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An Intersection Property of Balls and Relations with M-Ideals

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We investigate the following intersection property of balls in a Banach space X: for every $\varepsilon>0$ there is a finite family (x_i) in the open unit ball of X such that $\|x-x_i\|\leqq 1$ for all i implies $\|x\|\leqq\varepsilon$. It was introduced in [3] as an easy-to-check criterion in connection with M-ideals and it can considerably faciliate the study of the M-structure of a Banach space. Apart from the relations to M-ideals a close investigation of this property reveals several connections and applications to other areas in the geometrical theory of Banach spaces.

In *section 1* we give a general study of the intersection property (IP for short). Neither this property nor its negation pass in general to subspaces or quotients, but working with M-ideals we can prove some positive results. We give a new sufficient condition for the IP using the w*-closure of the extreme points of the dual unit ball. We show how the IP and non-IP behave with respect to dense and to separable subspaces. Finally, in Th. 1.7 and Prop. 1.8 we characterize the isomorphic versions of IP and non-IP by showing that any Banach space is isomorphic to a space with IP and a space is isomorphic to a space without IP iff it contains c_0.

We say a Banach space X can be a proper M-ideal if there is a space Y such that X is an M-ideal, but not an M-summand in Y. In [3] it was shown that in this case X fails the IP and it was asked whether the converse is also true. In *section 2* we negatively answer this by constructing a counterexample. A modification of this example together with the IP is used to show that the ultraproduct of C_Σ-spaces is in general not a C_Σ-space, thus giving a new approach to this problem, see [8, 9].

In *section 3* we sort of specialize the IP to one point and show the connection with the strong extreme points introduced in [11]. We give several examples of strong extreme points and classes of Banach spaces where all extreme points are strong extreme points. There are a few obvious relations of the IP with other geometric properties of the unit ball; yet, we give two examples which show that strict convexity or smoothness are not enough to guarantee the IP.

In the final *section 4* we give several conditions and examples which ensure that the space of compact operators on a Banach space has (resp. fails) the

IP. We also show that no M-summands can lie between the finite rank operators and all bounded operators.

Let us fix some *notation*. For Banach spaces X and Y we denote by $K(X, Y)$ [$K(X)$ in the case $X = Y$] the space of compact operators from X to Y and by $L(X, Y)$ [$L(X)$ if $X = Y$] the space of all bounded operators from X to Y. For $1 \leqq p \leqq \infty$ $X \oplus_p Y$ denotes the l^p-direct sum of X and Y. $B_X(a, r)$ is the closed ball with center a and radius r in X (we omit the X if the space is clear from the context). B_X is the unit ball of X and S_X stands for the unit sphere. We write ex A for the extreme points of a set A and $\mathring{A}$ stands for the interior of A. The underlying field $\mathbb{K}$ of all spaces can be either $\mathbb{R}$ or $\mathbb{C}$.

For the definition of an M-ideal and related terminology we refer to [2], for all other unexplained notation see [6, 12], and [16].

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1. The Intersection Property and M-ideals

In this section we study the intersection property of balls introduced in [3]. Although it has in general no nice stability behaviour we get some positive results in connection with M-ideals, for dense subspaces and for separable subspaces. We characterize the "isomorphic versions" of this property and its negation.

Let us first recall the definition and some examples

Definition. A normed space X is said to have the intersection property (IP for short) if for each $\varepsilon > 0$ there is a finite family (x_i), $i = 1, \ldots, n$ in the open unit ball of X such that whenever $\|x - x_i\| \leqq 1$ for all $i = 1, \ldots, n$ we have $\|x\| \leqq \varepsilon$.

It was shown in [3] that $C(K)$-spaces, any Banach space with the Radon-Nikodym property and Banach spaces that have a non-trivial L^p-projection ($1 \leqq p < \infty$) are some of the spaces with this property. On the other hand any Banach space which can be a proper M-ideal fails the IP ([3], Th. 4.3), so $C_0(L)$ with L locally compact, not compact, $K(l^p)$ with $1 < p < \infty$, and $C(\mathbb{T})/A$ provide examples of spaces without IP. We will give several new examples of spaces with IP/without IP throughout this paper.

Remarks. 1) With an obvious modification the definition of the IP makes sense in any metric space. However, with the exception of Lemma 1.5 and Cor. 1.6, we will use it only for Banach spaces.

2) a) If one requires only the existence of a finite family (x_i) in the closed unit ball of X, one gets a strictly weaker property.

b) Replacing the condition $\|x - x_i\| \leqq 1$ by $\|x - x_i\| < 1$ gives an equivalent form of the definition of the IP.

3) The spaces $c_0 \oplus_1 \mathbb{K}$ and $c_0 = c_0 \oplus_\infty \mathbb{K}$ show that neither the IP nor its negation pass to subspaces, quotients or ranges of contractive projections.

4) A straightforward calculation using the definition shows that *if two Banach spaces X and Y are such that their Banach-Mazur distance $d(X, Y) = 1$, then*

X has the IP iff Y has the IP. The renorming of Prop. 1.9 applied to the space c_0 shows that this is the best distance estimate to preserve the *IP*.

In addition to the third remark let us note that the *IP* doesn't pass from a space to an M-ideal in the space (e.g. c_0 is an M-ideal in c), however

Proposition 1.1. *Suppose X is a Banach space with the IP and $M \subset X$ is an M-ideal, then X/M has the IP.*

Proof. Let $\varepsilon > 0$ and get $x_1, \ldots, x_n$ as in the definition of the *IP*. We claim that $(\pi(x_i))_{i=1,\ldots,n}$ will do for X/M, where $\pi: X \to X/M$ denotes the quotient map. Suppose $\|\pi(x) - \pi(x_i)\| < 1$ (see Remark 2b above). Since there are $m_i \in M$ with $\|(x - x_i) - m_i\| < 1$ we have $M \cap \mathring{B}(x - x_i, 1) \neq \emptyset$. Since $\|(x - x_i) - x\| = \|x_i\| < 1$ we have $\bigcap_i \mathring{B}(x - x_i, 1) \neq \emptyset$. The characterisation of M-ideals with the n-ball property for open balls (see e.g. [2], Th. 2.17) now gives an $m \in M$ with $\|(x - x_i) - m\| < 1$ for $i = 1, \ldots, n$. Since X has the *IP* this implies $\|x - m\| \leq \varepsilon$. Therefore $\|\pi(x)\| \leq \|x - m\| \leq \varepsilon$. □

Remark. 1) It is easy to see that if M is an M-summand in X, then X has the *IP* iff both M and its complementary M-summand have the *IP*.

