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THE EQUIVALENCE OF BEST PLANE FIT GRADIENT WITH ROBERT's, PREWITT's AND SOBEL's GRADIENT FOR EDGE DETECTION AND A 4-NEIGHBOUR GRADIENT WITH USEFUL PROPERTIES

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Abstract. The best plane fit gradient edge detection technique is reexamined. It is shown that this technique for some different mask sizes and different point operator is equivalent to Robert's, Prewitt's, Sobel's and Huckel's gradient techniques. A 4-neighbour edge detection technique is defined and it is shown that this technique satisfies many desirable properties of an edge detector.

Zusammenfassung. Die Technik des "best plane fit gradient edge detection" wird untersucht. Es wird gezeigt dass diese Technik vergleichbar ist, für verschiedene Maskengrössen und verschiedene Punkt-Operatoren, zu den Methoden von Robert, Prewitt, Sobel and Huckel. Eine 4-Nachbarn Kantendetektionstechnik wird definiert, und es wird gezeigt dass diese Technik viele der wünschenswerten Eigenschaften eines Kantendetektors hat.

Résumé. La technique d'adaptation du meilleur plan pour la détection des contours de gradient est réexaminée. Il est montré que cette technique est équivalente aux techniques de Robert, Prewitt, Sobet et Huckel pour certaines dimensions de marques différentes et opérateurs ponctuels différents. Une détection de contour à 4 voisins est définie et il est montré que cette technique possède plusieurs propriétés souhaitables d'un détecteur de contour.

Keywords. Image processing, edge detection, best plane fit gradient, Robert's gradient, Prewitt's gradient, Sobel's gradient, 4-Neighbour gradient.

1. Introduction

It is well known that the visual information is concentrated at points of large spatial variation of light intensity in the picture. Thresholding of the spatial gradient of gray levels is one of the popular techniques of edge detection. The technique is useful in picture segmentation and description also.

The notion of gradient has been extended for digital image processing problems and a number of different gradients have been proposed in recent times. Some of the attractive proposals are due to Robert [1], Prewitt [2], Sobel [3] and Rosenfeld and Thurston [4]. Other techniques include the estimation of slope of best plane fit (bpf) or step edge fit [5] in the least mean square error sense and the template matching [6]. The different techniques are aimed at meeting various desirable properties the edge detector should possess. The desirable properties are: low sensitivity to edge orientation, rapidly declining edge gradient response for offset mask, low false edge detection rate in presence of noise, high figure of merit, versatility and high implementation efficiency in both hardware and software. Abdou and Pratt [7] as well as Deutsch and Fram [8] studied some of the properties for a few edge detection techniques. It is quantitatively shown [7] that Prewitt's and Sobel's gradient are among the best.

The present note reexamines the bpf gradient edge detection technique for its relation with other techniques. It is interesting to note in Section 2 that Robert's and Prewitt's gradients are equivalent to the bpf gradient for a 2×2 and 3×3 pel masks, respectively. Also, bpf gradient with sum of magnitude operator for 2×2 pel mask is equivalent to Hueckel operator. If some of the pels of a 3×3 pel mask is properly weighted, a similar relation between bpf gfadient and Sobel's gradient can also be found. Thus bpf can be considered as a general and powerful class of techniques deserving more attention.

In Section 3 a different 2×2 pel mask has been considered for bpf as well as its equivalent Robert-like difference gradient. It is shown that the gradient, called 4-neighbour gradient, satisfies many desirable properties including applicability to two-tone image problems. Test results to ideal edge image has also been presented.

