OPSEARCH, Vol. 21, No. 3, 1984

OPTIMAL RANDOMIZED DECISION RULE IN UNIVARIATE STOCHASTIC PROGRAMMING

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(*Received: February 1984, Revised: August 1984*)

ABSTRACT

The present work develops a rule for minimizing $(E(X)$ subject to $X \geqslant 0$, $P(X \geqslant b) \geqslant a$ $(0 < a < 1)$ where b is a random variable with known continuous probability distribution. The decision variable X is treated as random. The geometric significance of the result has also been pointed out.

1. Introduction

Consider the problem of minimizing $E(X)$ subject to the constraints $X \geqslant 0$ and $P(X \geqslant b) \geqslant a$, where $0 < a < 1$ and b is a random variable distributed independently of X with a known cumulative distribution function $F(b)$ which is nondecreasing and assumed to be everywhere continuous.

Under continuity of $F(b)$, for fixed a $(0 < a < 1)$, the set $\{b: F(b) = a\}$ is compact having minimum and maximum elements, say, z and w, respectively. Trivially if $z \leqslant 0$, the optimal solution is $X = 0$ with probability 1.

Now suppose $z > 0$. Then a nonrandomized solution is to take $X = z$. Vajda [3] and later Mukherjee [2] had shown through examples that if randomized decisions are admitted then the minimum $E(X)$ can sometimes be made still smaller. Following Mukerjee [1] it may be seen that randomization leads to solutions superior to the nonrandomized optimal if and only if there exist x_0, y_0 $(0 \leqslant x_0 < z,\ w < y_0)$ such that

$$(x_0 - z)/[F(x_0) - a] > (y_0 - z)/[F(y_0) - a] \tag{1}$$

Mukerjee (1982) also suggested two rules yielding optimal (randomized) solutions. These rules, however, are not universally applicable. Motivated by a geometric consideration presented in the last section, the present work develops a general rule applicable whenever randomization is superior. Incidentally, it may be remarked that the assumptions made in this paper regarding $F(b)$ are weaker than those in Mukerjee (1982).

2. Optimal Randomized Rule

Hereafter, suppose (1) holds for some x_0, y_0 $(0 \leqslant x_0 < z,\ w < y_0)$. Then to find the optimal solution, which must be a randomized one, define

$$S = \{ (x,y) : 0 \leqslant x \leqslant z,\ w \leqslant y < \infty \} \text{ and for } (x,y) \in S,$$

$$\left.\begin{aligned} g(x,y) &= [\,(y-z)\,F(x) + (z-x)F(y)\,]/(y-x), && \text{if } x < y \\ &= a && \text{, if } x = y \end{aligned}\right\}. \tag{2}$$

Note that the second possibility in (2) can arise only when $z = w$ and $x = y = z$. The continuity of $g(x,y)$ is trivial when $z < w$. Also when $z = w$, the fact that $\lim\limits_{\substack{x \to z- \\ y \to z+}} g(x,y) = a$ indicates the continuity of $g(x,y)$.

THEOREM 1. *There exists a pair* $(x_1, y_1) \in S$ *with* $x_1 < z,\ w < y_1$ *such that* $g(x_1, y_1) = \max\limits_{(x,y) \in S} g(x,y)$.

Proof. First note that $g(x,y)$ is bounded above by unity. Next observe that $(x_0,y_0) \in S$, $g(x_0,y_0) > a$ by (1) and $\lim\limits_{y \to \infty} g(x,y) = F(x) \leqslant a$ uniformly in x $(0 \leqslant x \leqslant z)$. Hence there exists y^* $(> y_0)$ such that defining $S^* = \{ (x,y) : 0 \leqslant x \leqslant z,\ w \leqslant y \leqslant y^* \}$ $(\subset S)$, one gets $g(x,y) < g(x_0,y_0)$ whenever $(x,y) \in S - S^*$, and consequently

$$\sup_{(x,y) \in S} g(x,y) = \sup_{(x,y) \in S^*} g(x,y). \tag{3}$$

Compactness of S^* and continuity of $g(x,y)$ imply the existence of (x_1,y_1) such that $g(x,y)$ is maximum in S^* at (x_1,y_1). Hence by (3), $g(x_1,y_1) = \max\limits_{(x,y) \in S} g(x,y)$. Clearly $g(x_1,y_1) \geqslant g(x_0,y_0) > a$. Since $g(z,y) \leqslant a$ for each $y(w \leqslant y < \infty)$ and $g(x,w) \leqslant a$ for each $x(0 \leqslant x \leqslant z)$, it follows that $x_1 < z,\ w < y_1$, completing the proof.

