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SHORT COMMUNICATIONS

A GENERALIZATION OF THE THREE MODULES THEOREM

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ABSTRACT

In this paper a new characterization for modular sets and a generalization of the well known Three Modules Theorem are presented.

1. Introduction

A *simple game* is conceptually equivalent to a *coherent* system in reliability theory ([4], [6]). The characterization of *modular sets* of coherent systems and simple games has been extensively investigated ([3], [4], [6] and [8]). In a recent paper, based on analogy with *graph theory* and *matroids*, an operation called *contraction* was introduced for simple games as well as coherent systems. Using contraction, a new characterization for modular sets was given [7]. In this paper, we again make use of contraction to obtain an yet another characterization for modular sets.

The *Three Modules Theorem* is well known in literature. Proofs of this theorem have been given in the context of switching functions [1], coherent systems [3] and simple games [8]. In this paper we give a generalized version of this Theorem.

Consistent with our earlier paper [7], we shall use the set up of simple games in this paper also. However, all the results hold true for coherent systems also.

2. Preliminaries and Notations

Let N denote a finite nonempty set. A simple game λ on N is a function $\lambda : 2^N \rightarrow \{0, 1\}$ satisfying : (*i*) $\lambda(\phi) = 0$, (*ii*) $\lambda(N) = 1$ and (*iii*) $\lambda(S) \leq \lambda(T)$, whenever $S \subseteq T$. Elements of N are called *players* and elements of 2^N are called *coalitions*. A coalition S is called *wining* (*losing*) if $\lambda(S) = 1$ (0). A coalition S is called blocking if $\lambda(N\text{-}S) = 0$. A winning (blocking) coalition S is called *minimal* if $T \subset S$ implies $\lambda(T)=0$ ($\lambda(N\text{-}T)=1$). We shall denote by $\alpha(\lambda)$ and $\beta(\lambda)$, respectively (the collections of minimal winning and blocking coalitions of λ. It is well known that a coalition S is winning (blocking) if and only if $S \cap T \neq \phi$ for all $T \in \beta(\lambda)$ ($\alpha(\lambda)$). A player i is called a dummy if $\lambda(S \cup \{i\}) = \lambda(S\text{-}\{i\})$ for all $S \subseteq N$. We shall assume throughout this paper that there are no dummies in the game λ or equivalently for any $i \in N$ there exists a $P \in \alpha(\lambda)$ and a $Q \in \beta(\lambda)$ such that $i \in P$ and $i \in Q$.

The dual λ^* of the simple game λ is again a simple game on N defined by $\lambda^*(S) = 1-\lambda(N\text{-}S)$ for all $S \subseteq N$. It is well known that $\alpha(\lambda^*) = \beta(\lambda)$ and $\beta(\lambda^*) = \alpha(\lambda)$.

Let λ_1 and λ_2 be two simple games on N_1 and N_2, respectively. We define the *composite* simple games $\lambda_1 \times \lambda_2$ and $\lambda_1+\lambda_2$ on $N_1 \cup N_2$ by

$$(\lambda_1 \times \lambda_2)(S) = \lambda_1(S \cap N_1)\,\lambda_2(S \cap N_2),$$
$$(\lambda_1+\lambda_2)(S) = 1-(1-\lambda_1(S \cap N_1))(1-\lambda_2(S \cap N_2)),$$

for all $S \subseteq N_1 \cup N_2$. It is easy to verify that $(\lambda_1 \times \lambda_2)^* = \lambda_1^* + \lambda_2^*$.

Let λ be a simple game on N and A be a nonempty subset of N. We call A a *modular set* of λ if exactly one of the following to assertions holds true for any $S \subseteq A$:

(*i*) $\lambda(S \cup X) = \lambda(X)$ for all $X \subseteq N-A$,

(*ii*) $\lambda(S \cup X) = \lambda(A \cup X)$ for all $X \subseteq N-A$.

The modular sets are also called *committees* [8]. We see that N itself and all singleton subsets of N are modular sets of λ. The *module* corresponding to a modular set A is a game θ on A defined by

$$\theta(S) = \begin{cases} 0 \text{ if } (i) \text{ holds true,} \\ 1 \text{ if } (ii) \text{ holds true,} \end{cases}$$

for all $S \subseteq A$.

