1}{2}}\{T_n(\mathbf{X}_n-\theta\mathbf{1}_n)-T_{n'}(\mathbf{X}_{n'}-\theta\mathbf{1}_{n'})\}]/B(F)+o(1).$$

Thus, it suffices to prove that under the hypothesis F is symmetric about 0,

$$\lim_{n\to\infty} P\{\sup_{|n'-n|<\delta n} n^{\frac{1}{2}}\,|T_n(\mathbf{X}_n)-T_{n'}(\mathbf{X}_{n'})| > \eta \mid \theta = 0\} < \varepsilon. \tag{5.12}$$

Define T_n^* as in Theorem 4.5. Then, routine computation yields that $E(T_n^*) = 0$, $E(T_n^{*2}) = nA_n^2/4$, where A_n^2 is defined in (2.9). As $\{T_n^*, \mathscr{B}_n\}$ forms a martingale

Hence (cf. Theorem 4.5), by using the Kolmogorov inequality for martingales (see Loève [14] page 386), we get

$$(5.13)\qquad \begin{aligned} &P\{\sup_{1\le k\le[\delta n]}|T^*_{n+k}-T_n^*|>t\}\\ &\le t^{-2}[E(T^{*2}_{n+[\delta n]}-T_n^{*2})]\\ &=\tfrac{1}{4}t^{-2}[(n+[\delta n])A^2_{n+[\delta n]}-nA_n^2].\end{aligned}$$

Put $t=\eta_1 n^{\frac{1}{2}}$ and note that $A_n^2=A^2+o(1)$. Then,

$$(5.14)\quad P\{\sup_{1\le k\le[\delta n]}|((n+k)/n)T_{n+k}-T_n+k\mu/2n|>\eta_1 n^{-\frac{1}{2}}\}\le\eta_1^{-2}[\delta A^2+o(1)].$$

Since $k/n<\delta$ and $0\le T_{n+k}\le\mu$, for all $k\ge 1$, we get

$$(5.15)\qquad P\{\sup_{n\le n'\le n+[\delta n]}n^{\frac{1}{2}}|T_n(\mathbf{X}_n)-T_{n'}(\mathbf{X}_{n'})|>\eta\mid\theta=0\}<\varepsilon/2.$$

Proceeding similarly when $n-[\delta n]\le n'\le n$, we get (5.12). The proof of (5.11) is analogous.

We may observe that by virtue of (2.14), (5.10) and (5.11) remain true for both the cases where the scores $J_n(i/(n+1))$, $i=1,\cdots,n$, are defined by (a) or (b) of Section 2.

LEMMA 5.4. $\lim_{n\to\infty}P\{n^{\frac{1}{2}}(\hat\theta_{L,n}-\theta)2B(F)/A+\tau_{\alpha/2}\le x\}=\Phi(x)$, *defined by* (2.13).

PROOF. See Sen [18].

Define $n_1(d)=[\nu(d)(1-\varepsilon)]$ and $n_2(d)=[\nu(d)(1+\varepsilon)+1]$. Then, we have the following:

LEMMA 5.5. $\lim_{d\to 0}\sum_{n=n_2(d)}P\{N(d)>n\}<\infty$.

PROOF. $\sum_{n=n_2(d)}^{\infty}P\{N(d)>n\}=\sum_{n=n_2(d)}^{\infty}P\{r^{\frac{1}{2}}(\hat\theta_{U,r}-\hat\theta_{L,r})>2dr^{\frac{1}{2}},$ for all $r=1,\cdots,n\}\le\sum_{n=n_2(d)}^{\infty}P\{n^{\frac{1}{2}}(\hat\theta_{U,n}-\hat\theta_{L,n})>2n^{\frac{1}{2}}d\}$. Since for $n\ge n_2(d)$, $2dn^{\frac{1}{2}}\ge 2d[n_2(d)]^{\frac{1}{2}}\ge[A\tau_{\alpha/2}/B(F)](1+\varepsilon')$, where $\varepsilon/3<\varepsilon'<\varepsilon/2$, and as by Lemma 5.2, $|n^{\frac{1}{2}}(\hat\theta_{U,n}-\hat\theta_{L,n})-A\tau_{\alpha/2}/B(F)|\ge c^*_{s1}n^{-\frac{1}{4}}(\log n)^3$ with probability $\le c^*_{s2}n^{-s}$ (as $n\to\infty$), by letting $d\to 0$ (i.e., $n_2(d)\to\infty$) and $s>1$, the lemma follows directly.

Proof of the main theorem. It follows from Lemma 5.2 and the definition of $N(d)$ that $N(d)$ is finite a.s. for all $d>0$ and is a non-increasing function of d. Lemma 5.5 and some simple manipulations prove that $EN(d)<\infty$, for all $d>0$. The remaining statements of (3.3) follow from the definition of $N(d)$ and the Monotone Convergence Theorem. We obtain (3.4) from Lemma 5.2 and the definition of $\nu(d)$. Again, Lemma 5.3 and Lemma 5.4 show that the two basic conditions of "uniform" continuity in probability with respect to $n^{-\frac{1}{2}}$' and asymptotic normality, as prerequisite for Theorem 1 of Anscome [1] are also satisfied. Hence, (3.5) follows from Theorem 1 of [1].

