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EXACT CHOICE AND FUZZY PREFERENCES

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Agents often have to make exact choices on the basis of vague preferences. Therefore analysis of the way in which exact choices are induced by vague preferences is of considerable interest. In this paper we use the model of vague preferences as fuzzy orderings. One objective of this paper is conceptual in nature: we discuss several alternative notions of exact choice sets generated by a fuzzy preference ordering and corresponding notions of rationalizability of exact choices in terms of fuzzy preference orderings. The second objective of this paper is to explore conditions for rationalizability of exact choices in terms of a fuzzy preference ordering, under alternative definitions of such rationalizability.

Key words: Fuzzy ordering; fuzzy preference; exact choice; rationalizability.

1. Introduction

The problem of choice by agents who have vague or fuzzy preference has received some attention recently.[1] In particular an elegant recent paper by Basu (1984) explores the problem of rationalizing exact choices (i.e. choices which have nothing fuzzy about them) in terms of fuzzy preference orderings. The investigation is similar to the one in the traditional literature on revealed preference which considers rationalizability of exact choices in terms of exact orderings.[2] The problem that Basu (1984) analyzes is clearly important. Our preferences are sometimes vague, although we may be constrained to make exact choices. Very little has been done

[1] See, for example, Basu (1984), Bezdek et al. (1978), Dubois and Prade (1980), Orlovsky (1978) and Ovchinnikov (1981).

[2] See, among others, Arrow (1959), Herzberger (1973), Houthakker (1950, 1965), Richter (1966, 1971, 1979), Samuelson (1938, 1948), Sen (1971) and Suzumura (1976, 1977).

so far to model such exact choice behaviour which may be based on vague preferences. In view of this, Basu's paper, which formulates the problem of choice in terms of fuzzy preference orderings, is indeed an important contribution.[3] However, the two central propositions of Basu in this context are rather discouraging. For, under one specific formulation of the concept of exact choice based on a fuzzy preference ordering, Basu arrives at the conclusion that exact choices are rationalizable in terms of a fuzzy preference ordering if and only if they are rationalizable in terms of an exact preference ordering, while under an alternative formulation Basu concludes that every conceivable pattern of exact choice behaviour is rationalizable in terms of a fuzzy preference ordering. The negative implications of these conclusions are clear. If these conclusions are accepted, then the hypothesis of a fuzzy preference ordering would have very little appeal as an explanation of the agent's exact choice behaviour: either it would have no more explanatory power than the hypothesis of an exact preference ordering, or alternatively, it would be able to account for all possible exact choice behaviour and therefore would be non-falsifiable by any conceivable empirical evidence derived from the observation of exact choices.

In this paper we re-examine the general conclusions of Basu under several alternative formulations of the problem. Thus, one objective of the paper is conceptual in nature: we consider several different notions (including the two considered by Basu) of rationalizability of exact choices in terms of fuzzy preference orderings which are linked to corresponding notions of exact choice sets generated by a fuzzy preference ordering. We argue that some of these alternative concepts of rationalizability are in many ways intuitively more appealing than the notions considered by Basu. The second objective of this paper is to show that propositions of the type proved by Basu are not valid under some of these intuitively more plausible alternative formulations of the problem of rationalizability of exact choices in terms of fuzzy preference orderings.

In Section 2 we introduce the notion of fuzzy preference orderings and show that the transitivity concepts which Basu uses in preference to the more usual transitivity concept in the literature on fuzzy relations, is a highly restrictive one. In Section 3 we discuss four different ways in which one can visualize exact choices as being induced by a fuzzy preference ordering, and in Section 4 we discuss four corresponding concepts of rationalizability of exact choices in terms of fuzzy preference orderings. In Section 5 we examine the two general conclusions of Basu in the context of the alternative concepts of rationalizability introduced in Section 4. We conclude in Section 6.

[3] Basu (1984) also discusses the problem of 'extent of rationality' of the agents. But in this paper we mainly discuss the problem of rationalizing exact choices in terms of fuzzy preference orderings.

2. The notion of fuzzy preference orderings

Let X be a set of alternatives $(3 \leq |X| < \infty)$ and let $\mathscr{X} = 2^X - \{\emptyset\}$. The elements of $\mathscr{X}$ are what may be called *exact subsets* of X as distinguished from fuzzy subsets of X. A *fuzzy subset* of X is a function $A: X \to [0, 1]$.[4] (Clearly, an exact subset of X can be considered to be a function $A: X \to [0, 1]$ such that $A(X) \subseteq \{0, 1\}$.) *A fuzzy binary weak preference relation* (FWPR) on X is a function $R: X^2 \to [0, 1]$ while an *exact binary weak preference relation* (EWPR) on X is an FWPR R such that $R(X^2) \subseteq \{0, 1\}$. Given an EWPR R, we write (using the more usual notation) xRy iff $R(x, y) = 1$.

Definition 2.1. An FWPR is *reflexive* iff for all $x \in X$, $R(x, x) = 1$, and it is *connected* iff for all distinct $x, y \in X$, $R(x, y) + R(y, x) \geq 1$.

When one tries to define the concept of transitivity for an FWPR, a variety of concepts seem to be available. We consider only three of them below.

Definition 2.2. An FWPR R satisfies

(a) *type* 1 *transitivity* (T_1) iff for all $x, y, z \in X$, $R(x, z) \geq \min[R(x, y), R(y, z)]$;

(b) *type* 2 *transitivity* (T_2) iff there exists $\beta \in]0, 1[$ such that for all distinct $x, y, z \in X$, if $[R(x, y) > 0 \;\&\; R(y, z) > 0]$, then $R(x, z) \geq \beta \max[R(x, y), R(y, z)] + (1 - \beta) \min[R(x, y), R(y, z)]$;

(c) *type* 3 *transitivity* (T_3) iff for all distinct $x, y, z \in X$ such that $R(x, y) > 0$ and $R(y, z) > 0$, $R(x, z) \geq \frac{1}{2}[R(x, y) + R(y, z)]$.[5]

T_1 constitutes the most generally accepted notion of transitivity for FWPRs. However, Basu (1984) uses the notion of T_3 and in general seems to prefer T_2 (which is a generalized version of T_3) to T_1. But we show below that T_2 as well as T_3 (which implies T_2) imposes rather severe restrictions on the FWPR under consideration.

