

ON A GENERALIZATION OF MARTINGALES DUE TO BLAKE

R. SUBRAMANIAN

It is shown that any uniformly integrable fairer with time game (stochastic process) converges in L_1 .

1. **Introduction.** Let (Ω, \mathcal{F}, P) be a probability space and $\{\mathcal{F}_n\}_{n \geq 1}$ an increasing family of sub σ -algebras of \mathcal{F} . Let $\{X_n\}_{n \geq 1}$ be a stochastic process adapted to $\{\mathcal{F}_n\}_{n \geq 1}$ (see, [2, p. 65]). Following Blake [1] we refer to $\{X_n\}_{n \geq 1}$ as a game and define

DEFINITION. The game $\{X_n\}_{n \geq 1}$ will be said to become *fairer with time* if for every $\varepsilon > 0$

$$P[|E(X_n/\mathcal{F}_m) - X_m| > \varepsilon] \rightarrow 0$$

as $n, m \rightarrow \infty$ with $n \geq m$. Any martingale is, trivially, a fairer with time game and thus this concept generalizes that of martingales. Blake, in [1], gave a set of sufficient conditions under which any uniformly integrable fairer with time game $\{X_n\}_{n \geq 1}$ is convergent in L_1 . We show that these sufficient conditions are not needed; in fact, we show that any uniformly integrable, fairer with time game converges in L_1 .

2. **THEOREM 2.1.** *Any uniformly integrable fairer with time game $\{X_n\}_{n \geq 1}$ converges in L_1 .*

Proof. To facilitate understanding, we break up the proof into a few important steps numbered (S1) through (S5). For every m and $n \geq m$ define $Y_{m,n} = E(X_n/\mathcal{F}_m)$. Let Γ stand for the family $\{Y_{m,n}$, for all m and $n \geq m\}$.

(S1) Γ is uniformly integrable.

Since $\{X_n\}_{n \geq 1}$ is uniformly integrable there exists a function f defined on the nonnegative real axis which is positive, increasing and convex, such that

$$\lim_{t \rightarrow \infty} \frac{f(t)}{t} = +\infty$$

and $\sup_n E[f \circ |X_n|] < \infty$. (See [2, II T 22].) Now,

$$\begin{aligned} E[f \circ |Y_{m,n}|] &= E[f \circ |E(X_n/\mathcal{F}_m)|] \\ &\leq E[f \circ E(|X_n|/\mathcal{F}_m)] \quad (\text{since } f \text{ is nondecreasing}) \\ &\leq E[E(f \circ |X_n|/\mathcal{F}_m)] \\ &= E[f \circ |X_n|]. \end{aligned}$$

Therefore,

$$\sup_{Y_{m,n} \in \Gamma} E[f \circ | Y_{m,n} |] \leq \sup_{X_n} E[f \circ | X_n |] < \infty.$$

Another application of II T 22 of [2] ensures that Γ is uniformly integrable. Hence (S1).

(S2) Given $\varepsilon > 0$, there exists M such that for all $m \geq M$, one has

$$E(|X_m - Y_{m,n}|) \leq 2\varepsilon \text{ for all } n \geq m.$$

Since Γ is uniformly integrable given $\varepsilon > 0$ there exists $\delta > 0$ such that $P(A) < \delta$ implies $\int_A |Y_{m,n}| dP \leq \varepsilon/2$, for all $Y_{m,n} \in \Gamma$. Choose M so large that $m \geq M$ and $n \geq m$ implies $P(|X_m - E(X_m/U_m)| > \varepsilon) < \delta$. Then, it is not difficult to see that

$$E[|X_m - Y_{m,n}|] \leq 2\varepsilon \text{ for all } m \geq M \text{ and } n \geq m.$$

(S3) For every fixed m , the sequence $\{Y_{m,n}\}$ converges in L_1 to an \mathscr{Z}_m measurable random variable Z_m .

Let $m \leq n < n'$.

$$\begin{aligned} E[|Y_{m,n} - Y_{m,n'}|] &= E[|E(X_n/\mathscr{Z}_m) - E(X_n'/\mathscr{Z}_m)|] \\ &= E[|E(X_n - X_n'/\mathscr{Z}_m)|] \\ &= E[|E\{E(X_n - X_n'/\mathscr{Z}_n)|\mathscr{Z}_m\}|] \\ &\leq E[|E\{(|E(X_n - X_n'/\mathscr{Z}_n)|)|\mathscr{Z}_m\}|] \\ &= E[|E(X_n - X_n'/\mathscr{Z}_n)|] \\ &= E[|X_n - Y_{m,n'}|]. \end{aligned}$$

Now from (S2) it follows that given $\varepsilon > 0$ for all sufficiently large n and n'

$$E[|Y_{m,n} - Y_{m,n'}|] \leq E[|(X_n - Y_{m,n'})|] \leq 2\varepsilon.$$

Hence, for m fixed, the sequence $\{Y_{m,n}\}$ is Cauchy in the L_1 -norm. So, there exists, an integrable random variable Z_m , such that, $Y_{m,n} \xrightarrow[n \rightarrow \infty]{L_1} Z_m$. Without loss of generality we can take Z_m to be \mathscr{Z}_m measurable. (Note that each $Y_{m,n}$ is \mathscr{Z}_m measurable and there is a subsequence $\{Y_{m,n'}\}$ converging almost surely to Z_m .)