2) If M is an M-ideal in X and both M and X/M have the *IP*, then so does X. [Use Th. 4.3 in [3] and part 1) for a proof.]

The most interesting part of the following result (i.e. if a Banach space X fails the *IP*, then $\overline{\text{ex}\, B_{X^*}}^{w^*}$ contains an interior point of the dual unit ball) was first observed by E. Behrends. If one views elements of X as continuous functions on $\overline{\text{ex}\, B_{X^*}}^{w^*}$, this part supports the intuition that spaces without the *IP* are "vanishing at infinity".

Theorem 1.2. *Let X be a Banach space, then $\overline{\text{ex}\, B_{X^*}}^{w^*} \subset S_{X^*}$ iff $(*)$ holds, where*

$(*)$ *For each $\varepsilon > 0$ there is a finite family $(x_i)_{i \leq n}$ in the open unit ball of X such that for all $p \in \text{ex}\, B_{X^*}$ we have $\max_{1 \leq i \leq n} |p(x_i)| > 1 - \varepsilon$.*

If X satisfies one (hence both) of the above conditions, then X has the IP.

Proof. Suppose $(*)$ fails for X. Denote by $\mathscr{F}$ the set of all finite subsets of $\mathring{B}_X$. For any $F \in \mathscr{F}$ we have by assumption a $p_F \in \text{ex}\, B_{X^*}$ with $\max_{x \in F} |p_F(x)| \leq 1 - \varepsilon$ for one universal $\varepsilon > 0$. Let f be a w^*-accumulation point of the net $(p_F)_{F \in \mathscr{F}}$. Since for $x \in \mathring{B}_X$ we have $\{x\} \in \mathscr{F}$, it follows easily that $|f(x)| \leq 1 - \varepsilon$. Hence $\|f\| < 1$ and $f \in \overline{\text{ex}\, B_{X^*}}^{w^*}$.

Conversely suppose $(*)$ holds. Let (p_α) be a net in $\text{ex}\, B_{X^*}$ converging in weak-$*$-topology to f. Let $\varepsilon > 0$ and choose $x_1, \ldots, x_n$ as in $(*)$. Then $\max_i |p_\alpha(x_i)| > 1 - \varepsilon$ for all α and we find α_0 such that for $\alpha \geq \alpha_0$ we have $|p_\alpha(x_i) - f(x_i)| < \varepsilon$ for $i = 1, \ldots, n$. Now $|f(x_i)| \leq 1 - 2\varepsilon$ for all i would imply

$$|p_\alpha(x_i)| \leq |p_\alpha(x_i) - f(x_i)| + |f(x_i)| \leq \varepsilon + 1 - 2\varepsilon = 1 - \varepsilon$$

for all i. Hence for some i $|f(x_i)| > 1 - 2\varepsilon$. As ε was arbitrary we get $\|f\| = 1$.

To prove the last statement we need the following fact

X has the IP iff for each $\varepsilon>0$ *there is a finite family* $(x_i)_{i\le n}$ *in* B_X *such that for all x there is a* $t\in S_{\mathbb{K}}$ *with* $\|x+t\,x_i\|\le 1$ *for all i implies* $\|x\|\le\varepsilon$.

To see that the latter condition implies IP, take $\varepsilon>0$ and get $(x_i)\subset B_X$ for $\varepsilon/2$. Choose δ with $0<\delta<1$ and $(1+\delta)\max\|x_i\|<1$. Choose a finite δ-net (t_j) in $S_{\mathbb{K}}$. The points $y_{ij}=-(1+\delta)t_j x_i$ now give the IP in X.

Given $\varepsilon>0$ take $x_1,\ldots,x_n$ as in (•). For $x\in X$ choose $p\in\operatorname{ex}B_{X^*}$ with $\|x\|=p(x)$. Choose $j\in\{1,\ldots,n\}$ and $t\in S_{\mathbb{K}}$ with $t\,p(x_j)=|p(x_j)|=\max_i|p(x_i)|>1-\varepsilon$. Now if $\|x+t\,x_i\|\le 1$ for all i, we get

$$\|x\|=p(x)=p(x+t\,x_j)-t\,p(x_j)\le 1-1+\varepsilon=\varepsilon.$$

Hence X has the IP. □

Remarks. 1) Examples of Banach spaces satisfying $\overline{\operatorname{ex}B_{X^*}}^{w^*}\subset S_{X^*}$ are Banach spaces with $\operatorname{ex}B_{X^*}$ w^*-closed (e.g. C_Σ-spaces, see [12], §10) and Banach spaces X for which there is an $x_0\in X$ with $|p(x_0)|=1$ for all $p\in\operatorname{ex}B_{X^*}$ (e.g. L^1-preduals which have an extreme point in the unit ball, see [15], Th. 6.1).

2) The condition $\overline{\operatorname{ex}B_{X^*}}^{w^*}\subset S_{X^*}$ is far from being equivalent to the IP: $X=l^2$ has IP, but satisfies $\overline{\operatorname{ex}B_{X^*}}^{w^*}=B_{X^*}$. (In example 1.3 we show that also spaces without IP can have the latter property.)

There are relations between "classical" geometric properties of the unit ball of a Banach space and the IP (e.g. uniform convexity implies IP). But because of the more "local" character of the IP it doesn't seem appropriate to emphasize these connections. However, it is remarkable that relatively "mild" global conditions – like smoothness (Ex. 1.3) and rotundity (remark 4 following Prop. 3.1) – are not enough to guarantee the IP.