We say that an FWPR R is *exact over* $\{x, y\} \subseteq X$ iff $([R(x, y) = 1 \;\&\; R(y, x) = 0]$ or $[R(x, y) = 0 \;\&\; R(y, x) = 1]$ or $[R(x, y) = R(y, x) = 1])$.

Proposition 2.1. *Let R be an FWPR which satisfies connectedness and T_2. Then for all distinct $x, y, z \in X$, either $[R(x, y) = R(y, x) = R(y, z) = R(z, y) = R(z, x) = R(x, z)]$ or R is exact over at least two out of the three two-element subsets of $\{x, y, z\}$.*

[4] For a discussion of fuzzy subsets see Basu (1984), Kaufmann (1975) and Orlovsky (1978).

[5] In terms of our notation, Basu (1984) defines this property for all $x, y, z \in X$ such that $y \notin \{x, z\}$, while we formulate our definition for all distinct $x, y, z \in X$. When x and z are identical, $R(x, z) = 1$ and hence $R(x, z)$ is necessarily at least as great as $\frac{1}{2}(R(x, y) + R(y, z))$. Hence, our definition of this property, despite the difference in formulation, is equivalent to Basu's definition.

Proof. Let R be a connected FWPR satisfying T_2 and let $\bar{\beta}$ be the value of β (see Definition 2.2(b)) with respect to (w.r.t.) which R satisfies T_2. Let $x, y, z \in X$ be distinct. Given T_2 and the definition of R, it follows that:

$$R(x, y) \geq \min [R(x, z), R(z, y)], \tag{2.1}$$

$$R(x, z) \geq \min [R(x, y), R(y, z)], \tag{2.2}$$

$$R(y, z) \geq \min [R(y, x), R(x, z)], \tag{2.3}$$

$$R(y, x) \geq \min [R(y, z), R(z, x)], \tag{2.4}$$

$$R(z, x) \geq \min [R(z, y), R(y, x)], \tag{2.5}$$

$$R(z, y) \geq \min [R(z, x), R(x, y)]. \tag{2.6}$$

Given this, there can be two possible cases:

either (I) for all distinct $a, b \in \{x, y, z\}$, $R(a, b) > 0$,
or (II) there exist distinct $a, b \in \{x, y, z\}$ such that $R(a, b) = 0$.

We show that in Case (I) $[R(x, y) = R(y, x) = R(y, z) = R(z, y) = R(x, z) = R(z, x)]$, and that in Case (II) R is exact over at least two out of the three two-element subsets of $\{x, y, z\}$.

(I) Suppose for all distinct $a, b \in \{x, y, z\}, R(a, b) > 0$. First we show that for all distinct $a, b, c \in \{x, y, z\}$,

$$(R(a, c) = \min [R(a, b), R(b, c)]) \rightarrow [R(a, c) = R(a, b) = R(b, c)]. \tag{2.7}$$

Let $a, b, c \in \{x, y, z\}$ be distinct. By T_2:

$$R(a, c) \geq \bar{\beta} \max [R(a, b), R(b, c)] + (1 - \bar{\beta}) \min [R(a, b), R(b, c)], \quad \text{where } \bar{\beta} \in]0, 1[. \tag{2.8}$$

Hence, if $R(a, b) \neq R(b, c)$, then $R(a, c) > \min [R(a, b), R(b, c)]$. So, given (2.8), (2.7) follows immediately.

Now, suppose at least one of the weak inequalities (2.1)–(2.6) is a strict inequality. Without loss of generality assume that (2.1) is a strict inequality, so that

$$R(x, y) > \min [R(x, z), R(z, y)]. \tag{2.9}$$

Suppose $\min [R(x, z), R(z, y)] = R(x, z)$. (Proof for the case where $\min [R(x, z), R(z, y)] = R(z, y)$ is similar and can be checked by the reader.) Then,

$$R(x, y) > R(x, z). \tag{2.10}$$

Now consider (2.2). By (2.10) and (2.7), $R(x, z) \neq \min [R(x, y), R(y, z)]$. Hence, given (2.10), by (2.2):

$$R(x, z) > R(y, z). \tag{2.11}$$

By following similar reasoning and taking up (2.3), (2.4), (2.5) and (2.6) in that order, we get, respectively:

$$R(y, z) > R(y, x), \tag{2.12}$$

$$R(y, x) > R(z, x), \tag{2.13}$$

$$R(z, x) > R(z, y), \tag{2.14}$$

$$R(z, y) > R(x, y), \tag{2.15}$$

and (2.10)-(2.15) give a contradiction. Hence, none of the weak inequalities given by (2.1)-(2.6) can be a strict inequality and all of them must hold with strict equality. Thus, given (2.1), we have $R(x, y) = \min [R(x, z), R(z, y)]$. Hence, by (2.7), $R(x, y) = R(x, z) = R(z, y)$. Applying similar reasoning also to (2.2)-(2.6), which have all been shown to hold with strict equalities, it is clear that

$$R(x, y) = R(y, x) = R(y, z) = R(z, y) = R(z, x) = R(x, z).$$

(II) Suppose there exist distinct $a, b \in \{x, y, z\}$ such that $R(a, b) = 0$. Without loss of generality assume that $R(x, y) = 0$. Then from (2.1), either $R(x, z) = 0$ or $R(z, y) = 0$. If $R(x, z) = R(x, y) = 0$, then by connectedness of R, $R(z, x) = R(y, x) = 1$. Then R is exact over $\{x, y\}$ and $\{x, z\}$. On the other hand, if $R(z, y) = R(x, y) = 0$, then by connectedness of R, $R(y, z) = R(y, x) = 1$ and hence R is exact over $\{x, y\}$ and $\{y, z\}$. Thus in case (II) R must be exact over at least two out of the three two-element subsets of $\{x, y, z\}$.