3. Some Results on Modular Sets

Throughout this section λ denotes a simple game on N (without dummies) and A is a nonempty subset of N. The *contraction* of λ to A is a simple game on A, denoted by $\lambda.A$ where for any $S \subseteq A$, $(\lambda.A)(S) = 0$ if and only if $\lambda(S \cup X) = 0$ for all $X \subseteq N\text{-}A$ such that $\lambda(X) = 0$ [7].

LEMMA 1. $(\lambda.A)(S) = \max\limits_{X \subseteq N\text{-}A} (\lambda(S \cup X) - \lambda(X))$ *for all* $S \subseteq A$.

Proof. By definition of $\lambda.A$ we note that $(\lambda.A)(S) = 1$ if and only if there exists a $X \subseteq N\text{-}A$ such that $\lambda(S \cup X) = 1$ and $\lambda(X) = 0$. The required assertion is immediate.

LEMMA 2. *For any $S \subseteq A$, $(\lambda.A)(S) = 1$ if and only if there exists a $P \in \alpha(\lambda)$ such that $P \cap A \neq \phi$ and $S \supseteq P \cap A$.*

Proof. Let S be such that $(\lambda.A)(S) = 1$. Then there exists a $T \subseteq N\text{-}A$ such that $\lambda(T) = 0$ and $\lambda(S \cup T) = 1$. This implies the existence of a $P \in \alpha(\lambda)$ such that $S \cup T \supseteq P$. We note that $P \cap A = \phi$ implies $T \supseteq P$ or equivalently $\lambda(T) = 1$; leading to a contradition. Hence we must have $P \cap A \neq \phi$. It follows that $S = (S \cup T) \cap A \supseteq P \cap A$. Conversely let $P \in \alpha(\lambda)$ be such that $P \cap A \neq \phi$. We note that $\lambda(P\text{-}A) = 0$ and $(P \cap A) \cup (P\text{-}A) = P$ or equivalently $\lambda((P \cap A) \cup (P\text{-}A)) = 1$. Therefore we have $(\lambda.A)(P \cap A) = 1$. It follows that $(\lambda.A)(S) = 1$ for all $S \supseteq P \cap A$.

LEMMA 3. $\alpha(\lambda.A) = \{S : S = P \cap A \neq \phi, P \in \alpha(\lambda)$ *and S is minimal with this property*$\}$.

Proof. The required assertion is a trivial consequence of Lemma 2. (see also [7, Lemma 1]).

LEMMA 4. $(\lambda^*.A)^*(S) = \min\limits_{X \subseteq N\text{-}A} (\lambda(S \cup X) - \lambda(A \cup X) + 1)$ *for all* $S \subseteq A$

Proof.
$$\begin{aligned}(\lambda^*.A)^*(S) &= 1 - (\lambda^*.A)(A\text{-}S),\\ &= 1 - \max_{X \subseteq N\text{-}A} (\lambda^*((A\text{-}S) \cup X)) - \lambda^*(X)),\\ &= \min_{X \subseteq N\text{-}A} (1 - \lambda^*((A\text{-}S) \cup X) + \lambda^*(X)),\\ &= \min_{X \subseteq N\text{-}A} (1 + \lambda(S \cup X) - \lambda(A \cup X)).\end{aligned}$$

LEMMA 5. $(\lambda.A)(S)^* = 0 \Rightarrow (\lambda^*.A)^*(S) = 0.$

Proof. Let $S \subseteq A$ be such that $(\lambda.A)(S) = 0$. Since there are no dummies, there exists a $P \in \alpha(\lambda)$ such that $P \cap A \neq \phi$. We note that $X = P\text{-}A \subseteq N\text{-}A$ and $\lambda(X) = 0$. Since $(\lambda.A)(S) = 0$, it follows from Lemma 1 that $\lambda(S \cup X) = 0$. We observe that $A \cup X = A \cup (P\text{-}A) \supseteq P$; that is $\lambda(A \cup X) = 1$. It follows from Lemma 4 that $(\lambda^*.A)^*(S) = 0$.

LEMMA 6. $\lambda.A = (\lambda^*.A)^*$ if and only if $(\lambda^*.A)^*(P \cap A) = 1$ for all $P \in \alpha(\lambda)$ such that $P \cap A \neq \phi$.

