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SCHEDULING ON A SINGLE PROCESSOR SUBJECT TO INTERRUPTIONS IN PROCESSING

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ABSTRACT

A problem of scheduling stochastic tasks on a processor which is subject to breakdowns is considered. It is assumed that the processing times of tasks and the time for which the processor operates continuously follow exponential distributions. We show that when the objective is to minimise the expected weighted sum of task completion times, the optimal schedule is not effected by breakdowns.

1. Introduction

It is generally assumed in the literature on scheduling theory that the processors operate continuously without interruptions. But, in practice, processing is interrupted occasionally due to breakdowns or maintenance operations especially when the processing times of tasks are quite large. Interruptions in processsing have been considered by Smith [5] and Glazebrook [2] for scheduling the stochastic tasks.

In his paper, Glazebrook [2] has considered a general problem of scheduling stochastic tasks on a single processor subject to breakdowns and task precedence constraints. The processing strategy can effect the breakdown process in this problem. He has dealt with the problem in discrete as well as continuous case, considering general discounted cost structure which includes the cost of repairing the processor.

We consider a continuous case in which task pre-emptions are allowed only at repair completion times and the objective is to minimise the expected weighted sum of task completion times. We assume that the processing times of tasks and continuous operating time of the processor follow exponential distributions. Though the above objective has not been explicitly considered by Glazebrook, it is possible to get the optimal policy from his results by appropriately modifying his objective function and letting the discount factor tend to zero. However, we give a simple and direct approach for this problem (see [3]).

2. Description of the Problem

We define the problem through the following assumptions :

(1) There are n tasks $1, 2, \ldots, n$ to be processed on a single processor.

(2) Only one task can be processed at a time.

(3) (*a*) The processor is subject to breakdowns and repairs.

(*b*) Initially at time $t = 0$ the processor is in operating condition.

(*c*) The length Y_i of i-th period over which the processor operates continuously follows exponential distribution with parameter. μ for $i \geq 1$ (i.e., all Y_i's are identically distributed exponential random variables).

(*d*) The length Z_i of i-th repair period, $i \geq 1$, is a r.v. with expectation θ ($< \infty$) and all Z_i's are identically distributed.

(4) The amount of processing time X_j that a job j, $1 \leq j \leq n$, requires is an exponential random variable with mean $1/\lambda_j$.

(5) Set-up times are assumed to be zero.

(6) All the random variables described above are mutually independent.

(7) Pre-emptions are allowed only at the time of completion of repair.

(8) Cost c_j is incurred on task j ($1 \leq j \leq n$) per unit time until the task j is completed.

The objective of the problem is to minimise the total expected cost.

3. Formulation as a Semi-Markov Decision Process

We formulate the problem as a semi-Markov decision process as follows: The state of the system at time t is the set U of uncompleted tasks at that time. At time $t = 0$ the state is $N = \{1, 2, \ldots, n\}$. The decision moments are time $t = 0$ and task and repair completion times. The set of actions $J(U)$ associated with state $U = \{i_1, i_2, \ldots, i_r\}$ is $\{0, i_1, i_2 \ldots i_r\}$. An action $k \in J(U)$, $k \neq 0$, is an assignment of task k to the processor. The action $k = 0$ is no assignment of tasks to the processor. Cost $\sum_{i \in U} c_i$ is incurred per unit time during the sojourn time in state U and the discount factor is zero.