Remark 2.1. By Proposition 2.1, given any triple (x, y, z) of distinct alternatives, an agent satisfying reflexivity connectedness and T_2 can have genuinely fuzzy preferences over more than one two-element subset of $\{x, y, z\}$ only if he satisfies the severe constraint that the 'degree of fuzziness' is the same for all six ordered pairs of distinct alternatives figuring in the triple.

From Proposition 2.1 and Remark 2.1 it is clear that both T_2 and T_3 (which implies T_2) are rather restrictive transitivity properties and the class of reflexive and connected FWPRs which satisfy T_2 (or alternatively T_3) does not seem to be a very interesting class. Thus, the more usually adopted transitivity concept of T_1 would seem to be considerably more interesting than either T_2 and T_3. Depending upon the specific notion of transitivity for FWPRs which we choose to combine with the concept of reflexivity and connectedness of FWPRs, we would get alternative notions of fuzzy preference orderings. However, in view of what we have said above, in the rest of this paper we use T_1 rather than T_2 or T_3. Hence the following definition.

Definition 2.3. *A fuzzy preference ordering* (FPO) is an FWPR which satisfies T_1 in addition to reflexivity and connectedness.

If R is a reflexive and connected EWPR, then for R, T_1, T_2 and T_3 turn out to be equivalent and coincide with the usual concept of transitivity of EWPRs (an EWPR R is *transitive* iff for all $x, y, z \in X$, $[xRy$ & $yRz]$ implies xRz). A reflexive, connected and transitive EWPR will be called an *exact preference ordering* (EPO).

3. Exact choice sets generated by fuzzy weak preference relations

Suppose an agent who has an FWPR R has to make exact choices. What are the plausible senses in which R can induce exact choices? We discuss below four different ways of inducing such exact choices given R.

For all $x, y \in X$ and for all $\alpha \in [0, 1]$, x *α-dominates* y iff $R(x, y) \geq \alpha$ and x is *pairwise optimal* vis-à-vis y iff $R(x, y) \geq R(y, x)$. In the discussion that follows, A is assumed to be a given exact subset of X.

(1) One possible hypothesis about the way in which exact choices may be generated by R for A is that the exact choice set generated by R is the set of all $x \in A$ which α-dominate every $y \in A$, where α is some number in the interval $[0, 1]$. The specific α chosen should carry plausibility as a confidence threshold in the sense that given an exact feasible set $\{x, y\}$, if $R(x, y) \geq \alpha$, then the agent would with a 'reasonable' degree of confidence specify x as an alternative chosen from $\{x, y\}$. Thus, if one can agree about what the threshold α should be, one can interpret

$$B_{D[\alpha]}(A, R) \equiv \{x \in A \mid R(x, y) \geq \alpha \text{ for all } y \in A\}$$

as the exact choice set generated by R given A. (Note that Basu, 1984, discussed the special case where $\alpha = 1$.)

(2) Under a second interpretation, the exact choice set induced by R given A is identified with

$$B_{PO}(A, R) \equiv \{x \in A \mid R(x, y) \geq R(y, x) \text{ for all } y \in A\}.$$

$B_{PO}(A, R)$ is clearly the set of all $x \in A$ which are pairwise optimal vis-à-vis all other alternatives in A. The intuition underlying this approach is as follows. Given a feasible set $\{x, y\}$, the agent determines his exact choice in a 'natural' way by comparing $R(x, y)$ and $R(y, x)$. Then for an exact set A containing more than two alternatives the agent specifies the exact choice set as the set of all alternatives x in A such that x is always chosen (following the rule described above) from every two-element subset of A containing x.

(3) Another route, which Basu (1984) follows in generating an exact choice set through R, can be described as follows.

Define the R-greatest set in A to be $G(A, R): X \to [0, 1]$ such that for all $x \in X - A$, $G(A, R)(x) = 0$ and for $x \in A$, $G(A, R)(x) = \min_{y \in A} R(x, y)$. (This is clearly the fuzzy counterpart of the notion of the R-greatest elements in A when R is exact.) Also define the *generalized Hamming distance* between any two fuzzy subsets A and D of X as

$$d(A, D) = \sum_{x \in X} |A(x) - D(x)|.$$

Finally, consider the class of exact sets which are nearest to $G(A, R)$ in terms of the generalized Hamming distance and denote it by $\mathrm{NE}[G(A, R)]$. Thus,

$$\mathrm{NE}[G(A, R)] = \{E \in 2^X \mid d(G(A, R), E) \le d(G(A, R), E') \text{ for all } E' \in 2^X\}.$$

It can be easily checked that if R is an FPO, then for all $E \in 2^X$, $E \in \mathrm{NE}[G(A, R)]$ iff

$$\{x \in X \mid G(A, R)(x) > 0.5\} \subseteq E \subseteq \{x \in X \mid G(A, R)(x) \ge 0.5\}. \qquad (3.1)$$

Then under this interpretation, the exact choice set induced by R for A can be taken to be any non-empty exact set E belonging to $\mathrm{NE}[G(A, R)]$.

(4) Under our last interpretation, we again start with the fuzzy set $G(A, R)$. However, now we identify the exact choice set generated by R not with some exact set nearest to $G(A, R)$ but with the set of all $x \in A$ which score the highest with the function $G(A, R)$, i.e. the choice set is identified with

$$B_{\mathrm{H}}(A, R) \equiv \{x \in A \mid G(A, R)(x) \ge G(A, R)(y) \text{ for all } y \in A\}.$$

We first note the following proposition (the proof is omitted).

Proposition 3.1.