Example 1.3. There is an equivalent norm $|.|$ *on* c_0 *such that* $(c_0,|\,|)^*$ *is strictly convex, hence* $(c_0,|\,|)$ *is smooth, and* $(c_0,|\,|)$ *is an M-ideal in its bidual. In particular* $(c_0,|\,|)$ *is a smooth space without IP.*

Proof. Let $X=c_0$ and take a dense sequence (x_n) in B_X. Define an operator $T\colon l^2\to X$ by $T(\lambda_n)=\sum_n\lambda_n x_n/2^n$. It is easy to see that $T^*\colon X^*\to l^2$ is given by $T^*f=(f(x_n/2^n))_n$ and that T^* is one-to-one. Put for $f\in X^*$ $|f|^*=\|f\|_{X^*}+\|T^*f\|_{l^2}$. Standard arguments show that $|.|^*$ is an equivalent strictly convex dual norm on X^*. Let $|.|$ denote the equivalent norm on X whose dual norm is $|.|^*$.

To show that $(X,|\,|)$ is an M-ideal in its bidual we use the following trick to calculate the new norm $|.|^{***}$ on X^{***} without using $|.|^{**}$. The operator $S\colon(X^*,|.|^*)\to X^*\oplus_1 l^2$ defined by $Sf=(f,T^*f)$ is an isometry, hence also $S^{**}\colon(X^{***},|.|^{***})\to X^{***}\oplus_1 l^2$ is an isometry and $S^{**}F=(F,T^{***}F)$ for $F\in X^{***}$. Since $T^{**}(\lambda_n)=T(\lambda_n)$ (we will identify all Banach spaces with their canonical images in their biduals) we get for $F=f+\varphi\in X^*\oplus X^\perp$ that $T^{***}F=T^*f$. Since X is an M-ideal in its bidual we have with respect to the original norms (see [7], Prop. 3.1) $X^{***}=X^*\oplus_1 X^\perp$. Hence

$$\begin{aligned}|F|^{***}=|f+\varphi|^{***}&=\|f+\varphi\|+\|T^{***}F\|_{l^2}\\&=\|f\|+\|\varphi\|+\|T^*f\|_{l^2}\\&=|f|^*+\|\varphi\|\\&=|f|^{***}+|\varphi|^{***}\end{aligned}$$

i.e. $X^\perp$ is (with respect to the new norm) still an L-summand in X^{***}. ☐

Remarks. 1) The proof shows that the statement in Example 1.3 is true for every separable Banach space X which is an M-ideal in its bidual.

2) The proof also shows that the two norms agree on $X^\perp$. This gives that the w*-closed L-summands in $X^\perp$, i.e. the M-ideals in X^{**}/X, i.e. the M-ideals in X^{**} lying between X and X^{**} remain unchanged. For $X=c_0$ we thus obtain a new (separable!) example for the situation discussed in remark 2) on p. 259 in [7].

The next two results prepare the isomorphic characterization of non-IP in Theorem 1.7.

Lemma 1.4. *Let X be a Banach space and A a dense subset of $\mathring{B}_X$. If there is an $\alpha>0$ such that for all finite families $(x_i)_{i\leq n}$ in A there exists a y in A with $\|y\|>\alpha$ and $\|y-y_i\|\leq 1$, $i=1,\ldots,n$, then X fails the IP.*

Proof. We claim that X fails the IP with $\alpha'=\alpha/2$. Let $x_1,\ldots,x_n\in X$ with $\|x_i\|<1$, $i=1,\ldots,n$. Choose $0<\delta<(1-\max\|x_i\|)/2$. Get $y_i\in A$ with $\|y_i-x_i\|<2\delta$. By hypothesis there is a y with $\|y\|>\alpha$ and $\|y-y_i\|\leq 1$ for $1\leq i\leq n$. Now

$$\begin{aligned}\left\|\frac{y}{2}-x_i\right\|&\leq\tfrac{1}{2}\|y-y_i\|+\tfrac{1}{2}\|y_i-x_i\|+\tfrac{1}{2}\|x_i\|\\&\leq\tfrac{1}{2}+\delta+\tfrac{1}{2}\max\|x_i\|<1\end{aligned}$$

and $\left\|\frac{y}{2}\right\|>\frac{\alpha}{2}=\alpha'$. Hence X fails the IP. ☐

Corollary 1.5. *If Y is a dense subspace of X, then X has the IP iff Y has the IP.*

Proof. If Y fails the IP, then so does X, follows from the above lemma. The other implication is obvious if one uses the remark 2b at the beginning of this section. ☐

The proof of the following proposition is inspired by Thm. 4.4.a in [15].

Proposition 1.6. *If X fails the IP, then for every separable subspace Y of X there exists a separable space Z failing the IP with $Y\subset Z\subset X$.*

Proof. Let (y_n) be a dense sequence in $\mathring{B}_Y$. Get $\alpha>0$ from the definition of non-IP in X. Denote by $\mathcal{G}_1$ the set of all linear combinations of elements of (y_n) with rational coefficients. For each finite family $G\subset\mathcal{G}_1\cap\mathring{B}_X$, choose x_G with $\|x_G\|>\alpha$ and $x_G\in\bigcap_{x_i\in G}B(x_i,1)$. Let $\mathcal{G}_2$ be the set of all linear combinations of elements of $\mathcal{G}_1\cup\{x_G\mid G\subset\mathcal{G}_1\cap\mathring{B}_X,\ G\text{ finite}\}$ with rational coefficients. Repeat the preceding step with finite subsets of $\mathcal{G}_2\cap\mathring{B}_X$. Continuing this way we get a countable collection $\mathscr{G}=\bigcup_n \mathscr{G}_n$ of elements in X such that for any finite subset G of $\mathscr{G}\cap \mathring{B}_X$, there is a $y\in\mathscr{G}$ with $\|y\|>\alpha$ and y is in the intersection of balls with centers from G and radius 1. Put $Z=\overline{\operatorname{span}\mathscr{G}}\,(=\bar{\mathscr{G}})$. Z is a separable subspace which fails the IP by Lemma 1.4. □

The next two results give the characterization of the isomorphic versions of IP and non-IP. Theorem 1.7 has been improved meanwhile by D. Yost [personal communication], since he showed that for a Banach space X containing an isomorphic copy of c_0 is the same as having an equivalent norm with which X can be a proper M-ideal.