Proof. In view of Lemma 5, we note that $\lambda.A = (\lambda^*.A)^*$ if and only if $(\lambda.A)(S) = 1 \Rightarrow (\lambda^*.A)^*(S) = 1$. By Lemma 2, $(\lambda.A)(S) = 1$ if and only if $S \supseteq P \cap A$ for some $P \in \alpha(\lambda)$ such that $P \cap A \neq \phi$. The required assertion is immediate.

LEMMA 7. *If B is a nonempty subset of A, $\lambda.B = (\lambda.A).B$.*

Proof. Trivial consequence of Lemma 3.

In Lemma 8, we give a simple proof of the main result in [7].

LEMMA 8. *A is a modular set of λ if and only if $\lambda.A = (\lambda^*.A)^*$. In this case the corresponding module is given by $\lambda.A$.*

Proof. We recall that A is a modular set of λ if and only if for all $S \subseteq A$:

$$\lambda(S \cup X) - \lambda(X) \neq (=) \, 0 \text{ for some (all) } X \subseteq N\text{-}A \Leftrightarrow \lambda(A \cup X) - \lambda(S \cup X) = (\neq) \, 0 \text{ for all (some) } X \subseteq N\text{-}A.$$

The required assertions follow from Lemmas 1 and 4.

Let A be a modular set. Butterworth ([4, p. 594] has shown that $P \cap Q \cap A \neq \phi$ for all $P \in \alpha(\lambda)$, $Q \in \beta(\lambda)$ such that $P \cap A \neq \phi$ and $Q \cap A \neq \phi$. This result also easily follows from Lemmas 3 and 8. In Theorem 1 we prove that the converse is also true.

THEOREM 1. *A is a modular set of λ if and only if $P \cap Q \cap A \neq \phi$ for all $P \in \alpha(\lambda)$ and $Q \in \beta(\lambda)$ such that $P \cap A \neq \phi$ and $Q \cap A \neq \phi$.*

Proof. We need to prove only the *if* part of the theorem. In view of Lemmas 6 and 8, we need only to prove that the conditions stated in the Theorem imply $(\lambda^*.A)^*(P \cap A) = 1$ for all $P \in \alpha(\lambda)$ such that $P \cap A \neq \phi$. We note from Lemma 2, that $S \subseteq A$ is winning in $\lambda^*.A$ if and only if $S \supseteq Q \cap A$ for some $Q \in \alpha(\lambda^*)$ or equivalently $Q \in \beta(\lambda)$. Let $P \in \alpha(\lambda)$ be such that $P \cap A \neq \phi$. By hypothesis we have $(P \cap A) \cap (Q \cap A) \neq \phi$ for all $Q \in \beta(\lambda)$ such that $Q \cap A \neq \phi$. It follows that $P \cap A$ is blocking in $\lambda^*.A$ or equivalently winning in $(\lambda^*.A)^*$.

We require the results of the following three lemmas for the purpose of generalization of the Three Modules Theorem. Lemma 10 which is called the *Two Modules Lemma* is due to Butterworth.

LEMMA 9. *Let A be a modular set of λ and B be a nonempty subset of A. Then B is a modular set of λ if and only if it is a modular set of $\lambda.A$ (see [2, p. 18] and [4, p. 595]).*

Proof. Since A is a modular set of λ, we conclude from Lemma 8 that $\lambda.A = (\lambda^*.A)^*$ or equivalently $(\lambda.A)^* = \lambda.A$. By making use of Lemma 7, we get

$$\lambda.\hat{B} = (\lambda^*.B)^* \Leftrightarrow ((\lambda.A).\ B) = ((\lambda^*.A).\ B)^* \Leftrightarrow (\lambda.A)\ .\ B = ((\lambda.A)^*\ .\ B)^*$$

LEMMA 10. *If A is proper subset of N then A and N-A are both modular sets of λ if and only if either $\lambda = (\lambda.A) \times (\lambda.(N\text{-}A))$ or $\lambda = (\lambda.A) + (\lambda.(N\text{-}A))$.*

Proof. See [4, p. 596].