(a) *For every* $A \in \mathcal{T}$ *and every fuzzy preference ordering* R, $B_{\mathrm{H}}(A, R) \ne \emptyset$; $B_{\mathrm{PO}}(A, R) \ne \emptyset$; and for some $E \in \mathrm{NE}[G(A, R)]$, $E \ne \emptyset$;

(b) *for every* $A \in \mathcal{T}$ *and every EWPR* R, $\mathrm{NE}[G(A, R)] = \{G(A, R)\}$; *and* $B_{\mathrm{D}[\alpha]}(A, R) = B_{\mathrm{PO}}(A, R) = B_{\mathrm{H}}(A, R) = G(A, R)$ *for all* $\alpha \in]0, 1]$.

It is not clear that one of the four alternative approaches discussed above has unambiguously more intuitive appeal than any the others. But it seems to us that as candidates for the status of the exact choice set generated by R, $B_{\mathrm{PO}}(A, R)$ and $B_{\mathrm{H}}(A, R)$ have some intuitive edge over a non-empty exact set arbitrarily picked from $\mathrm{NE}[G(A, R)]$.

First, note that $\mathrm{NE}[G(A, R)]$ may not be a singleton and therefore the approach which can adopt any set in $\mathrm{NE}[G(A, R)]$ as the choice set suffers from some ambiguity. Consider a case where $X = \{x, y, z\}$, $R(x, y) = 0.9$, $R(y, x) = R(y, z) = R(z, y) = R(z, x) = 0.5$ and $R(x, z) = 0.6$. Here the nearest exact set approach will permit each of the exact sets $\{x\}$, $\{x, z\}$, $\{x, y\}$, $\{x, y, z\}$ to qualify as the exact choice set for X generated by R. Apart from the ambiguity as such, it seems unreasonable to include y, for example, in the exact choice set while excluding z (which would be the case if $\{x, y\}$ is taken to be the exact choice set). The example given above, of course, violates the transitivity property T_3 postulated by Basu (1984) (see our Proposition 2.1), but consider another case where $R(a, b) = 0.5$ for all $a, b \in \{x, y, z\}$. In this case any non-empty exact subset of X – say $\{x, z\}$ – will qualify as the exact choice set for X under R. Here again, one can ask whether it is reasonable to include z or x while excluding y. It seems to us that in this second case it would be unreasonable to take $\{x, z\}$ as the exact choice set for X generated by the FWPR R, in the same way as it would be unacceptable to take $\{x, z\}$ as the exact choice set for X generated by an exact preference ordering R^* which shows 'indifference' over all pairs of alternatives in $\{x, y, z\}$.

One way to get rid of the ambiguity inherent in the third approach outlined above is to modify the approach by requiring that $\{x \in A \mid G(A, R)(x) \geq 0.5\}$ be the choice set. It is easy to see that $\{x \in A \mid G(A, R)(x) \geq 0.5\} = \{x \in A \mid R(x, y) \geq 0.5$ for all $y \in A\}$ and hence this modification will lead us to the first approach in terms of $B_{D[\alpha]}(A, R)$, where $\alpha = 0.5$.

There seem to be further intuitive problems involved in the nearest exact set approach. Suppose $A = \{x, y\}$, $R(x, y) = 0.9$ and $R(y, x) = 0.6$. It would seem reasonable to interpret this as implying that the strength of the agent's feeling that x is at least as good as y for him, is considerably greater than the strength of his feeling that y is at least as good as x. Now, faced with the question of what he will choose given the exact feasible set $\{x, y\}$, it is not clear that a rational individual should indicate that either of x and y will do equally well (which is what we take to mean when one says that given the feasible exact set $\{x, y\}$, the exact choice set generated by his preferences (irrespective of whether it is fuzzy or not) is $\{x, y\}$). It seems more plausible to us that $\{x\}$ should emerge as the exact choice set in the above example and this is what happens under the pairwise optimality approach in terms of $B_{PO}(A, R)$ [as well as under the highest-scoring-alternatives approach in terms of $B_H(A, R)$], while under the approach in terms of the exact sets nearest to $G(A, R)$, $\{x, y\}$ will be the exact choice set given the feasible set $\{x, y\}$.

The interpretation in terms of $B_{D[\alpha]}(A, R)$ also suffers from the intuitive problem outlined in the preceding paragraph. If α is 0.7, say, then in the case where $R(x, y) = 0.9$ and $R(y, x) = 0.7$, $B_{D[\alpha]}(\{x, y\}, R) = \{x, y\}$, but for reasons considered above it seems more plausible to say that $\{x\}$ should be the exact choice set. Besides, for every $\alpha > 0.5$, it is possible to have an FWPR R such that $B_{D[\alpha]}(A, R)$ is empty, since given $\alpha > 0.5$, it is possible to have $R(x, y) < \alpha$ and $R(y, x) < \alpha$. If $\alpha = 1$ (the special case considered by Basu, 1984), this problem would arise under every non-exact FWPR so that when $\alpha = 1$, a non-exact FWPR would never induce non-empty exact choice sets in the sense of $B_{D[1]}(A, R)$, for all possible exact subsets of X.

4. Alternative concepts of rationalizability of exact choice functions

Suppose the choice behaviour of an agent is given by a function $C: \mathscr{X} \to \mathscr{X}$ such that for all $A \in \mathscr{X}$, $C(A) \subseteq A$. We call such a function an exact choice function (ECF). Given an agent's ECF one may like to know whether the agent is one whose choice behaviour could possibly have been induced by a fuzzy preference ordering. This is the problem of rationalizability of an ECF in terms of a fuzzy preference ordering, which is the exact counterpart of the much discussed problem of rationalizability of exact choice functions in terms of exact preference orderings.[6] Clearly, depending upon one's notion of how a fuzzy preference ordering induces exact choice behaviour (a problem considered in Section 3), we would have alternative notions of rationalizability of an ECF. Given our discussion in Section 3, we can therefore define four possible notions of rationalizability. (We formulate the definitions in a somewhat general fashion by considering any given FWPR R.)