Theorem 1.7. *A Banach space X can be renormed to fail the IP iff X contains a subspace isomorphic to c_0.*

Proof. If $T: c_0 \to Y\subset X$ is an isomorphism, renorm first Y by taking $|y|=\|T^{-1}y\|$. Since c_0 fails the IP, $(Y,|\ |)$ fails the IP. Now let $K=(\mathring{B}_{(Y,|\ |)}+\mathring{B}_X)^-$. K is a convex body in X. Denote by $|||.|||$ the equivalent norm on X whose closed unit ball is K. To show that $(X,|||.|||)$ fails the IP we claim that the dense set $A=\mathring{B}_{(Y,|\ |)}+\mathring{B}_X$ satisfies the hypothesis of Lemma 1.4.

Note that since $(Y,|\ |)$ fails the IP there is $\alpha>0$ such that for all finite families $y_1,\dots,y_n$ in $\mathring{B}_{(Y,|\ |)}$ there is a $y\in Y$ with $|y-y_i|<1$ and $|y|>\alpha$. Now for any finite family y_i+x_i in A, i.e. $y_i\in\mathring{B}_{(Y,|\ |)}$ and $x_i\in\mathring{B}_X$ we have with the y corresponding to the y_i's: $y-(y_i+x_i)=(y-y_i)-x_i\in A\subset K$, hence $|||y-(y_i+x_i)|||\leqq 1$ and $|||y|||>C\alpha$, where C is the constant coming from the equivalence of the norms. Consequently $(X,|||.|||)$ fails the IP. The other implication is Th. 4.4 in [3]. □

Proposition 1.8. *Any Banach space X can be renormed (with an arbitrarily small change of the norm) so that the unit ball in the new norm has a strongly exposed point. Consequently X with the new norm has the IP.*

Proof. We call a point x_0 of a closed, bounded, convex, non-empty subset D of a Banach space X strongly exposed if there is an $f\in X^*$ such that for all $\varepsilon>0$ there is a $\delta>0$ such that $x\in D$ and $\operatorname{Re}f(x)>m-\delta$ imply $\|x-x_0\|\leqq\varepsilon$ where $m=\sup\{\operatorname{Re}f(x)\,|\,x\in D\}$. Note that this gives the usual definition for real spaces and it implies (as in the real case) that x_0 is a denting point of D. So if D is unit ball of the new norm, it has the IP by Prop. 4.2 (i) in [3].

To see the renorming take $\rho\geqq 1+1/k>1$ and fix $x_0\in S_X$, $f\in S_{X^*}$ with $f(x_0)=1$. Let $K=\overline{\operatorname{co}}(B_X\cup\rho\,x_0\,B_{\mathbb{K}})$. We have $B_X\subset K\subset\rho\,B_X$ and K is the unit ball of an equivalent norm on X. For $x\in\operatorname{co}(B_X\cup\rho\,x_0\,B_{\mathbb{K}})$, i.e. $x=(1-\lambda)\,y+\lambda\,\alpha\,\rho\,x_0$ with $y\in B_X$, $\alpha\in B_{\mathbb{K}}$ it is easily seen that $\operatorname{Re}f(x)>\rho-\varepsilon$ implies $\rho-\varepsilon<1-\lambda+\lambda\rho\operatorname{Re}\alpha$. But this gives $1-\lambda<\varepsilon/(\rho-1)\leqq k\varepsilon$ and $1-\operatorname{Re}\lambda\alpha\leqq(1-\lambda+\varepsilon)/\rho=:c$, hence $|1-\lambda\alpha|\leqq\sqrt{2c}$. So we get

$$\|\rho\,x_0-x\|=\|\rho(1-\lambda\alpha)\,x_0-(1-\lambda)\,y\|\leqq\rho\,|1-\lambda\alpha|+(1-\lambda)\leqq\rho\sqrt{2(k+1)\varepsilon}+k\varepsilon$$

and $\rho\,x_0$ is a strongly exposed point in K. □

2. A Counterexample

In this section we give an example of a Banach space that fails the IP and can not be a proper M-ideal. As an application we obtain some results on the stability of C_Σ-spaces. (We refer to [12] for the relevant definitions and properties of L^1-preduals.)

Proposition 2.1. *Let (X_n) be a sequence of Banach spaces such that each X_n can not be a proper M-ideal. Then $X = (\oplus \sum X_n)_{l^\infty}$ can not be a proper M-ideal.*

Proof. We consider X_n as naturally embedded in X and denote by P_n the corresponding coordinate projection, which is an M-projection on X with range X_n. Now suppose that X is an M-ideal in a Banach space Z. Since X_n is an M-ideal in X, we have that X_n is an M-ideal in Z ([2], Prop. 2.9). Since X_n is not a proper M-ideal there is an M-projection $Q_n\colon Z \to Z$ with range X_n. Now $Q_n|_X$ is an M-projection, so that by the uniqueness of the range of an M-projection ([7], Prop. 2.1) we have $Q_n|_X = P_n$. Define $Q\colon Z \to X$ by $Q z = (Q_n z)_n$. Clearly Q is a contractive, linear map and for $x \in X$ we have since $x = (P_n x)_n$

$$Q x = (Q_n x)_n = (P_n x)_n = x.$$

Therefore Q is a contractive projection on Z with range X. Since X is an M-ideal in Z, it follows ([7], Cor. 2.2) that X is an M-summand in Z, i.e. X can not be a proper M-ideal. □

Theorem 2.2. *The Banach space $X = (\oplus \sum_{n \in \mathbb{N}} C_\sigma(S^n))_{l^\infty}$, where $S^n = S_{l^2(n+1)}$, $\sigma\colon S^n \to S^n$ is defined by $\sigma(x) = -x$ and $C_\sigma(S^n) = \{f \in C(S^n) \mid f \circ \sigma = -f\}$, satisfies*

a) *X can not be a proper M-ideal*

b) *X fails the IP.*

Proof. ad a): Since $C_\sigma(S^n)$ is a C_Σ-space, the extreme points of the dual unit ball are w^*-closed (see [12], §10). Hence it follows from Theorem 1.2 that the component spaces of X have the IP, thus can not be proper ideals ([3], Th. 4.3). Proposition 2.1 now finishes the proof.

ad b): Let us introduce an auxiliary definition

A Banach space X is said to fail the IP for n, if there is an $\alpha > 0$ such that for any family (x_i) of n points of the open unit ball in X there exists a y with $\|y\| > \alpha$ and $\|y - x_i\| \leq 1$ for $i = 1, \ldots, n$.

