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Shape similarity measures for open curves

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Abstract: The shape of an open curve is defined which does not depend on its position, size and orientation. Two distance measures between shapes are constructed on the basis of which two shape similarity measures are defined.

Key words: Open curve, starting point, end point, shape, directional codes, slope, distance, shape similarity.

1. Introduction

In an earlier paper (Parui and Dutta Majumder, 1982) we proposed some measure of shape similarity for 2-dimensional regions. The present paper deals with open curves in the 2-dimensional plane. An open curve is described as a finite sequence of equispaced points on the curve. This description is information preserving in the sense that it is possible to reconstruct any reasonable approximation of the curve from the descriptor. We define the shape of an open curve in Section 2, propose two definitions of shape similarity of analogue open curves in Section 3 and give the computational techniques for digital curves in Section 4.

Algorithms for shape analysis of open curves have been reviewed by Pavlidis (1980).

2. Shape of an open curve

Definition 2.1. A subset $A \subset \mathbb{R}^2$ (2-dimensional Euclidean plane) is called *an open curve* if

(i) A is a closed, bounded and connected set, and

(ii) there exist only two points Q_1 and Q_2 in A such that $A - \{Q_i\}$ is not disconnected ($i = 1, 2$), that is, for all other points Q in A, $A - \{Q\}$ is disconnected.

Q_1 and Q_2 are the two *extreme points* of the open curve A.

Now, an open curve can be looked upon as a continuous function f from a closed interval, say, $[0, 1]$ into $\mathbb{R}^2$. Again, $f(t)$ can be written as $(x(t), y(t))$ where x and y are two continuous functions from $[0, 1]$ into $\mathbb{R}$ (the real line). Thus we can represent an open curve as $\{(x(t), y(t)): t \in [0, 1]\}$. From the definition of an open curve, for no two t_1, t_2 $(t_1 \neq t_2)$ belonging to $[0, 1]$, $x(t_1) = x(t_2)$ and $y(t_1) = y(t_2)$.

$(x(0), y(0))$ and $(x(1), y(1))$ are the two extreme points of A. $(x(0), y(0))$ is called the *starting point* and $(x(1), y(1))$ is called the *end point* of the open curve $\{(x(t), y(t)): t \in [0, 1]\}$.

Now, the representation $\{(x(t), y(t)): t \in [0, 1]\}$ of an open curve is not unique. For example, if $x_1(t) = 0$, $y_1(t) = t$, $x_2(t) = 0$ for all t, $y_2(t) = 2t$ for t belonging to $[0, \frac{1}{2}]$ and $y_2(t) = \frac{1}{2}(2t + 1)$ for t belonging to $[\frac{1}{2}, 1]$, then $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ represent the same open curve, that is, the line segment joining the two points $(0, 0)$ and $(0, 1)$. A unique representation of an open curve will be provided below.

Definition 2.2. The *length* of an open curve $A = \{(x(t), y(t)): t \in [0, 1]\}$ is

$$L = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Definition 2.3. Suppose

$$\bar{x} = \frac{1}{L(A)} \int_0^1 x(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

and

$$\bar{y} = \frac{1}{L(A)} \int_0^1 y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt,$$

then the point $(\bar{x}, \bar{y})$ is called the *centroid* of the open curve A.

For each $s \in [0, L]$ there exists a $t \in [0, 1]$ such that

$$s = \int_0^t \sqrt{\left(\frac{dx}{dt_1}\right)^2 + \left(\frac{dy}{dt_1}\right)^2}\, dt_1.$$

There is in fact a one-to-one correspondence between t and s. Let $u(s) = x(t)$ and $v(s) = y(t)$. The representation $\{(u(s), v(s)): s \in [0, L]\}$ of an open curve is unique. The functions $u(s)$ and $v(s)$ are continuous such that for no two s_1, s_2 $(s_1 \neq s_2)$ belonging to $[0, L]$, $u(s_1) = u(s_2)$ and $v(s_1) = v(s_2)$. The centroid $(\bar{x}, \bar{y})$ can be rewritten as $(\bar{u}, \bar{v})$ where

$$\bar{u} = \frac{1}{L} \int_0^L u(s)\,ds, \qquad \bar{v} = \frac{1}{L} \int_0^L v(s)\,ds.$$

Let $\{(u(s), v(s)): s \in [0, L]\}$ be an open curve. Then $\{(u_1(s), v_1(s)): s \in [0, L]\}$ is also an open curve where $u_1(s) = u(L - s)$ and $v_1(s) = v(L - s)$.

Definition 2.4. The *inverse* of an open curve $\{(u(s), v(s)): s \in [0, L]\}$ is defined as $\{(u_1(s), v_1(s)): s \in [0, L]\}$ and is denoted by A^{-1}.