Definition 4.1. Let C be an ECF. Let R be an FWPR. C is respectively
(i) D[α]-rationalizable,
(ii) PO-rationalizable,
(iii) NE-rationalizable,
(iv) H-rationalizable,
in terms of R iff for all $A \in \mathscr{X}$, $C(A)$ coincides with
(i′) $B_{D[\alpha]}(A, R)$,
(ii′) $B_{PO}(A, R)$,
(iii′) some $E \in NE[G(A, R)]$,
(iv′) $B_H(A, R)$,
respectively.

Note that when R is an EWPR, in view of Proposition 3.1(b), all the four concepts of rationalizability introduced in Definition 4.1 coincide (assuming that $\alpha > 0$). This provides the motivation for the following definition.

Definition 4.2. Let R be an EWPR. An ECF C is *rationalizable* in terms of R iff for all $A \in \mathscr{X}$, $C(A) = \{x \in A \mid xRy \text{ for all } y \in A\}$.

5. Conditions for rationalizability of exact choice functions in terms of fuzzy preference orderings

In this section we explore the conditions for alternative types of rationalizability of an ECF in terms of an FPO. In doing so we consider particularly the relationship between rationalizability of an ECF in terms of an EWPR and different types of rationalizability in terms of an FPO.

Proposition 5.1. *For every $\alpha (0 < \alpha \le 1)$, an exact choice function is D[α]-rationalizable in terms of a fuzzy preference ordering iff it is rationalizable in terms of an exact preference ordering.*

Proof. Let $\alpha \in]0, 1]$. Suppose C is D[α]-rationalizable in terms of an FPO R. Then, $C(A) = B_{D[\alpha]}(A, R)$ for all $A \in \mathscr{X}$. Define an EWPR $\hat{R}$ on X as follows:

[6] See footnote 2 for an account of the literature.

for all $x, y \in X$, $x\hat{R}y$ iff $R(x, y) \geq \alpha$.

We show that $\hat{R}$ is an EPO and that C can be rationalized in terms of $\hat{R}$.

Reflexivity of $\hat{R}$ follows directly from the reflexivity of R. To prove connectedness of R, consider distinct $x, y \in X$. By assumption, $B_{D[\alpha]}(\{x, y\}, R) = C(\{x, y\})$; and by definition, $C(\{x, y\}) \neq \emptyset$. Hence, $R(x, y) \geq \alpha$ or $R(y, x) \geq \alpha$, which implies $x\hat{R}y$ or $y\hat{R}x$. Thus, $\hat{R}$ is connected. To show transitivity of $\hat{R}$, consider any $x, y, z \in X$ and assume that $x\hat{R}y$ and $y\hat{R}z$. Then by definition of R, $R(x, y) \geq \alpha$ and $R(y, z) \geq \alpha$. Since R satisfies T_1, it follows that $R(x, z) \geq \alpha$. Therefore $x\hat{R}z$ and $\hat{R}$ is transitive.

Now consider $C(A)$ where $A \in \mathscr{X}$. Since C is D[α]-rationalizable, $C(A) = \{x \mid R(x, y) \geq \alpha$ for all $y \in A\}$ and hence $C(A) = \{x \mid x\hat{R}y$ for all $y \in A\}$. Hence, C is rationalizable in terms of $\hat{R}$.

Now suppose C is rationalizable in terms of an EPO R^*. Then,

$$\text{for all } A \in \mathscr{X},\ C(A) = \{x \in A \mid R^*(x, y) = 1 \text{ for all } y \in A\}. \tag{5.1}$$

Since for all $x, y \in X$, $R^*(x, y) = 1$ or $R^*(x, y) = 0$ and since $\alpha \in]0, 1]$,

$$\begin{aligned}\text{for all } A \in \mathscr{X},\ &\{x \in A \mid R^*(x, y) = 1 \text{ for all } y \in A\} \\ &= \{x \in A \mid R^*(x, y) \geq \alpha \text{ for all } y \in A\}.\end{aligned} \tag{5.2}$$

By (5.1) and (5.2), C is D[α]-rationalizable in terms of R^* which, being an EPO, is also an FPO.

Remark 5.1. Basu (1984) shows that an ECF is D[1]-rationalizable in terms of a reflexive and connected FWPR satisfying T_3 iff it is rationalizable in terms of an EPO. (Since T_3 implies T_1 the necessity part of Basu's proposition follows from the necessity part of Proposition 5.1.)

Before we state our next proposition we require the concept of quasi-transitive EWPRs. An EWPR R is *quasi-transitive* iff or all $x, y, z \in X$, $[(xRy \ \& \ \sim yRx) \ \& \ (yRz \ \& \ \sim zRy)]$ implies $[xRz \ \& \ \sim zRx]$.

Proposition 5.2. *An exact choice function is PO-rationalizable in terms of a fuzzy preference ordering iff it is rationalizable in terms of a reflexive, connected and quasi-transitive EWPR.*

Proof.

I. Necessity. Suppose C is PO-rationalizable in terms of an FPO R. Then, for all $A \in \mathscr{X}$, $C(A) = B_{PO}(A, R)$. Define an EWPR $\hat{R}$ on X as follows: for all $x, y \in X$, $x\hat{R}y$ iff $R(x, y) \geq R(y, x)$. It is clear that $\hat{R}$ is reflexive and connected and that for all $A \in \mathscr{X}$, $C(A) = \{x \mid x\hat{R}y$ for all $y \in A\}$. So to show that C is rationalizable in terms of a reflexive, connected and quasi-transitive EWPR, it is enough to show that $\hat{R}$ is quasi-transitive.

Suppose $\hat{R}$ violates quasi-transitivity. For all $a, b \in X$, let $a\bar{P}b$ iff $a\hat{R}b \ \& \ \sim b\hat{R}a$.