To prove that $C_\sigma(S^n)$ fails the IP for n, we will invoke a consequence of a theorem of Borsuk-Ulam (see [1], p. 485, Satz VIII) which says that given $f_1, \ldots, f_n \in C_\sigma(S^n)$, there is an $x_0 \in S^n$ such that $f_i(x_0) = 0$ for $i = 1, \ldots, n$.

Take $\alpha = 1/3$ and let $f_1, \ldots, f_n \in C_\sigma(S^n)$ with $\|f_i\| < 1$ for $i = 1, \ldots, n$. Let $x_0 \in S^n$ be such that $f_i(x_0) = 0$ for $i = 1, \ldots, n$. Choose an open neighbourhood U of x_0 such that $-U \cap U = \phi$ and $|f_i(x)| < 1/2$ for all $x \in U$ and $i = 1, \ldots, n$. Take a continuous function $\tilde{g}\colon S^n \to [0, 1]$ with $\tilde{g}(x_0) = 1$ and $\tilde{g}$ vanishes on $S^n \setminus U$. Put $g(x) = [\tilde{g}(x) - \tilde{g}(-x)]/2$ and $V = -U \cup U$. We have $g \in C_\sigma(S^n)$, $\|g\| = 1/2 > \alpha$, and g vanishes on $S^n \setminus V$. It is easy to verify that $\|g - f_i\| \leq 1$ for $i = 1, \ldots, n$, hence $C_\sigma(S^n)$ fails the IP for n (with $\alpha = 1/3$).

To see that X fails the IP, take $\alpha = 1/3$ and a finite family (f_i), $i = 1, \dots, k$ in $\mathring{B}_X$, $f_i = (f_i(n))$. Since for $n \geqq k$ we have "E fails the IP for $n \Rightarrow E$ fails the IP for k" (with the same α) and $C_\sigma(S^n)$ fails the IP for n, we get for these n a $g(n) \in C_\sigma(S^n)$ with $\|g(n)\| > 1/3$ and $\|g(n) - f_i(n)\| \leqq 1$ for $i = 1, \dots, k$. Put $g(j) = 0$ for $j < k$. The $g \in X$ defined this way satisfies $\|g\| > 1/3$ and $\|g - f_i\| \leqq 1$ for $i = 1, \dots, k$. Hence X fails the IP. □

Remarks. 1) E. Behrends was the first to use the space $C_\sigma(S^n)$ to show that the number n of points x_i in the definition of the IP can be arbitrarily large, thus answering the question in the note on p. 167 in [3].

2) We will see below that the space X in Th. 2.2 is a C_σ-space, but not a C_Σ-space. By part a) of the theorem and [3], Th. 3.4 it contains no proper pseudoball. Therefore it gives the answer to the correct formulation of the problem in [3], p. 167 "... whether C_σ-spaces which are not C_Σ-spaces *always* have pseudoballs which are not balls".

3) With Prop. 2.1 and Th. 2.2 we have: the property "not a proper M-ideal" is stable under l^∞-products, the property "IP" not.

The next proposition is known and only included to show how the IP can be used to distinguish C_σ- and C_Σ-spaces. Part IIb goes back to a question of S. Heinrich [8], which was first answered with topological arguments in [9], Prop. 2.5.

Proposition 2.3.

I a) *The l^∞-product of C_σ-spaces is a C_σ-spaces.*

b) *The ultraproduct of C_σ-spaces is a C_σ-space.*

II a) *The l^∞-product of C_Σ-spaces need not be a C_Σ-space.*

b) *The ultraproduct of C_Σ-spaces need not be a C_Σ-space.*

Proof. Ia) This follows from the characterization of C_σ-spaces as norm-one-complemented subspaces of $C(K)$-spaces (see [12]) and the well-known fact that the l^∞-product of $C(K)$-spaces is again a $C(K)$-space.

Ib) see Prop. 2.1 in [8].

IIa) The space X in Th. 2.2 is the l^∞-product of C_Σ-spaces and fails the IP. Hence by Th. 1.2 the set of extreme points of the dual unit ball can not be w^*-closed, so X is not a C_Σ-space.

IIb) Let X be as in Th. 2.2. For a free ultrafilter U on $\mathbb{N}$ denote by N the subspace of X formed by those $(f(n))$ for which $\lim_U \|f(n)\| = 0$. We claim that X/N fails the IP.

Take $\alpha = 1/3$ and a finite family $(\dot{f}_i)$, $i = 1, \dots, k$ in the open unit ball of X/N. Find $n_i \in N$ with $\|f_i + n_i\| < 1$. Construct a $g \in X$ for $(f_i + n_i)$ as in the last part of the proof of Th. 2.2, i.e. $\|g\| > 1/3$, $\|g - (f_i + n_i)\| \leqq 1$ and $\|g(n)\| > 1/3$ for all $n \geqq k$. Since we work with a free ultrafilter we have $1/3 \leqq \lim_U \|g(n)\| = \|\dot{g}\|$ and of course $\|\dot{g} - \dot{f}_i\| \leqq \|g - f_i - n_i\| \leqq 1$. □

3. Strong Extreme Points and the IP

The easiest way a Banach space X can have the IP is with one point x_0 which has the property (∗) in Prop. 3.1 below. This turns out to be equivalent to

Definition [11]. A point $x_0 \in B_X$ is called a strong extreme point if for every $\varepsilon > 0$ there is a $\delta > 0$ so that if y, z belong to B_X and $\|(y+z)/2 - x_0\| < \delta$, then $\|y-z\| < 2\varepsilon$.