The starting point of A^{-1} is the end point of A and vice-versa. Note that $(A^{-1})^{-1} = A$.

Notation. $\mathscr{S}$ is the collection of all open curves where A and A^{-1} are treated as different elements of $\mathscr{S}$.

Definition 2.5. An open curve $A = \{(u(s), v(s)): s \in [0, L]\}$ is said to be *antisymmetric* if $u(s) + u(L - s) = 2u(L/2)$ and $v(s) + v(L - s) = 2v(L/2)$ for all $s \in [0, L]$.

Definition 2.6. $B = \{(u(s) + a, v(s) + b): s \in [0, L]\}$ is said to be a *translation* of the open curve $A = \{(u(s), v(s)): s \in [0, L]\}$ where a and b are any two real numbers. Note that B also is an open curve.

Definition 2.7. S_1 is a relation in $\mathscr{S}$ such that for A and B belonging to $\mathscr{S}$, $(A, B) \in S_1$ if B is a translation of A.

Proposition 2.1. *S_1 is an equivalence relation in $\mathscr{S}$.*

Definition 2.8. $B = \{(cu(s), cv(s)): s \in [0, L]\}$ is said to be a *dilation* of the open curve $A = \{(u(s), v(s)): s \in [0, L]\}$ where c is any positive real number. Note that B also is an open curve.

Definition 2.9. S_2 is a relation in $\mathscr{S}$ such that for A and B belonging to $\mathscr{S}$, $(A, B) \in S_2$ if B is a dilation of A.

Proposition 2.2. *S_2 is an equivalence relation in $\mathscr{S}$.*

Definition 2.10. The curve

$$B = \{(u(s)\cos\alpha - v(s)\sin\alpha,$$
$$u(s)\sin\alpha + v(s)\cos\alpha): s \in [0, L]\}$$

is said to be a *rotation* of the open curve $A = \{(u(s), v(s)): s \in [0, L]\}$ where α is an angle belonging to $[0, 2\pi)$. Note that B is also an open curve.

Definition 2.11. S_3 is a relation in $\mathscr{S}$ such that for A and B belonging to $\mathscr{S}$, $(A, B) \in S_3$ if B is a rotation of A.

Proposition 2.3. *S_3 is an equivalence relation in $\mathscr{S}$.*

Definition 2.12. S is a relation in $\mathscr{S}$ such that for A and B belonging to $\mathscr{S}$, $(A, B) \in S$ if there exist C and D belonging to $\mathscr{S}$ such that $(A, C) \in S_1$, $(C, D) \in S_2$ and $(D, B) \in S_3$.

Proposition 2.4. *S is an equivalence relation in $\mathscr{S}$.*

Definition 2.13. The shape of an open curve is defined to be an equivalence class generated by S in $\mathscr{S}$. Let $\mathscr{A}$ be the collection of all equivalence classes, that is, of all shapes.

Within each such equivalence class any two open curves can be made coincide with each other by translation, dilation and rotation. On the other hand no two open curves from two different equivalence classes can be made coincide by translation, dilation and rotation. That is, a shape is what remains of an open curve after disregarding its position, size and orientation in $\mathbb{R}^2$.

Note that the shapes of A and A^{-1} are not, in general, the same.

Proposition 2.5. *The shapes of A and A^{-1} are the same if and only if A is antisymmetric.*

3. Definitions of shape similarity

We shall now propose two definitions of shape distance and shape similarity for open curves.

Before describing this method, let us first define directional codes. The directional codes $1, 2, \ldots, 8$ are shown in Fig. 1. Each pair of adjacent directions makes an angle of 45° in between them. Now this definition of directional codes can be extended. let d be any real number belonging to $(0, 8]$. Note that $i-1 < d \le i$ for some $i \in \{1, 2, \ldots, 8\}$. Then the directional code d defines a direction that makes an angle of $(i-d)45°$ with direction i on the anticlockwise side (Fig. 2). Thus there is a one-to-one correspondence between S_8 and the set of all directions in $\mathbb{R}^2$.

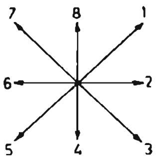

Fig. 1. Directional codes.

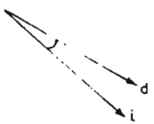

Fig. 2. Angle between d and i is $(i-d)45°$.

In the present method we find the tangent at every point of an open curve and then its directional code d in S_8.