Given that $\hat{R}$ violates quasi-transitivity, there exist $x, y, z \in X$ such that $x\hat{P}y$, $y\hat{P}z$ but $z\hat{R}x$. Then by definition of $\hat{R}$, $R(x,y) > R(y,x)$, $R(y,z) > R(z,y)$ and $R(z,x) \geq R(x,z)$. Since R is an FPO,

$$R(x,y) > R(y,x) \geq \min\,[R(y,z), R(z,x)], \tag{5.3}$$

$$R(y,z) > R(z,y) \geq \min\,[R(z,x), R(x,y)], \tag{5.4}$$

and

$$R(z,x) \geq R(x,z) \geq \min\,[R(x,y), R(y,z)], \tag{5.5}$$

Suppose $\min\,[R(y,z), R(z,x)] = R(y,z)$. Then by (5.3), $R(x,y) > R(y,z)$ and hence by (5.4), $R(y,z) > R(z,x)$, which contradicts our assumption that $(\min\,[R(y,z), R(z,x)] = R(y,z))$.

Now suppose, $\min\,[R(y,z), R(z,x)] \neq R(y,z)$. Then $R(y,z) > \min\,[R(y,z), R(z,x)] = R(z,x)$, and, noting (5.3), $[R(x,y) > R(z,x)]$. Then, given $[R(y,z) > R(z,x)]$, (5.5) cannot hold. This completes the proof of quasi-transitivity of $\hat{R}$.

II. Sufficiency. Suppose C is rationalizable in terms of a reflexive, connected and quasi-transitive EWPR R^*. Then,

$$\text{for all } A \in \mathscr{X}, \; C(A) = \{x \in A \mid xR^*y \text{ for all } y \in A)\}. \tag{5.6}$$

Define an FWPR R as follows. Let $g, h \in]0,1]$ be such that $g > h \geq 0.5$. For all $x, y \in X$,

$$\text{if } x = y, \text{ then } R(x,y) = 1, \tag{5.7}$$

$$\text{if } x \neq y \text{ and } C(\{x,y\}) = \{x\}, \text{ then } R(x,y) = g \text{ and } R(y,x) = h, \tag{5.8}$$

and

$$\text{if } x \neq y \text{ and } C(\{x,y\}) = \{x,y\}, \text{ then } R(x,y) = R(y,x) = h. \tag{5.9}$$

Since C is rationalizable in terms of R^* and R^* is quasi-transitive, $[C(\{x,y\}) = \{x\}$ & $C(\{y,z\}) = \{y\}]$ implies $[C(\{x,z\}) = \{x\}]$. It is then easy to check that R, as defined by (5.7), (5.8) and (5.9), is an FPO. We show that C is PO-rationalizable in terms of R.

From (5.6)–(5.9) it follows that

$$\text{for all } x, y \in X, \; xR^*y \text{ iff } R(x,y) \geq R(y,x). \tag{5.10}$$

So, by (5.6) and (5.10) it follows that $C(A) = \{x \in A \mid R(x,y) \geq R(y,x) \text{ for all } y \in A\}$ for all $A \in \mathscr{X}$. Hence C is PO-rationalizable in terms of R.

Remark 5.2. It is well known that rationalizability of an exact choice function in terms of a reflexive, connected and quasi-transitive EWPR does not necessarily imply its rationalizability in terms of an EPO. Hence, by Proposition 5.2 it follows that PO-rationalizability of an exact choice function in terms of an FPO does not necessarily imply its rationalizability in terms of an exact preference ordering (though it is obvious that an exact choice function which is rationalizable in terms of an EPO must also be PO-rationalizable in terms of an FPO). It is also well known that not every exact choice function is rationalizable in terms of an EWPR. Hence, by Proposition 5.2 it follows that not every exact choice function is PO-rationalizable in terms of a fuzzy preference ordering.

Proposition 5.3. *Every exact choice function is NE-rationalizable in terms of a fuzzy preference ordering R such that $R(x, y) = 0.5$ for all distinct $x, y \in X$.*

Proof. The proof of Proposition 5.3 is exactly the same as the proof of Basu's (1984) Theorem 2 which states that every ECF is NE-rationalizable in terms of a reflexive and connected FWPR satisfying $\mathbf{T}_3$.

Remark 5.3. What is crucial in deriving Proposition 5.3 is the non-uniqueness of the exact set nearest to the fuzzy greatest set $G(A, R)$ generated by an FWPR R for A. When $R(x, y) = 0.5$ for all distinct $x, y \in X$, $\{x \in X \mid G(A, R)(x) > 0.5\} = \emptyset$ and noting (3.1), $\mathrm{NE}[G(A, R)] = 2^A$. Hence, whatever $C(A)$ may be, there will exist $E \in \mathrm{NE}[G(A, R)]$ such that $C(A) = E$.

The remaining results in this section are concerned with H-rationalizability of ECFs. We first introduce some definitions.

For every ECF C, the base relation $\bar{R}_C$ of C is defined as follows:

$$\text{for all } x, y \in X,\ x\bar{R}_C y \text{ iff } x \in C(\{x, y\}).$$

Note that for every ECF C, $\bar{R}_C$ is reflexive and connected. For all $x, y \in X$, let $[x\bar{P}_C y \text{ iff } \{x\} = C(\{x, y\})]$ and $[x\bar{I}_C y \text{ iff } \{x, y\} = C(\{x, y\})]$. When there is no ambiguity about C, we write simply $\bar{R}$, $\bar{P}$ and $\bar{I}$ instead of $\bar{R}_C$, $\bar{P}_C$ and $\bar{I}_C$, respectively.

The following condition, first proposed by Bordes (1976), is well known in the literature on 'rational' social choice.

Definition 5.1. An ECF C satisfies *Property* (β_+) iff for all $x, y \in X$, and all $A, B \in \mathcal{X}$, $[x, y \in A \subseteq B \ \&\ y \in C(A) \ \&\ x \in C(B)] \to [y \in C(B)]$.

Proposition 5.4. *Let C be an ECF which is H-rationalizable in terms of a fuzzy preference ordering. Then*

(a) *C satisfies Property (β_+);*

(b) *$\bar{R}$ is quasi-transitive, i.e. $\bar{P}$ is transitive;*

(c) *for all $x, y, z \in X$, if $(x\bar{P}y \ \&\ y\bar{I}z \ \&\ x\bar{P}z)$, then $C(\{x, y, z\}) = \{x\}$.*

Proof. Let C be an ECF and let R be an FPO which H-rationalizes C.