To prove (3.6), consider

$$(5.16)\qquad \nu^{-1}(d)E[N(d)]=\nu^{-1}(d)[\textstyle\sum_1+\sum_2+\sum_3 nP\{N(d)=n\}],$$

where $\sum_1$ extends over all $n\le n_1(d)$, $\sum_2$ over all $n: n_1(d)<n<n_2(d)$ and $\sum_3$ over all $n\ge n_2(d)$. Since $\lim_{d\to 0}\nu(d)=\infty$ and $\lim_{d\to 0}N(d)/\nu(d)=1$ a.s., for every $\varepsilon(>0)$, there exists a value of d, say d_0, such that for all $0 < d \leqq d_0$, $P\{n_1(d) < N(d) < n_2(d)\} \geqq P\{|N(d)/v(d)-1| < \varepsilon\} \geqq 1-\eta$, η being arbitrarily small. Hence, for $d \leqq d_0$,

$$v^{-1}(d)\sum_1 nP\{N(d) = n\} \leqq (1-\varepsilon)P\{N(d) \leqq n_1(d)\} \leqq \eta(1-\varepsilon). \tag{5.17}$$

Also, for all $n_1(d) < n < n_2(d)$, $|n/v(d)-1| < \varepsilon$, and hence,

$$|v^{-1}(d)\sum_2 nP\{N(d) = n\}-1| \leqq \varepsilon\sum_2 P\{N(d) = n\}+\eta \leqq \varepsilon+\eta. \tag{5.18}$$

Finally,

$$\begin{aligned} &v^{-1}(d)\sum_3 nP\{N(d) = n\} \\ &\quad = v^{-1}(d)\sum_3 P\{N(d) > n\}+v^{-1}(d)n_2(d)P\{N(d) \geqq n_2(d)\} \\ &\quad \leqq v^{-1}(d)\sum_3 P\{N(d) > n\}+(1+\varepsilon+v^{-1}(d))P\{N(d) \geqq n_2(d)\}. \end{aligned} \tag{5.19}$$

Since $v(d) \to \infty$ as $d \to 0$, using Lemma 5.5, both the terms on the r.h.s. of (5.19) converge to 0 as $d \to 0$. Thus, (3.6) follows from (5.17), (5.18) and (5.19). □

REMARK. Since $P_\theta\{\theta \in I_n\} = 1-\alpha_n(\to 1-\alpha$ as $n \to \infty)$, for every non-random n, one can naturally ask whether (3.5) holds for a general "stopping variable" $N(t)$ (a positive integer valued rv, not necessarily defined in the same way as in our Section 2). The answer is in the affirmative when there exists a sequence $n(t)$ of positive integers such that

$$\lim_{t\to\infty} n(t) = \infty \quad \text{but} \quad \lim_{t\to\infty} N(t)/n(t) = 1, \qquad \text{in probability.} \tag{5.20}$$

In such a case, one can use Theorem 2 of Pyke and Shorack [17] to represent $T_{N(t)}$ as a functional of an empirical process (related to the tied down Wiener process), where our Theorem 4.3 or Brillinger's [4] theorem gives us the access to prove the asymptotic (a.s.) linearity of this empirical process, and hence, Theorem 1 of Sen [18] readily extends to random sample sizes, under (5.20). However, the proof of (3.6) for such a general stopping variable demands the details of the order of the tail probabilities as have been provided throughout Section 4 and Section 5.

6. Asymptotic relative efficiency (ARE). Suppose we have two bounded length confidence interval procedures A and B for estimating the median of a symmetric distribution by means of an interval of length $\leqq 2d$, $d > 0$; if $N_A(d)$ and $N_B(d)$ denote the stopping variables and $P_A(d)$ and $P_B(d)$ the coverage probabilities of the procedures A and B respectively, then we define the ARE of the procedure A with respect to the procedure B by

$$e_{A,B} = \lim_{d\to 0}\{E[N_B(d)]/E[N_A(d)]\}, \tag{6.1}$$

provided $\lim_{d\to 0} P_A(d) = \lim_{d\to 0} P_B(d)$ and either of the limits exists.

Let S and C stand for the procedures suggested by us and that by Chow and Robbins [6]. Using (3.5) and (3.6) of Theorem 3.1 and the corresponding results of [6], we get that under the assumption $\sigma^2 = \text{Var}(X_1) < \infty$,

$$e_{S,C} = 4\sigma^2 B^2(F)/A^2. \tag{6.2}$$

The above is the Pitman-efficiency of a general rank order test with respect to Student's t-test. In the particular case of the normal scores statistic where $J(u) = \Phi^{-1}((1+u)/2)$, ($\Phi$ being the standard normal df), (2.4) holds and it follows from the results of [5] that $e_{S,C} \geqq 1$, for all df F with a density function f and a finite second moment, the equality sign being attained iff F is itself normal $(0, \sigma^2)$. Hence, in that case, the ARE of our proposed procedure with respect to the Chow–Robbins procedure is $\geqq 1$, the equality being attained iff the parent df is normal.

In the particular case, when $J(u) = u$ (i.e., Wilcoxon scores), $e_{S,C} = 12\sigma^2[\int_{-\infty}^{\infty} f^2(x)\,dx]^2$ and this includes Geertseema's [9] $e(W, M)$ expression as a particular case of the sequential procedure suggested by us.

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