2. Robert's, Prewitt's, Sobel's operators and their relationship with bpf operator

Let $x(i, j)$ denote the gray level of the candidate pel at ith row and jth column. For brevity let $x(i, j)$ be written as x_0 and the gray levels of its neighbouring pels $x(i-1, j)$, $x(x-1, j-1)$, $x(i, j-1)$, $x(i+1, j-1)$, $x(i+1, j)$, $x(i+1, j+1)$, $x(i, j+1)$ and $x(i-1, j+1)$ be written as $x_1, x_2, \ldots x_8$, respectively (Fig. 1). Then,

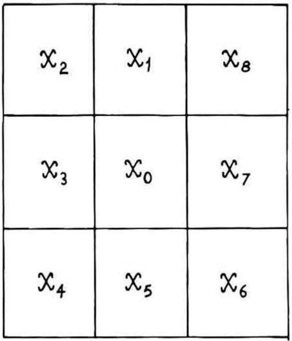

Fig. 1. A 3×3 mask used for finding gradient of candidate pel x_0.

the ordinary gradient, Robert's gradient and Prewitt's and Sobel's gradient functions are given, respectively, by

$$G_1 = x_0 - x_1, \quad G_2 = x_0 - x_3. \tag{1}$$

$$G_1 = x_0 - x_2, \quad G_2 = x_1 - x_3, \tag{2}$$

$$G_1 = \frac{1}{2+w}(x_4 + wx_5 + x_6) - (x_2 + wx_1 + x_8),$$

$$G_2 = \frac{1}{2+w}(x_6 + wx_7 + x_8) - (x_2 + wx_3 + x_4), \tag{3}$$

where $w=1$ for Prewitt's and $w=2$ for Sobel's gradient. The gradient A is given by a point operator O_p on G_k, $k=1,2$. The point operator may be root mean square (rms), magnitude average or maximum of magnitudes. For example, if O_p signify magnitude average then $A=O_p\{G_1, G_2\}=\frac{1}{2}[|G_1|+|G_2|]$. Because of the nature of point operators, G_1 and G_2 in (1–3) can be defined interchangeably without affecting the value of A. An edge is deemed present if A exceeds a predefined threshold, say, t.

The gradient functions can also be found as a two-dimensional spatial convolution or vector product of image array $X(i,j)$ with mask functions $H_k(i,j)$

$$G_k(i,j)=X(i,j)\circledast H_k(i,j), \tag{5}$$

where $\circledast$ denotes the convolution or vector product operation. For example, the Robert's gradient mask functions for convolution are

$$H_1=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix},\quad H_2=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}.$$

The maximum value of k depends on the specific edge detector. However, $k>4$ is uncommon and for many detectors a value of $k=2$ is used.

In the bpf technique a plane $Z=ax+by+c$ is fitted to the pel gray levels under consideration. An error of fit is defined and the error is minimized with respect to a, b and c. The solutions a and b give the slope of the plane. For example, let x_0, x_1, x_2 and x_3 be the gray levels of the pels to be fitted and let the error be squared Euclidean distance given by

$$e=\{ai+bj+c-x_0\}^2+\{a(i-1)+bj+c-x_1\}^2+\{a(i-1)+b(j-1)+c-x_2\}^2 \\ +\{ai+b(j-1)+c-x_3\}^2.$$

Setting the partial derivatives of e with respect to a, b and c equal to zero, we get the solutions

$$a=\frac{x_0+x_3}{2}-\frac{x_1+x_2}{2},\quad b=\frac{x_0+x_1}{2}-\frac{x_2+x_3}{2}. \tag{6}$$

The gradient A is given by a slightly different point operator O'_p. Here $A=O'_p\{a,b\}$ signify square root of squared sum (s.r.s.s.) (i.e., $A=\sqrt{[a^2+b^2]}$) instead of r.m.s. or sum of magnitude (i.e., $A=|a|+|b|$) instead of magnitude average in O_p. The max operator, however, is the same in both cases. It is seen that a and b are similar to G_1 and G_2, respectively.