(i) To prove Proposition 5.4(a) we first show that for all $x, y \in X$ and for all $A \in \mathcal{X}$,

$$\text{if } x \in C(A) \text{ and } y \in A - C(A), \text{ then } C(\{x, y\}) = \{x\}. \tag{5.11}$$

Suppose $x \in C(A)$ and $y \in A - C(A)$ but $C(\{x, y\}) \neq \{x\}$. Then $y \in C(\{x, y\})$, which implies $R(y, x) \geq R(x, y)$. Since $x \in C(A)$ and $y \in A - C(A)$, $G(A, R)(x) > G(A, R)(y)$. Let $G(A, R)(y) = R(y, z)$, where $z \in A$. Then $G(A, R)(x) > R(y, z)$. Given $R(y, x) \geq R(x, y)$ and $G(A, R)(x) > R(y, z)$, and noting that $R(x, y) \geq G(A, R)(x) \leq R(x, z)$, we have $[R(y, x) > R(y, z) < R(x, z)]$. This contradicts $(R(y, z) \geq \min [R(y, x), R(x, z)])$ which follows from properties T_1 of R. This proves (5.11).

Now suppose C violates Property (β_+) so that for some $x, y \in X$ and some $A, B \in \mathcal{X}$, $x, y \in A \subseteq B$, $y \in C(A)$, $x \in C(B)$, and $y \in B - C(B)$. Then $G(B, R)(x) > G(B, R)(y)$. Then, letting $G(B, R)(y) = R(y, z)$, where $z \in B$,

$$G(B, R)(x) > R(y, z). \tag{5.12}$$

Hence, noting $R(x, z) \geq G(B, R)(x)$, $[R(x, z) > R(y, z)]$. Given $x \in C(B)$ and $y \in B - C(B)$, by (5.11), $C(\{x, y\}) = \{x\}$. Hence, given $y \in C(A)$, by (5.11) again, $x \in C(A)$. Since $x, y \in C(A)$, $G(A, R)(y) = G(A, R)(x)$. Hence, noting $[R(y, x) \geq G(A, R)(y)$; $G(A, R)(x) \geq G(B, R)(x)$; and (5.12)], $[R(y, x) > R(y, z)]$. Thus, we have $R(x, z) > R(y, z) < R(y, x)$. This, as in the preceding paragraph, leads to a contradiction which completes the proof of Proposition 5.4(a).

(ii) Given $[x\bar{R}y$ iff $R(x, y) \geq R(y, x)$, for all $x, y \in X]$ and that R is an FPO, the proof of Proposition 5.4(b) is similar to the proof of the necessity part of Proposition 5.2.

(iii) To prove Proposition 5.4(c), suppose for some $x, y, z \in X$, $[x\bar{P}y \ \& \ x\bar{P}z \ \& \ y\bar{I}z]$. Let $A = \{x, y, z\}$. By Proposition 5.4(a), C satisfies Property β_+.

Without loss of generality, assume $G(A, R)(x) = R(x, y)$. Then, noting that $R(x, y) > R(y, x)$ (which follows from $x\bar{P}y$), and that $R(y, x) \geq G(A, R)(y)$, we have $G(A, R)(x) > G(A, R)(y)$. Hence $y \notin C(A)$. However, by Property (β_+) and $y\bar{I}z$, $C(A) \cap \{y, z\} = \emptyset$ or $C(A) \cap \{y, z\} = \{y, z\}$. Given $y \notin C(A)$, it follows that $C(A) = \{x\}$.

Remark 5.4. As the reader can easily check, the three necessary conditions (given in Proposition 5.4) for C to be H-rationalizable in terms of an FPO are independent of one another. Indeed, Property (β_+) is a requirement of consistency when the menu of feasible choices is expanded. It states that if y is chosen from A when x is available, then in an expanded set B, y must always be chosen if x is. It is known that Property (β_+) does not even imply the absence of a $\bar{P}$-cycle. Another way of viewing Property (β_+) is that it is an *inclusion* condition - consistent choice requires that certain elements must be included in the expanded set. In contrast, the third of the necessary conditions in Proposition 5.4 is a very weak *rejection* condition.

Remark 5.5. We do not know whether the three necessary conditions in Proposition 5.4, together, are also sufficient for H-rationalizability. An obvious problem seems to be that the requirement $[(x\bar{P}y \ \& \ y\bar{I}z \ \& \ x\bar{P}z) \rightarrow (C(\{x, y, z\}) = \{x\})]$ is a very weak rejection condition. However, as the following example shows, even a slightly stronger condition ceases to be necessary for H-rationalizability in terms of an FPO.

Example 5.1. Let $X = \{x, y, z\}$. Let $C(\{x, y\}) = C(\{x, z\}) = \{x\}$; $C(\{y, z\}) = \{y\}$; and $C(\{x, y, z\}) = \{x, y\}$. Then C is H-rationalizable by the following FPO R: $R(x, y) = 0.9$; $R(y, x) = 0.6$; $R(x, z) = 0.6$; $R(z, x) = 0.4$; $R(y, z) = 0.6$; and $R(z, y) = 0.4$. However, C violates the following condition: $[\{a \in X | a\bar{P}b$ for all $b \in X - \{a\}\} \neq \emptyset] \rightarrow [C(X) = \{a \in X | a\bar{P}b$ for all $b \in X - \{a\}\}]$.

Proposition 5.2 showed that rationalizability in terms of a reflexive, connected and quasi-transitive EWPR is necessary and sufficient for H-rationalizability in terms of an FPO. However, rationalizability in terms of a reflexive, connected and quasi-transitive EWPR is neither necessary nor sufficient for H-rationalizability in terms of an FPO. Indeed, an ECF which is H-rationalizable in terms of an FPO must fall into one of two distinct categories. Either it is not rationalizable in terms of any EWPR, or it is rationalizable in terms of an EPO. This is shown by Proposition 5.5 below. Note that the ECF in Example 5.1 is not rationalizable in terms of any EWPR, although it is H-rationalizable in terms of an FPO.