(S4) $\{Z_m, \mathscr{Z}_m\}_{m \geq 1}$ is a uniformly integrable martingale.

The fact that $\{Z_m\}_{m \geq 1}$ is uniformly integrable follows trivially because the closure in L_1 of a uniformly integrable collection is uniformly integrable. (See, [2, II T20].) To show $\{Z_m, \mathscr{Z}_m\}$ is a martingale it is enough to show that for every m , $E(Z_{m+1}/\mathscr{Z}_m) = Z_m$ a.s. Since

$$\begin{aligned}
 E[|E(Y_{m+1,n}/\mathcal{Z}_m) - E(Z_{m+1}/\mathcal{Z}_m)|] \\
 &= E[|E((Y_{m+1,n} - Z_{m+1})/\mathcal{Z}_m)|] \\
 &\leq E[|E((Y_{m+1,n} - Z_{m+1})/\mathcal{Z}_m)|] \\
 &= E[|Y_{m+1,n} - Z_{m+1}|] \longrightarrow 0 \text{ as } n \longrightarrow \infty,
 \end{aligned}$$

there exists a subsequence n' of $\{n: n \geq m\}$ such that

$$E(Y_{m+1,n'}/\mathcal{Z}_m) \xrightarrow{\text{a.s.}} E(Z_{m+1}/\mathcal{Z}_m).$$

We can assume (— if necessary, by choosing a further subsequence, —) that $Y_{m,n} \xrightarrow{\text{a.s.}} Z_m$. Now,

$$\begin{aligned}
 E(Z_{m+1}/\mathcal{Z}_m) &= \lim_{n' \rightarrow \infty} E(Y_{m+1,n'}/\mathcal{Z}_m) \text{ a.s.} \\
 &= \lim_{n' \rightarrow \infty} E((E(X_{n'}/\mathcal{Z}_{m-1}))/\mathcal{Z}_m) \text{ a.s.} \\
 &= \lim_{n' \rightarrow \infty} E(X_{n'}/\mathcal{Z}_m) \text{ a.s.} \\
 &= \lim_{n' \rightarrow \infty} Y_{m,n'} \text{ a.s.} \\
 &= Z_m \text{ a.s.}
 \end{aligned}$$

Hence (S4). (S5) $\{X_n\}_{n \geq 1}$ converges in L_1 .

Since $\{Z_n, \mathcal{Z}_n\}_{n \geq 1}$ is a uniformly integrable martingale, there exists an integrable random variable Z_∞ such that $Z_n \xrightarrow[n \rightarrow \infty]{L_1} Z_\infty$. We shall show that $X_n \xrightarrow[n \rightarrow \infty]{L_1} Z_\infty$. From (S3) and (S2) it is easy to check that given $\varepsilon > 0$ there exists M such that for all $m \geq M$

$$\int |X_m - Z_m| dP \leq 2\varepsilon.$$

Therefore, for sufficiently large m ,

$$\int |X_m - Z_\infty| dP \leq \int |X_m - Z_m| dP + \int |Z_m - Z_\infty| dP \leq 3\varepsilon,$$

say. Hence (S5) and the theorem.

Since any game (stochastic process) $\{X_n\}_{n \geq 1}$ converging in L_1 can be taken to be a game fairer with time, by setting $\mathcal{Z}'_n \equiv \mathcal{Z}$ in n , we get the following corollary.

COROLLARY 2.1. *Let $\{X_n\}_{n \geq 1}$ be a game. It converges in L_1 if and only if it is uniformly integrable and fairer with time with respect to some increasing family of sub σ -algebras $\{\mathcal{Z}'_n\}_{n \geq 1}$ to which it is adapted.*

Let $p > 1$.

THEOREM 2.2. *Let $(X_n)_{n \geq 1}$ be a fairer with time game with $(\|X_n\|^p)_{n \geq 1}$ uniformly integrable. Then $(X_n)_{n \geq 1}$ converges in L_p .*

Proof. Noting that the function f defined on the nonnegative real axis by $f(t) = t^p$ is positive, increasing and convex and $\lim_{t \rightarrow \infty} (f(t)/t) = +\infty$, in view of II T 22 of [2], it is clear that $(X_n)_{n \geq 1}$ is uniformly integrable. Hence by Theorem 2.1 it converges in L_1 ; in particular, $(X_n)_{n \geq 1}$ converges in probability. Therefore, $(X_n)_{n \geq 1}$ converges in L_p . (See Proposition II 6.1 of [3].)

COROLLARY 2.2. *The game $(X_n)_{n \geq 1}$ converges in L_p if and only if $(\|X_n\|^p)_{n \geq 1}$ is uniformly integrable and $(X_n)_{n \geq 1}$ is fairer with time with respect to some increasing family of sub σ -algebras $(\mathcal{Z}_n)_{n \geq 1}$ to which it is adapted.*

REMARK. In view of our Theorem 2.1, the second convergence theorem of Blake in [1] becomes redundant.

REFERENCES

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INDIAN STATISTICAL INSTITUTE
CALCUTTA-35, INDIA