Proposition 1. *The* bpf *gradient in* s.r.s.s. *sense is equal to Robert's gradient in* r.m.s. *sense while the* bpf *gradient in sum of magnitude sense is equal to Robert's gradient in max sense and vice-versa.*

Proof. From (6)

$$\sqrt{[a^2+b^2]}=\left[\left(\frac{x_0+x_3}{2}-\frac{x_1+x_2}{2}\right)^2+\left(\frac{x_0+x_1}{2}-\frac{x_2+x_3}{2}\right)^2\right]^{1/2} \\ =[\tfrac{1}{2}\{(x_0-x_2)^2+(x_1-x_3)^2\}]^{1/2},$$

which is Robert's gradient in r.m.s. sense.

Again,

$$|a|+|b| = \left|\frac{x_0+x_3}{2} - \frac{x_1+x_2}{2}\right| + \left|\frac{x_0+x_1}{2} - \frac{x_2+x_3}{2}\right|$$

$$= \tfrac{1}{2}[|(x_0-x_2)-(x_1-x_3)| + |(x_0-x_2)+(x_1-x_3)|]$$

$$= \tfrac{1}{2}\cdot 2\max\{|x_0-x_2|, |x_1-x_3|\},$$

which is the Robert's gradient in the max sense. Similarly,

$$\max[|a|, |b|] = \tfrac{1}{2}[|a+b| + |a-b|]$$

$$= \tfrac{1}{2}[|x_0-x_2| + |x_1-x_3|],$$

which is the Robert's gradient in the magnitude average sense. Q.E.D.

Now, Rosenfeld [9] showed that Robert's gradient with max operator is a Hueckel-type edge detector. Hence, from Proposition 1, bpf gradient in sum of magnitude sense is a Hueckel-type edge operator.

Consider now the bpf through the gray level of pels x_i for all $i = 0, 8$. In a similar manner as above it can be seen that

$$a = \frac{1}{2}\left[\frac{x_4+x_5+x_6}{3} - \frac{x_1+x_2+x_8}{3}\right], \quad b = \frac{1}{2}\left[\frac{x_6+x_7+x_8}{3} - \frac{x_2+x_3+x_4}{3}\right]. \tag{7}$$

Proposition 2. *The* bpf *gradient in* s.r.s.s. *sense is equal to* $1/\sqrt{2}$ *times the Prewitt's gradient in* r.m.s. *sense while the* bpf *gradient in magnitude sum and max sense are, respectively, equal to and* 1/2 *times the Prewitt's gradient in magnitude average and max sense.*

Proof. For Prewitt's gradient we get from (3) and (7)

$$a = \tfrac{1}{2}G_1, \quad b = \tfrac{1}{2}G_2.$$

Hence

$$\sqrt{[a^2+b^2]} = \tfrac{1}{2}\sqrt{[G_1^2+G_2^2]}, \quad |a|+|b| = \tfrac{1}{2}[|G_1|+|G_2|] \quad \text{and} \quad \max[|a|, |b|] = \tfrac{1}{2}\max[|G_1|, |G_2|].$$

Q.E.D.

A similar relationship between Sobel's gradient and the bpf gradient can also be found if the gray level x_1, x_3, x_5 and x_7 are weighted by 2. In that case, we get

$$a = \frac{2G_1}{3}, \quad b = \frac{2G_2}{3},$$

and the following proposition can be proved directly.

Proposition 3. *The* bpf *gradient of weighted gray levels in* s.r.s.s. *sense is equal to* $2\sqrt{2}/3$ *times the Sobel's gradient in* r.m.s. *sense while the* bpf *gradient of weighted gray levels in magnitude sum and max sense are, respectively, equal to* 4/3 *and* 2/3 *times the Sobel's gradient in magnitude average and max sense.*

Therefore, it is seen that the bpf gradient technique is quite general and it is equivalent to other useful techniques except sometimes for the constant factor that may be absorbed in the threshold for edge detection. Here we have examined the bpf with pels under 2×2 and 3×3 masks. Larger masks can also be used, but they may lead to thick edge. However, the edge orientation sensitivity is reduced and the edge dedection capability in presence of noise is improved, which is desirable.