LEMMA 11. *Let $A_1, A_2, \ldots, A_k$ be k ($\geqslant 2$) disjoint nonempty subsets of N such that all the 2^k—1 sets obtained by considering the union of one or more sets in the collection $\{A_1, A_2, \ldots, A_k\}$ are modular sets of λ. If $A = A_1 \cup A_2 .. \cup A_k$ then $\lambda.A$ is either equal to $(\lambda.A_1) \times (\lambda.A_2) \times \ldots \times (\lambda.A_k)$ or $(\lambda.A_1) + (\lambda.A_2) + \ldots + (\lambda.A_k)$.*

Proof. We verify first that the assertion is true for $k = 2$. By hypothesis A_1, A_2 and $A_1 \cup A_2$ are modular sets of λ. By Lemma 9 we see that A_1 and A_2 are modular sets of $\lambda.(A_1 \cup A_2)$. The required assertion for the case $k = 2$ follows from Lemma 10. We shall use induction to show that the assertion is true for all k such that $2 \leqslant k \leqslant |N|$. Let the assertion be true for $k = r$ where r is such that $2 \leqslant r < |N|$. Consider now the case $k = r+1$. Let $B = A_1 \cup A_2 \ldots \cup A_r$. We note that $A = B \cup A_{r+1}$. The hypothesis of the Lemma states that A, B and A_{r+1} are modular sets of λ. By Lemma 9 we conclude that B and A_{r+1} are modular sets of $\lambda.A$. Since $A = B \cup A_{r+1}$, it follows from Lemmas 9 and 10 that either $\lambda.A = (\lambda.B) \times (\lambda.A_{r+1})$ or $(\lambda.A) = (\lambda.B) + (\lambda.A_{r+1})$ Since all of the 2^r-1 sets obtained form the collection $\{A_1, A_2, \ldots A_r\}$ are modular sets, it follows from the induction hypothesis that $\lambda.B$ is either equal to $(\lambda.A_1) \times (\lambda.A_2) \times \ldots \times (\lambda.A_r)$ or $(\lambda.A_1) + (\lambda.A_2) + \ldots + (\lambda.A_r)$. It is enough to show that

$$\lambda.A = (\lambda.B \times (\lambda.A_{r+1}) \Rightarrow \lambda.B = (\lambda.A_1) \times (\lambda.A_2) \times \ldots \times (\lambda.A_r),$$

$$\lambda.A = (\lambda.B) + (\lambda.A_{r+1}) \Rightarrow \lambda.B = (\lambda.A_1) + (\lambda.A_2) \ldots + (\lambda.A_r).$$

Suppose $\lambda.B = (\lambda.A) \times (\lambda.A_{r+1})$ and $\lambda.B = (\lambda.A) + (\lambda.A_2) + \ldots + (\lambda.A_r)$. We shall show that this will lead to a contradiction. We note that

$$(\lambda.A = [(\lambda.A_1) + (\lambda.A_2) + \ldots + (\lambda.A_r)] \times (\lambda.A_{r+1}).$$

Consider the set $D = A_r \cup A_{r+1}$. We note that D is a modular set of λ and by Lemma 9, it is also a modular set of $\lambda.A$. Since $A_{r+1} \subseteq D$, exactly one of the following two assertions must be satisfied:

(*i*) $(\lambda.A)(A_{r+1} \cup X) = (\lambda.A)(X)$ for all $X \subseteq A\text{-}D$.

(*ii*) $(\lambda.A)(A_{r+1} \cup X) = (\lambda.A)(D \cup X)$ for all $X \subseteq A\text{-}D$.

Let $C = A_1 \cup A_2 \cup \ldots \cup A_{r-1}$. We note that $C \subseteq A-D$. It is easy to verify that $(\lambda.A)\ (C) = 0$, $(\lambda.A)\ (A_{r+1}) = 0$, $(\lambda.A)\ (A_{r+1} \cup C) = 1$ and $(\lambda.A)\ (D) = 1$. We see that (*i*) is violated for $X = C$ and (*ii*) is violated for $X = \phi$. Hence D is not a modular sets of $\lambda.A$ leading to a contradiction. The proof of the implication

$$(\lambda.A) = (\lambda.B) + (\lambda.A_{r+1}) \Rightarrow \lambda.B = (\lambda.A_1) + (\lambda.A_2) + \ldots + (\lambda.A_k),$$

is similar.

We now state the *Three Modules Theorem* and for its proof we refer to [3] or [4].