Proposition 5.5 *Let C be an ECF and let C be H-rationalizable in terms of an FPO. If C is rationalizable in terms of an EWPR, then C is rationalizable in terms of an EPO (this EPO must be the base relation of C).*

Proof. Suppose an ECF C is H-rationalizable in terms of an FPO. Suppose C is also rationalizable in terms of an EWPR. Then by a well-known result (see Herzberger, 1973) this EWPR must be the base relation $\bar{R}$ of C. It is then enough to show that $\bar{R}$ is an ordering. Since $\bar{R}$ is clearly reflexive and connected, we have only to show that $\bar{R}$ is transitive. Suppose $\bar{R}$ is not transitive. Then, noting that $\bar{R}$ is connected, $(x\bar{R}y$ & $y\bar{R}z$ & $z\bar{P}x)$ for some $x, y, z \in X$. Since by Proposition 5.4(b), $\bar{P}$ is transitive, it is easy to check that $(x\bar{R}y$ & $y\bar{R}z$ & $z\bar{P}x)$ implies $(x\bar{I}y$ & $y\bar{I}z$ & $z\bar{P}x)$. Since C is rationalizable in terms of $\bar{R}$, given $(x\bar{I}y$ & $y\bar{I}z$ & $z\bar{P}x)$, we have $x \notin C(\{x, y, z\}) = \{y, z\}$. However, given $(x\bar{I}y$ & $y \in C(\{x, y, z\}))$, by Proposition 5.4.1, $x \in C(\{x, y, z\})$. This contradiction completes the proof.

Remark 5.6. Although by Proposition 5.4(b) quasi-transitivity of the base relation is necessary for H-rationalizability of an ECF in terms of a FPO, it is clear from the proof of Proposition 5.5 that even the joint condition of quasi-transitivity of the base relation and rationalizability in terms of the base relation is not sufficient for H-rationalizability in terms of an FPO.

Our results on H-rationalizability in terms of an FPO suggest that this concept imposes fairly severe restrictions on the nature of choice over two-element sets (since $\bar{R}$ must be quasi-transitive). H-rationalizability also implies Property (β_+), which is an inclusion condition. So far we have not been able to find any necessary *exclusion* condition with some 'bite'. Hence, ECFs with rather 'large' choice sets can be H-rationalized in terms of FPOs. We conclude this section with an example which illustrates this point.

Example 5.2. Let C^* be an ECF such that $\bar{R}$ is quasi-transitive. Moreover, for all $A \subseteq X$ with $|A| \geq 3$:

(i) if $A^* \equiv \{x \in A | x\bar{P}y$ for all $y \in A - \{x\}\} \neq \emptyset$, then $C^*(A) = A^*$;

(ii) $C^*(A) = A$, otherwise.

Choose g, h with $g > h \geq \frac{1}{2}$. For all distinct $x, y \in X$, let $R(x, y) = g$ iff $x\bar{P}y$ and $R(x, y) = h$ iff $y\bar{R}x$. That R satisfies T_1 has been proved in Proposition 5.2. We show that R also H-rationalizes C^*.

Choose any $A \subseteq X$ with $|A| \geq 3$. Suppose for some $x \in A$, $x\bar{P}y$ for all $y \in A - \{x\}$. Then $G(A, R)(x) = g$ and $G(A, R)(y) = h$ for all $y \in A - \{x\}$.

Suppose that $\{x \in A | x\bar{P}y$ for all $y \in A - \{x\}\} = \emptyset$. Then, for every $x \in A$, there is $y \in A - \{x\}$ such that $y\bar{R}x$. Hence, for every $x \in A$, $G(A, R)(x) = h$.

Hence, R H-rationalizes C^*.

6. Concluding remarks

We may now sum up the main points of our discussion in the earlier sections.

(1) As a rationality property of fuzzy preferences T_1 seems to be more interesting than T_3. Also, PO-rationalizability and H-rationalizability seem to be considerably more plausible and interesting notions of rationalizability than NE-rationalizability and D[1]-rationalizability (and possibly than D[α]-rationalizability in general).

(2) Basu's (1984) conclusion that all exact choice functions are NE-rationalizable need not constitute a damning verdict on the hypothesis of fuzzy preference orderings. This conclusion seems to arise from the inherent intuitively unsatisfactory nature of the concept of NE-rationalizability. No such conclusion can be derived with the notions of D[α]-rationalizability, PO-rationalizability and H-rationalizability (see Propositions 5.1, 5.2 and 5.4 and Remarks 5.2 and 5.4).

(3) Does the hypothesis of a fuzzy preference ordering offer greater mileage than the hypothesis of an exact preference ordering so far as explanation of the observed behaviour of an agent is concerned? We have shown (see Proposition 5.1) that there is no gain (as well as no loss) in terms of explanatory power if one adopts D[α]-rationalizability as the relevant notion; this is true not only for the extreme case of $\alpha = 1$ considered by Basu (1984) but in general for every positive α not greater than one. Given the notion of PO-rationalizability, the hypothesis of a fuzzy preference ordering explains a wider range of choice behaviour than the hypothesis of an exact preference ordering. However, even under the notion of PO-rationalizability, fuzzy preference orderings do not explain any wider range of choice phenomena than reflexive, connected and quasi-transitive EWPRs since PO-rationalizability in terms of a fuzzy preference ordering is equivalent to rationalizability in terms of a

reflexive, connected and quasi-transitive EWPR. On the other hand, H-rationalizability does not seem to bear any direct relation to rationalizability in terms of any particular well-known category of EWPRs. Further investigation into the conditions (formulated in terms of properties of the exact choice functions) for H-rationalizability is required before one can comment on the explanatory potential of the concept.

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