A different 2×2 mask can also be used for the edge detection. We consider 4-neighbours i.e., x_1, x_3, x_5 and x_7 of the candidate x_0 as the mask and examine the behaviour of bpf or difference gradient. It is shown and emphasized below that this gradient, henceforth called the 4-neighbour gradient shows many desirable properties an edge detection technique should possess.

3. 4-Neighbour gradient and its properties

If represented as difference the gradient functions of 4-neighbour gradient is given by

$$G_1 = x_5 - x_1, \quad G_2 = x_7 - x_3. \tag{8}$$

On the other hand, if a best plane is fitted through x_1, x_3, x_5 and x_7 we have

$$a = \frac{x_5 - x_1}{2}, \quad b = \frac{x_7 - x_3}{2}. \tag{9}$$

The relation between (8) and (9) is evident.

Let us now examine the properties of 4-neighbour gradient. Firstly, the value of gradient does not depend on the choice of origin and orientation of the axis (i.e., left handed or right handed). Secondly, the gradient is insensitive to edge orientation if max operator is chosen.

The fact is explained through Fig. 2 where an ideal step edge of height h is considered. The edge is inclined about the vertical axis passing through the centre of the candidate pel by an angle θ. The pel

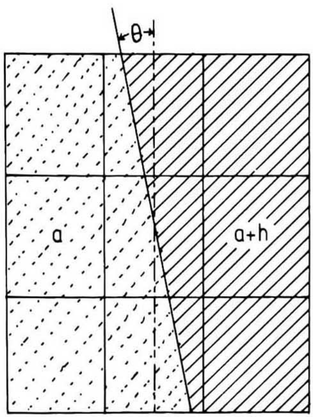

Fig. 2. Model of a step edge with height h and inclination θ about vertical axis.

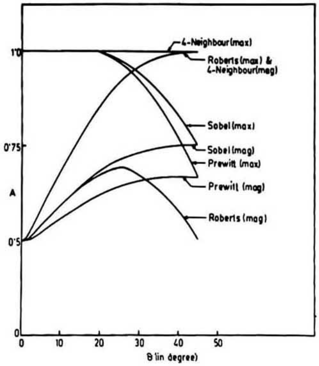

Fig. 3. Edge amplitude response as a function of edge orientation.

gray levels vary as a function of edge orientation as a result of the inherent averaging associated with discretization of the sampled image array. However, since the candidate pel x_0 is not accounted in 4-neighbour gradient, it is easy to see that the gradient value is h irrespective of θ if a max operator is used. Robert's, Sobel's and Prewitt's gradients do not hold this desirable property. The edge orientation sensitivity of different gradients with the model of Fig. 2 are given in Fig. 3 for comparison.

Thirdly, the 4-neighbour gradient provides rapidly declining edge gradient response as the detector mask moves away from a central edge. In this respect, however, the performance of Robert's gradient is identical with that of the 4-neighbour gradient for slant edge. The edge gradient amplitude response for the different gradients are given in Fig. 4 for two extreme cases namely, vertical and slant edges, respectively.

Fourthly, unlike other gradients, the 4-neighbour gradient with max operator can be used directly for edge detection of two-level pictures. Rosenfeld [10] proposed 8-neighbour and 4-neighbour for connectivity in two level pictures and showed that if the object is 8-connected, its background is 4-connected.

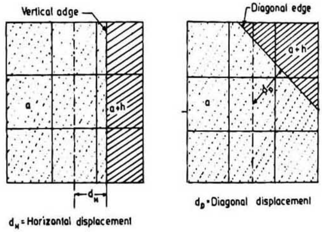

Fig. 4(a). Model of (i) vertical edge and (ii) diagonal edge displaced from the centre of the candidate pel.