Three Modules Theorem: *Let A_1, A_2 and A_3 be disjoint nonempty subsets of N. If $A_1 \cup A_2$ and $A_2 \cup A_3$ are modular sets of λ then*

(*i*) *A_1, A_2, A_3, $A_1 \cup A_3$ and $A_1 \cup A_2 \cup A_3$ are all modular sets of λ,*

(*ii*) *$\lambda.(A_1 \cup A_2 \cup A_3)$ is equal to either $(\lambda.A_1) \times (\lambda.A_2) \times (\lambda.A_3)$ or $(\lambda.A_1) + (\lambda.A_2) + \lambda.A_3)$.*

In Theorem 2 we state and prove a generalization of the Three Modules Theorem.

THEOREM 2. *Let $A_1, \ldots, A_2 \ldots, A_k$ be ($k \geqslant 3$) disjoint nonempty subsets of N such that $A_1 \cup A_2$, $A_2 \cup A_3, \ldots, A_{k-1} \cup A_k$ are modular sets of λ. If $A = A_1 \cup A_2 \cup \ldots \cup A_k$, then*

(*i*) *All the $2^k - 1$ sets obtained by considering the union of one or more sets in the collection $\{A_1, A_2, \ldots, A_k\}$ are modulur sets of λ.*

(*ii*) *$\lambda.A$ is either equal to $(\lambda.A_1) \times (\lambda.A_2) \times \ldots \times (\lambda.A_r)$ or $(\lambda.A_1) + (\lambda.A_2) + \ldots + (\lambda.A_r)$.*

Proof. Using Lemma 11 we note that (*i*) *implies* (*ii*). Hence it is sufficient to establish (*i*). By the Three Modules Theorem we see that the assertion is true for $k = 3$. We shall use induction to prove the general assertion. Suppose the assertion is true for $k = r$ where $3 \leqslant r < |N|$. We first show that for any i and j such that $1 \leqslant i \neq j \leqslant r+1$, $A_i \cup A_j$ is a modular of λ. Without loss of generality let $j > i$. If $j = i+1$, there is nothing to prove. So let $j > i+1$. By hypothesis of the theorem A_i, A_{i+1}, A_{i+2} are disjoint nonempty subsets of N such that $A_i \cup A_{i+1}$ and $A_{i+1} \cup A_{i+2}$ are modular sets of λ. By the Three Moudules Theorem we conclude that $A_i \cup A_{i+2}$ is a modular set of λ. Now consider disjoint nonempty subsets A_i, A_{i+2} and A_{i+3}. Since $A_i \cup A_{i+2}$ and $A_{i+2} \cup A_{i+3}$ are modular sets of λ, we conclude from the Three Modules Theorem that $A_i \cup A_{i+3}$ is a modular set of λ and so on. From this result and the induction hypothesis we can conclude that all of the $2^{r+1}-2$ sets obtained by considering the union of r or less sets in the collection $\{A_1, A_2, \ldots, A_{r+1}\}$ are modular sets of λ. Thus we need only show that $A_1 \cup A_2 \cup \ldots \cup A_{r+1}$ is a modular set. For this purpose consider the nonempty disjoint sets A_r, A_{r+1} and

$B = A_1 \cup A_2 \cup \ldots \cup A_{r-1}$. We note that $B \cup A_r$ and $A_r \cup A_{r+1}$ are modular sets of λ. By the Three Modules Theorem we see that $A_1 \cup A_2 \cup \ldots \cup A_r \cup A_{r+1}$ is a modular set of λ.

COROLLARY. *All the nonempty subsets of N are modular sets of* λ *if and only if either* $\alpha(\lambda) = \{N\}$ *or* $\beta(\lambda) = \{N\}$.

Proof. Suppose $\alpha(\lambda) = \{N\}$ and $A = \{j_1, j_2, \ldots, j_r\} \subseteq N$. It follows that $\alpha(\lambda.A) = \{A\}$ and $\alpha(\lambda^*.A) = \{\{j_1\}\ \{j_2\}, \ldots, \{j_r\}\}$. It now follows that $\lambda.A = (\lambda^*.A)^*$; that is A is a modular set of λ. The proof for the case when $\beta(\lambda) = \{N\}$ is similar. This proves the *if* part of the assertion. The *only if* part follows from Theorem 2.

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