Let $A = \{(u(s), v(s)) \colon s \in [0, L]\}$ be an open curve. For all s belonging to $(0, L)$ we define the following:

$$e(s,h) = \begin{cases} 1 & \text{if } u(s) - u(s-h) > 0, \\ 0 & \text{if } u(s) - u(s-h) = 0, \quad h > 0, \\ 1 & \text{if } u(s) - u(s-h) < 0, \end{cases}$$

$$\delta(s,h) = \begin{cases} 1 & \text{if } u(s) - u(s+h) > 0, \\ 0 & \text{if } u(s) - u(s+h) = 0, \quad h > 0. \\ 1 & \text{if } u(s) - u(s+h) < 0, \end{cases}$$

Let $e(s) = \lim_{h\downarrow 0} e(s,h)$ and $\delta(s) = \lim_{h\downarrow 0} \delta(s,h)$.

Definition 3.1. A point $(u(s), v(s))$, $s \in (0, L)$, on the open curve A is a *corner point* if (i)

$$\lim_{h\downarrow 0} \frac{v(s) - v(s-h)}{u(s) - u(s-h)} \ne \lim_{h\downarrow 0} \frac{v(s) - v(s+h)}{u(s) - u(s+h)}$$

and (ii) not both $e(s)$ and $\delta(s)$ are zero.

Assumption. There are only finitely many corner points on an open curve.

For the slope at the s-th point we take only the left limit because the left and the right limits are not always the same, that is, let

$$f(s) = \lim_{h\downarrow 0} \frac{v(s) - v(s-h)}{u(s) - u(s-h)} \quad \text{for } s \in (0, L).$$

Let

$$g(s) = \frac{\frac{1}{2}\pi - \tan^{-1} f(s)}{\frac{1}{4}\pi} \quad \text{when } |f(s)| < \infty.$$

That is, $0 < g(s) < 4$. Define

$$w(s) = \begin{cases} g(s) & \text{iff } e(s) = 1, \ |f(s)| < \infty, \\ g(s) + 4 & \text{iff } e(s) = -1, \ |f(s)| < \infty, \\ 4 & \text{iff } f(s) = -\infty, \\ 8 & \text{iff } f(s) = +\infty. \end{cases}$$

Note that the value of $w(s)$ belongs to S_8 and $w(s)$ gives the directional code of the tangent at $(u(s), v(s))$.

Definition 3.2. $\{w(s) \colon s \in (0, L)\}$ is defined to be the *slope* of the open curve $\{(u(s), v(s)) \colon s \in [0, L]\}$.

Proposition 3.1. *The slope $\{w(s): s \in (0, L)\}$ determines an open curve uniquely if the starting point or the end point is provided.*

Proposition 3.2. *The slope of an open curve depends on the size and orientation but not on the position of the curve.*

Proposition 3.3. *Suppose the slopes of two open curves A and B are $\{w_1(s): s \in (0, L)\}$ and $\{w_2(s): s \in (0, L)\}$. Then $w_2(s) = w_1(s) + k \pmod 8$ for all $s \in (0, L)$ and for fixed k if and only if B is a rotation of A.*

Proposition 3.4. *If the slopes of an open curve A and its inverse A^{-1} are $\{w_1(s): s \in (0, L)\}$ and $\{w_2(s): s \in (0, L)\}$, then $w_2(L-s) = w_1(s) + 4$* (mod 8) *for all $s \in (0, L)$ unless the s-th point is a corner point of A.*

Definition 3.3. D_1 is a distance function on S_8 such that for d_1 and d_2 belonging to S_8,

$$D_1(d_1, d_2) = \min\{|d_1 - d_2|, 8 - |d_1 - d_2|\}. \quad (1)$$

$\frac{1}{4}\pi D_1(d_1, d_2)$ in fact gives the angle between the directions d_1 and d_2.

Proposition 3.5. *D_1 is a metric on S_8.*

Definition 3.4. D_2 is a distance function on $\mathscr{L}$ such that for A, B belonging to $\mathscr{L}$,

$$D_2(A, B) = \int_0^1 D_1(w_1(\mu L_1), w_2(\mu L_2))\mathrm{d}\mu \quad (2)$$

where $\{w_1(s): s \in (0, L_1)\}$ and $\{w_2(s): s \in (0, L_2)\}$ are the slopes of A and B respectively. L_1 and L_2 are the lengths of A and B respectively.

Proposition 3.6. (i) $D_2(A, B) \geq 0$,
(ii) $D_2(A, B) = 0$ *if and only if B is a translation and/or dilation of A,*
(iii) $D_2(A, B) = D_2(B, A)$,
(iv) $D_2(A, B) + D_2(B, C) \geq D_2(A, C)$.

Notation. Let B_α denote the open curve obtained after rotating an open curve B by an angle α in the anticlockwise direction.