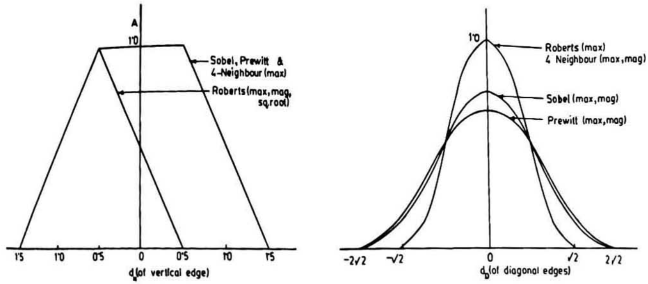

Fig. 4(b). Edge amplitude response as a function of displacement of (i) vertical edge and (ii) diagonal edge.

It is evident that a suitable threshold with max operated 4-neighbour gradient functions of (8) and (9) can detect the edges of thickness two. Also the technique automatically discards the isolated holes employing a simple check.

The 4-neighbour gradient is computationally as efficient as any other efficient edge detection technique. It requires two integer additions and one comparison at each candidate pel for max-type operator. Ordinary difference and Robert's gradient is equivalent to 4-neighbour in computer efficiency. Also, 4-neighbour gradient can be implemented in array processors quite conveniently.

Another important property that an edge detector should possess is the low sensitivity to random noise. The study of this property is complicated by the fact that the edges depend on the choice of appropriate gradient as well as the threshold. With noisy images the threshold selection becomes a tradeoff between the missing of valid edges and the creation of noise induced false edges. The notion of valid edges is somewhat fuzzy. Although a statistical procedure can be formulated as in Abdou and Pratt [7], we do not follow it for the reason that the conditional probability densities $p(A|edge)$ and $p(A|no\ edge)$ are to be modelled rather than estimated and we think the probability is conditioned on fuzzy variables '*edge*' and '*no edge*'. We study the noise sensitivity qualitatively as follows.

Robert [1] suggested that the effect of quantisation noise seems to be reduced if symmetric gradient is used. It is seen that the 4-neighbour gradient is symmetric and hence it is as effective as the other symmetric gradients towards quantisation noise. To see the effect of random noise, a picture of ideal vertical edge as shown in Fig. 5(a) is chosen and random noise has been added to it. The picture is subject to Prewitt's, Sobel's, Robert's and 4-neighbour edge detection techniques and the results are shown in Fig. 5(b–e). It is seen that Prewitt's technique is the most competent. Sobel's and 4-neighbour techniques have intermediate performance followed by Robert's technique. A number of simulations show that Robert's and 4-neighbour techniques have similar performance on an average while Sobel's technique is superior to Prewitt's for slant edge.

Although Robert's and 4-neighbour techniques have similar average performance in noisy picture, the later technique is emphasized here because, as discussed above, it satisfies other desirable conditions in a better way than Robert's gradient.

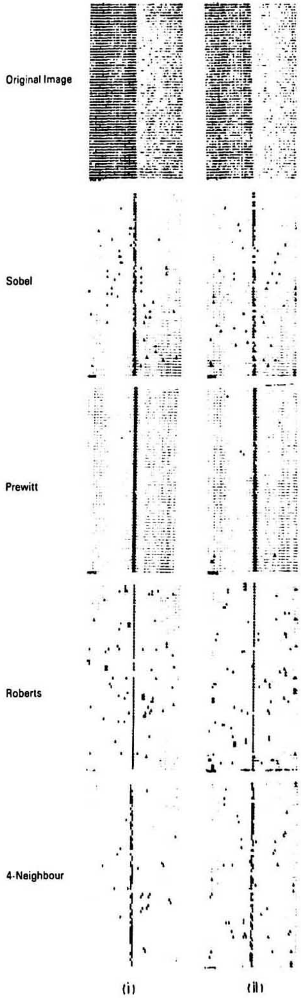

Fig. 5. Results showing noise sensitivity of different edge detection operators. (a) Original image with (i) S.N.R. ≈ 29.5, (ii) S.N.R. ≈ 25.6. (b) Sobel operator. (c) Prewitt operator. (d) Roberts operator. (e) 4-neighbour operator.

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