For a semi-Markov decision process, it is enough to consider the stationary policies in order to minimise the total expected cost. For reference, see [4], Chapter 7. So, we restrict our attention to the set (C_s) of stationary policies. It is obvious that a stationary policy gives infinite total expected cost if it takes action 0 in a state $U \neq \phi$. Therefore we consider only the stationary policies which do not take the action 0 in any state $U \neq \phi$. Let C_s^* represent the set of such stationary policies. The policies in C_s^* can be divided into two classes: (1) permutation policies and (2) non-permutation policies. A permutation policy f is a stationary policy to which there corresponds a permutation $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ of the elements of N such that when the elements $\pi_1, \pi_2, \ldots, \pi_n$ are renumbered as $1, 2, \ldots, n$, respectively, the policy f takes action $k = \min_{j \epsilon U} j$ in any state $U \neq \phi$. A permutation policy can be represented by the corresponding permutation. The policies other than the permutation policies in C_s^* are called non-permutation policies. Under a policy of C_s^*, the system moves from a set U of cardinalty $r(\geqslant 1)$ to a set U of cardinality $(r\text{-}1)$ uniquely. So, for any non-permutation policy f, we can find a corresponding permutation policy f' such that the processes under f and f' are identical. Therefore, we consider only the permutation policies in order to minimise the total expected cost.

Let $G_\pi(U)$ represent the total expected cost under a permutation (permutation policy) π from a decision moment at which the system enters the state U. We simply represent $G_\pi(N)$ by G_π. In the theorem given below, we obtain an optimal permutation that minimises G_π over the set (S) of all permutations of $1, 2, \ldots, n$.

LEMMA 1. *Let T_j be the time at which task j is completed when the task j is processed from time $t = 0$ until its completion without pre-emption. Then* $E(T_j) = (1 + \mu\theta)/\lambda_j$.

Proof. Since x_j follows exponential distribution and Y_i's and Z_i's are i.i.d. random variables, we can write, using renewal concepts,

$$E(T_j) = E(X_j \mid X_j \leqslant Y_1)\, P(X_j \leqslant Y_1) + [E(Y_1 + Z_1 \mid X_j > Y_1) + E(T_j)]P(X_j > Y_1) = E[\min(X_j, Y_1)] + [E(Z_1) + E(T_j)]\, P(X_j > Y_1),$$

i.e., $P(X_j \leqslant Y_1)\, E(T_j) = E[\min(X_j, Y_1)] + E(Z_1)\, P(X_j > Y_1)$,

$$\lambda_j(\lambda_j + \mu)^{-1} E(T_j) = (\lambda_j + \mu)^{-1} + \theta\mu(\lambda_j + \mu)^{-1},$$

$$E(T_j) = (1 + \mu\theta)/\lambda_j.$$

THEOREM 1. *G_π is minimum when the tasks are processed without pre-emption in the non-increasing order of $c_i\lambda_i$.*

Proof. Consider an arbitrary fixed permutation $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$. Without loss of generality we assume that $\pi = (1,2,\ldots,n)$. We can write $G_\pi = [\sum_{i \in N} c_i]E[T_1] + G_\pi(N-\{1\})$ since the state of the system at completion time T_1 of task 1 is $N-\{1\}$. Applying this argument recursively and using Lemma 1, we get

$$G_\pi = (1+\mu\theta)\left[\frac{c_1}{\lambda_1} + c_2\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right) + \ldots + c_n\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \ldots + \frac{1}{\lambda_n}\right)\right].$$

Therefore, for any permutation $s = (s_1, s_2, \ldots, s_n)$, we have

$$G_\pi = (1+\mu\theta)\left[\sum_{i=1}^{n} c_i\left(\frac{1}{\lambda_{s_1}} + \frac{1}{\lambda_{s_2}} + \ldots + \frac{1}{\lambda_{s_i}}\right)\right].$$

Now, the required result follows from Smith [6].

From the earlier arguments and the above theorem, we conclude that the total expected cost will be minimised if the tasks are processed without pre-emptions in the non-increasing order of $c_j\lambda_j$.

Remark. The optimal policy that minimises the total expected cost is independent of the nature of repair time and the values of parameters θ and μ. If the tasks are processed without pre-emptions in the non-increasing order of $w_j \lambda_j$, where w_j, $1 \leqslant j \leqslant n$, is the weight associated with job j, then the expected weighted sum of completion times would be minimised.

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