THEOREM 2. *The optimal randomized decision rule is given by a two-point distribution defined as*

$$\begin{aligned} P(X = x_1) &= [F(y_1) - a]/[F(y_1) - F(x_1)] \\ P(X = y_1) &= [a - F(x_1)]/[F(y_1) - F(x_1)] \end{aligned} \tag{4}$$

Proof. With (x_1, y_1) as in Theorem 1, clearly (4) represents a feasible solution. To prove optimality, observe that for each fixed $x(0 \leqslant x < z)$, $g(x,y_1) \leqslant g(x_1,y_1)$ while for each fixed x $(> w)$, $g(x_1, x) \leqslant g(x_1, y_1)$ and hence on simplification, for each fixed x ($x \geqslant 0$, $x \notin [z,w]$)

$$x \geqslant [(y_1 - x_1) F(x) + x_1 F(y_1) - y_1 F(x_1)]/[F(y_1) - F(x_1)]. \qquad (5)$$

Also $g(x_1,y_1) \geqslant g(x_0,y_0) > a$, which yields

$$z > [(y_1 - x_1) a + x_1 F(y_1) - y_1 F(x_1)]/[F(y_1) - F(x_1)],$$

so that (5) holds also when $x \in [z,w]$. Thus (5) holds for each x ($\geqslant 0$).

For each feasible solution, randomized or not, $P(X \geqslant 0) = 1$ and $P(X \geqslant b) \geqslant a$, *i.e.* $E[F(X)] \geqslant a$, and hence, noting that (5) holds for each x ($\geqslant 0$),

$$E(X) \geqslant [(y_1 - x_1)a + x_1 F(y_1) - y_1 F(x_1)]/[F(y_1) - F(x_1)]. \qquad (6)$$

Since the right hand member of (6) equals $E(X)$ under (4), the result follows.

3. Concluding Remarks

The optimal solution (4) may be given a geometric interpretation. For any $(x,y) \in S$, $g(x,y)$ is the ordinate at z of the straight line segment joining $(x,F(x))$ and $(y,F(y))$. Since (1) can be rewritten as $g(x_0,y_0) > a$, there exists a combination (x_0,y_0) for which this ordinate exceeds $a = F(z)$. In fact the combination (x_1,y_1), as in Theorem 1, maximizes this ordinate. This is illustrated in Fig. 1.

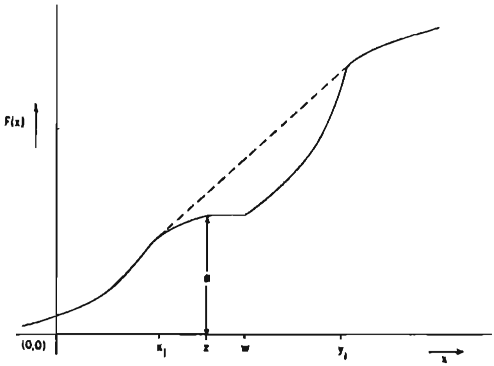

Fig. 1

Hence if $T(x)$ denotes the concave hull of $F(x)$, $x \geqslant 0$, it is enough to find x_1, y_1 such that

$$x_1 = \max_x \{x : T(x) = F(x), x \leqslant z\},$$

$$y_1 = \min_x \{x : T(x) = F(x), x \geqslant w\}.$$

If $x_1 < z$, $w < y_1$, the optimal solution is randomized while otherwise it is deterministic. In particular if $F(x)$, $x \geqslant 0$, is strictly convex (concave) the optimal solution is always randomized (deterministic).

In practice, to obtain x_1, y_1, one may have to employ numerical methods. The referee suggests that a modified form of linear programming where, to find the reduced costs and to check for optimality some simple one-dimensional optimization techniques are used, is one such method.

The procedure suggested here may be extended for the more general problem of minimizing $E[h(X)]$ subject to $X \geqslant 0$ and $P(X \geqslant b) \geqslant a$, where a,b are as before and $h(X)$ is any strictly increasing unbounded function of X. As before, the nonrandomized optimal solution is to take $X = z$ with probility 1 and keeping analogy with (1), some randomized solution will be superior to this if and only if there exist x_0, y_0 $(0 \leqslant x_0 < z$, $w < y_0)$ such that

$$[h(x_0) - h(z)]/[F(x_0) - a] > [h(y_0) - h(z)]/[F(y_0) - a] \tag{7}$$

When (7) holds, to find the optimal (randomized) solution, define in analogy with (2) for any $(x,y) \in S$,

$$\phi(x,y) = [\{h(y) - h(z)\} F(x) + \{h(z) - h(x)\} F(y)]/[h(y) - h(x)], \text{ if } x < y$$
$$= a \qquad , \text{ if } x = y$$

As before, there exist $(x_1, y_1) \in S$ such that $\phi(x_1, y_1) = \max_{(x,y) \in S} \phi(x,y)$. Then as in Theorem 2, the two point distribution (4) based on (x_1, y_1) will give the optimal randomized decision rule.

Acknowledgement

The author is thankful to Professor S.P. Mukherjee, University of Calcutta, for his interest in the work. Thanks are also due to the referee for his highly constructive suggestions.

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