In this section we also give several examples of Banach spaces whose unit balls have strong extreme points.

Proposition 3.1. *Let X be a Banach space and $x_0 \in B_X$. Then x_0 is a strong extreme point iff we have*

(•) *For every $\varepsilon > 0$ there is a $\rho < 1$ such that for any $y \in X$ $\|y \pm \rho x_0\| \leqq 1$ implies $\|y\| \leqq \varepsilon$.*

In particular: Banach spaces which have strong extreme points have the IP.

Proof. Suppose x_0 fails (•). Let $0 < \rho_n < 1$ with $\lim \rho_n = 1$. By hypothesis there is an $\alpha > 0$ and $y_n \in X$ with $\|y_n\| > \alpha$ and $\|y_n \pm \rho_n x_0\| \leqq 1$ for $n \in \mathbb{N}$. For any $\delta > 0$ choose n so that $1 - \rho_n < \delta$. Take $y = \rho_n x_0 + y_n$, $z = \rho_n x_0 - y_n$. Then y, $z \in B_X$ and $(y+z)/2 = \rho_n x_0$, $y - z = 2y_n$. Hence

$$\left\|\frac{y+z}{2} - x_0\right\| = 1 - \rho_n < \delta, \quad \text{but} \quad \|y-z\| > 2\alpha$$

i.e. x_0 is not a strong extreme point.

Conversely suppose we have (•) for x_0. Let us first note that for any $\varepsilon > 0$ the ρ appearing in (•) satisfies $\rho \geqq 1 - \varepsilon$. [To see this observe that $|(1-\rho) \pm \rho| \leqq 1$. Using $y = (1-\rho)x_0$ we get $\|g\| = 1 - \rho \leqq \varepsilon$]. Now let $\varepsilon > 0$ and take $\varepsilon' = \min\{\varepsilon/2, 1/2\}$. Get ρ as in (•) for ε'. Then $\rho \geqq 1 - \varepsilon' \geqq 1/2$. Let $\delta = 1 - \rho$. For any y, $z \in B_X$ with $\|(y+z)/2 - x_0\| < \delta$ we have by taking $d = (y-z)/2$

$$\|\rho x_0 \pm \rho d\| \leqq \left\|\rho x_0 - \rho\left(\frac{y+z}{2}\right)\right\| + \left\|\rho\left(\frac{y+z}{2}\right) \pm \rho d\right\|$$
$$\leqq \rho(1-\rho) + \rho \leqq 1.$$

Therefore $\|\rho d\| \leqq \varepsilon'$, hence $\|y-z\| \leqq 2\varepsilon'/\rho \leqq 2\varepsilon$, i.e. x_0 is a strong extreme point. □

Remarks. 1) It follows from the proof above that if a Banach space X is such that B_X has an ε-strong extreme point for each $\varepsilon > 0$ (see [11], i.e. X has the AKMP in their notation), then X has the *IP*.

2) It is well-known that midpoint locally uniform rotundity (MLUR) (see [20] for relevant definitions – and for an answer to question 4 on p. 174 in [11]) is equivalent to saying that every point on the surface of the unit ball is a strong extreme point. Consequently any such space has the *IP*.

3) It is easy to see that we have the implications: x_0 denting point $\Rightarrow x_0$ strong extreme point $\Rightarrow x_0$ extreme point and none of the arrows can be reversed. (Below we give a striking example for the last claim.) Still it seems that strong extreme points are closer to extreme points than to denting points. This is supported by Prop. 3.5 and the following observation (which can easily be deduced from the proof of Th. 6.1 in [15]): *Every extreme point in an L^1-predual space is a strong extreme point.*

4) The space $C(\mathbb{T})/A$, where $\mathbb{T}$ denotes the unit circle in the complex plane and A the disc algebra, is a strictly convex Banach space which is an M-ideal in its bidual. (The M-ideal part of the last statement is in [17] and follows also from Ex. 3.3a and Th. 3.4.a in [7] with the identification of $C(\mathbb{T})/A$ and the space of compact Hankel operators. To see the strict convexity, show that $H_0^1=(C(\mathbb{T})/A)^*$ is smooth by using the description of smooth points in $L^1(\mu)$-spaces and the F. and M. Riesz theorem.) Consequent $C(\mathbb{T})/A$ is a strictly convex space that fails the IP, in particular it has no strong extreme points.

A weaker form of the next result for the more general situation of closed bounded convex sets was proved in [11]:

Proposition 3.2. *If x_0 is a strong extreme point of B_X, then it is also a strong extreme point of $B_{X^{**}}$.*

Proof. Let $\varepsilon>0$ and put $\varepsilon'=\varepsilon/2$. By hypothesis and Prop. 3.1 there is a $\rho<1$ such that $\|y\pm\rho x_0\|\leq 1$ implies $\|y\|\leq\varepsilon'$. Choose $\delta>0$ so that $\rho'=(1+\delta)\rho<1$ and $\delta\leq\sqrt{2}-1$. Now if $y^{**}\in X^{**}$ and $\|y^{**}\pm\rho' x_0\|\leq 1$, using the principle of local reflexivity, we can get an operator T: $\operatorname{span}\{x_0, y^{**}\}\to X$ so that $Tx_0=x_0$ and $\|T\|$, $\|T^{-1}\|\leq 1+\delta$. Taking $y=Ty^{**}$ we have $\|y\pm\rho' x_0\|\leq 1+\delta$ so that $\|y/(1+\delta)\pm\rho x_0\|\leq 1$, hence $\|y\|\leq\varepsilon'(1+\delta)$. Therefore $\|y^{**}\|=\|T^{-1}y\|\leq(1+\delta)^2\varepsilon'\leq 2\varepsilon'=\varepsilon$, i.e. x_0 is a strong extreme point of $B_{X^{**}}$. □

The following proposition is essentially known ([4], p. 38). Note, however, that the crucial step in the proof – "the numerical radius is an equivalent norm" – is only true for complex Banach algebras; but it is not hard to circumvent this difficulty:

Proposition 3.3. *If A is a Banach algebra with identity e, then e is a point of local uniform convexity of B_A, in particular a strong extreme point.*

Proof. The complex case is Th. 5 on p. 38 in [4]. If A is real, consider A in its complexification $A_{\mathbb{C}}$ (see [18], Th. 1.3.2). Since e is still the identity and the restriction of the norm of $A_{\mathbb{C}}$ to A is the original norm of A, it follows from the definition of a point of local uniform convexity that e has this property also in A. □

Corollary 3.4. *For any Banach space X, the identity operator id_X and every isometric isomorphism T of X are strong extreme points of $B_{L(X)}$.*

Proof. See the remark on p. 38 in [4].

Proposition 3.5. *Let X be a Banach space, $x_0\in S_X$, and $1\leq p<\infty$. If $X=Y\oplus_p\mathbb{K}x_0$, then x_0 is strongly exposed in B_X – in particular a strong extreme point. Consequently every extreme point of the unit ball of an $L^1(\mu)$-space is a strong extreme point.*

Proof. The second statement comes from the fact that extreme points of $L^1(\mu)$-spaces generate 1-dimensional L-summands. To see the first write $X^*=Y^*\oplus_q\mathbb{K}f$ with $Y=\ker f$, $f(x_0)=1$, and $\|f\|=1$. If we have for $\varepsilon>0$ that $\operatorname{Re} f(x)>1-\varepsilon$ for an $x\in B_X$ we get with $\|x\|^p=\|y+\alpha x_0\|^p=\|y\|^p+|\alpha|^p\leq 1$, with $\operatorname{Re} f(x)=\operatorname{Re}\alpha$, and some calculation that $\|x_0-x\|^p\leq(2\varepsilon)^{p/2}+1-(1-\varepsilon)^p$. Since the right hand side of the last inequality can be made arbitrarily small, we get the claim. □

4. *IP* for Spaces of Compact Operators

In this section we give several conditions and examples which ensure that $K(X)$, the space of compact operators on a Banach space X, has (rep. fails) the IP.

Theorem 4.1. *Suppose X is a Banach space such that $\overline{\operatorname{ex} B_{X^*}}^{w^*} \subset S_{X^*}$ and X^* has the IP, then M has the IP for any space M with $X^* \otimes X \subset M \subset L(X)$.*

Proof. Let $\varepsilon > 0$. Since X^* has the IP choose $f_1, \ldots, f_n$ as in the definition and put $\eta = \max \|f_i\|$. Choose a δ with $0 < \delta < 1/\eta - 1$. Since $\overline{\operatorname{ex} B_{X^*}}^{w^*} \subset S_{X^*}$ we get by Theorem 1.3 $x_1, \ldots, x_m$ in $\mathring{B}_X$ such that $\max_j |p(x_j)| > 1 - \delta$ for all $p \in \operatorname{ex} B_{X^{**}}$. The operators $(f_i \otimes x_j)_{\substack{i=1,\ldots,n\\ j=1,\ldots,m}}$ are in the open unit ball of M. Now assume we have for $T \in M$ $\|T - f_i \otimes x_j\| \leq 1$ for $i = 1, \ldots, n$, $j = 1, \ldots, m$. Fix $p \in \operatorname{ex} B_{X^*}$ and choose j_0 such that $|p(x_{j_0})| = \max_j |p(x_j)| > 1 - \delta$. Passing to $\bar{p} = e^{it} p$ for some t, we can assume that $r = p(x_{j_0}) = |p(x_{j_0})|$. Then $1 - \delta < r \leq 1 < 1/\eta$, so that

$$1 = \lambda r + (1 - \lambda)\, 1/\eta \quad \text{where} \quad \lambda = \frac{1/\eta - 1}{1/\eta - r}.$$

Now $f_i = \lambda r f_i + (1 - \lambda) f_i/\eta$ and since

$$\|T^* p - p(x_{j_0}) f_i\| = \|T^* p - (f_i \otimes x_{j_0})^* p\| \leq 1 \qquad \text{for all } i$$

we have

$$\|\lambda T^* p - f_i\| \leq \lambda \|T^* p - r f_i\| + (1 - \lambda) \|f_i\|/\eta \leq 1 \qquad \text{for all } i.$$

By the choice of the f_i's this implies $\|\lambda T^* p\| \leq \varepsilon$. But since $1 - \delta < r$ we get

$$\frac{1}{\lambda} = \frac{1/\eta - r}{1/\eta - 1} < \frac{1/\eta - 1 + \delta}{1/\eta - 1} = 1 + \frac{\delta}{1/\eta - 1} < 2.$$

Therefore $\|T^* p\| \leq 2\varepsilon$. Since this is true for any $p \in \operatorname{ex} B_{X^*}$ we get $\|T^*\| \leq 2\varepsilon$ and consequently M has the IP. □

With a similar proof one gets

Corollary 4.2. *If $x_0 \in X$ is such that $|p(x_0)| = 1$ for all $p \in \operatorname{ex} B_{X^*}$ and $f \in B_{X^*}$ is a strong extreme point, then $f \otimes x_0$ is a strong extreme point of the unit ball of $K(X)$.*

Remarks. 1) It is not trivial (and sometimes not possible!) to exhibit extreme points of $B_{K(X)}$. Corollary 4.2 gives the existence at least for L^1-preduals X with $\operatorname{ex} B_X \neq \emptyset$.

2) It is known that the unit ball of $K(l^2)$ has no extreme points [we get the non-existence of strong extreme points easily from the fact that $K(l^2)$ fails the IP as being an M-ideal in $L(l^2)$]. Therefore the assumption about x_0 in Cor. 4.2 can not be weakened to "x_0 a strong extreme point in B_X".

3) The assumption in Cor. 4.2 can be modified to "$x_0 \in B_X$ is a strong extreme point and $f \in X^*$ is such that $|x^{**}(f)| = 1$ for all $x^{**} \in \mathrm{ex}\, B_{X^{**}}$". In the next proposition this gives the existence of strong extreme points in $B_{K(X)}$ if X is an $L^1(\mu)$-space with $\mathrm{ex}\, B_X \neq \emptyset$ (cf. Prop. 3.5 and Prop. 3.2).

Proposition 4.3. *For any $L^1(\mu)$-space X, $K(X)$ has the IP.*

Proof. Since X has the approximation property and $X^* = C(T)$ for some compact space T we have $K(X) = X^* \tilde{\otimes}_\varepsilon X = C(T) \tilde{\otimes}_\varepsilon X = C(T, X)$ (the space of X-valued continuous functions on T), see [6], Chap. VIII. However, it is easy to see that if E is a Banach space with the IP and S a compact space, then $C(S, E)$ has the IP. In the present situation since X has non-trivial L-projections, it has the IP (cf. Prop. 4.2 (vi) in [3]). Therefore $K(X)$ has the IP. □

The next proposition is an improvement, due to D. Werner, of one of our former results. Note that (with the notation of Prop. 4.4) $K(X)$ is not an M-ideal in $L(X)$ if L is not discrete (combine the remark following Th. 5 in [14] and Th. 2 in [14]). See also the open problems at the end of this paper.

Proposition 4.4. *Let L be a locally compact, not compact space, αL the one-point compactification of L, Y any Banach space, and $X = C_0(L) \oplus_\infty Y$. Then $K(X)$ is a proper M-ideal in $K(X, C(\alpha L) \oplus_\infty Y)$, in particular $K(X)$ fails the IP.*

Proof. Using the M-projections in X and $C(\alpha L) \oplus_\infty Y$ we get with Prop. 6.1 in [13] the decompositions $K(X) = K(X, C_0(L)) \oplus_\infty K(X, Y)$ and $K(X, C(\alpha L) \oplus_\infty Y) = K(X, C(\alpha L)) \oplus_\infty K(X, Y)$. Since $C_0(L)$ and $C(\alpha L)$ have the approximation property, we know $K(X, C_0(L)) = X^* \tilde{\otimes}_\varepsilon C_0(L) = C_0(L, X^*)$ and $K(X, C(\alpha L)) = X^* \tilde{\otimes}_\varepsilon C(\alpha L) = C(\alpha L, X^*)$. It is known that $C_0(L, X^*)$ is a proper M-ideal in $C(\alpha L, X^*)$ ([2], Cor. 10.2) and being a proper M-ideal is stable with respect to the $\oplus_\infty$-sum with an arbitrary Banach space. □

The proof of the following result is inspired by Prop. 2 in [10].

Proposition 4.5. *Let X be a Banach space and M be a space such that $X^* \otimes X \subset M \subset L(X)$. For $T \in L(X)$ denote by $P_M(T) = \{S \in M \mid \|S - T\| = d(T, M)\}$ the set of best approximants for T from M.*

Then $P_M(T)$ has empty interior (relative to M) for all $T \in L(X)$, in particular: if $X^* \otimes X \subset M \subsetneq L(X)$, then M is not an M-summand in $L(X)$. Consequently if such an M is an M-ideal in $L(X)$, it fails the IP.

Proof. The last statement follows from the second and Th. 4.3 in [3]. The second statement is an immediate consequence of the first, since we have: if M is an M-summand in E and $e \in E$, then $P_M(e) = B_M(Qe, d(e, M))$, where Q denotes the M-projection from E onto M.

If the first claim is not true we can assume (after a suitable translation) that 0 is an interior point in $P_M(T)$ for some $T \in L(X) \setminus M$, i.e. there is a $\delta > 0$ such that $S \in M$ and $\|S\| < \delta$ implies $\|T + S\| = \|T\|\ [= d(T, M)]$. Now choose $x \in S_X$ with $\|Tx\| > \|T\| - \delta/4$ and $f \in S_{X^*}$ with $f(x) = 1$. Let $T_\delta = (\delta/2) f \otimes Tx/\|Tx\|$. Then $T_\delta \in X^* \otimes X \subset M$, $\|T_\delta\| = \delta/2$, and

$$\|T+T_\delta\| \geq \|(T+T_\delta)x\| = \left\|Tx + \frac{\delta}{2}\frac{Tx}{\|Tx\|}\right\|$$
$$= \|Tx\| + \frac{\delta}{2} > \|T\| + \frac{\delta}{4} > \|T\|$$

So $S = T_\delta$ gives a contradiction and the claim is proved. □

Remarks. 1) In some situations Prop. 4.5 combined with arguments using the IP give complete information about the M-ideals between $K(X)$ and $L(X)$:

- *If X is infinite dimensional and $K(X)$ has the IP, then $K(X)$ is not an M-ideal in $L(X)$ [Prop. 4.5]*
- *If in addition X is such that $K(X)^{**} = L(X)$, then there are no M-ideals between $K(X)$ and $L(X)$* [A proof similar to the one for Prop. 3.2 shows: If a Banach space E has the IP, then E^{**} has the IP (which points from E!), i.e. any space F with $E \subset F \subset E^{**}$ has the IP. Apply this and for a second time Prop. 4.5]

2) a) The following problem from [3] is still open: Does every dual space have the IP? The answer is affirmative for separable duals by Prop. 4.2.v in [3]. Let us give another partial result: *If X^* is such that for any separable subspace Y of X^* there is a separable subspace Z with $Y \subset Z \subset X^*$ and Z complemented in X^*, then X^* has the IP.* (The assumption about X^* is satisfied e.g. if X^* is weakly compactly generated, [5], p. 149). Proof: If X^* fails the IP, so does a separable subspace Y (Prop. 1.6). Hence $c_0 \hookrightarrow Y \subset Z \subset X^*$ with Z complemented and separable (Th. 1.7). From Th. 1.3 in [19] it follows that Z contains a subspace isomorphic to l^∞, which gives the contradiction.

b) We don't know whether $K(X)$ has the IP implies X has the IP. Proposition 4.4 gives a positive answer for a special case.

Added in proof. Proposition 4.5 already appeared as Theorem 2.9 in K. Saatkamp – Best Approximation in the Space of Bounded Operators and its Applications, Math. Ann. 250, 35–54 (1980)

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