Definition 3.5. D_3 is a distance function on $\mathscr{L}$ such that for A, B belonging to $\mathscr{L}$,

$$D_3(A, B) = \inf_\alpha D_2(A, B_\alpha).$$

Proposition 3.7. (i) $D_3(A, B) \geq 0$,
(ii) $D_3(A, B) = 0$ *if and only if B is a translation, dilation and rotation of A,*
(iii) $D_3(A, B) = D_3(B, A)$,
(iv) $D_3(A, B) + D_3(B, C) \geq D_3(A, C)$.

Proposition 3.8. *D_3 defines a metric on $\mathscr{S}$ i.e. on shapes.*

Proposition 3.9. $\sup_{A,B} D_3(A, B) = 2$.

Proposition 3.10. *$D_3(A, A^{-1}) = 0$ if and only if A is antisymmetric.*

So far we have distinguished between A and A^{-1}, that is, the shapes of A and A^{-1} have not been the same in general. But in many cases it may be required to disregard which extreme point we scan an open curve from when characterizing its shape. That is, for any open curve A, we say A and A^{-1} have the same shape. Below we modify the definition of distance between the two shapes.

Definition 3.6. D_4 is a distance function on $\mathscr{L}$ such that for A, B in $\mathscr{L}$,

$$D_4(A, B) = \min\{D_3(A, B), D_3(A, B^{-1})\}.$$

Here the slope of B^{-1} is taken to be $\{w(L-s)+4$ (mod 8): $s \in (0, L)\}$ where the slope of B is $\{w(s): s \in (0, L)\}$.

Proposition 3.11. $D_4(A, A^{-1}) = 0$.

Proposition 3.12. $\sup_{A,B} D_4(A, B) = 2$.

Definition 3.7. μ_1 is a similarity measure between two shapes such that $\mu_1 = 1 - \frac{1}{2}D_4$.

Method 2

In this method we find the Euclidean distance between every two corresponding points on two open curves. Since we are going to define a distance between shapes, and shapes are invariant under translation and dilation, we shall consider only the open curves with centroid at the origin (0, 0) and with length, say, L. Let $\mathscr{L}_1$ be the collection of all such open curves. Let A and B belong to $\mathscr{L}_1$ such that $A=\{(u_1(s), v_1(s)): s\in[0,L]\}$ and $B=\{(u_2(s), v_2(s)): s\in[0,L]\}$.

Definition 3.8. D_5 is a distance function on $\mathscr{L}_1$ such that

$$D_5(A,B)=\sqrt{\int_0^L\{(u_1(s)-u_2(s))^2+(v_1(s)-v_2(s))^2\}\mathrm{d}s}.$$

Proposition 3.13. *D_5 is a metric on $\mathscr{L}_1$, that is,*

(i) $D_5(A,B)\geq 0$,

(ii) $D_5(A,B)=0$ *if and only if* $A=B$,

(iii) $D_5(A,B)=D_5(B,A)$,

(iv) $D_5(A,B)+D_5(B,C)\geq D_5(A,C)$.

Definition 3.9. D_6 is a distance function on $\mathscr{L}_1$ such that $D_6(A,B)=\inf_\alpha D_5(A,B_\alpha)$.

Proposition 3.14. (i) $D_6(A,B)\geq 0$,

(ii) $D_6(A,B)=0$ *if and only if B is a rotation of A*,

(iii) $D_6(A,B)=D_6(B,A)$,

(iv) $D_6(A,B)+D_6(B,C)\geq D_6(A,C)$.

Two open curves A and B belonging to $\mathscr{L}_1$ have the same shape if and only if B is a rotation of A. Thus D_6 (like D_3) is a metric on $\mathscr{S}$ (the set of all shapes).

Proposition 3.15. $D_6(A,B)\leq L\sqrt{L}/\sqrt{2}$.

Proposition 3.16. $D_6(A,A^{-1})=0$ *if and only if A is antisymmetric.*

Now, if we want the shapes of A and A^{-1} to be the same, we can modify D_6 as before.

Definition 3.10. D_7 is a distance function on $\mathscr{L}_1$ such that $D_7(A,B)=\min\{D_6(A,B), D_6(A,B^{-1})\}$.

Proposition 3.17. $D_7(A,A^{-1})=0$.

Proposition 3.18. $D_7(A,B)\leq L\sqrt{L}/\sqrt{2}$.

Definition 3.11. μ_1 is a similarity measure between two shapes such that $\mu_2=1-\sqrt{2}D_7/L\sqrt{L}$.

4. Computations of similarity measures

Input here is open digital curves which are sequences of pixels (i_k, j_k), $k=0,1,\dots,N$, such that (i_{k-1}, j_{k-1}) is an 8-neighbour of (i_k, j_k) for $k=1,\dots,N$ and (i_0, j_0) is not an 8-neighbour of (i_N, j_N). With the help of an earlier algorithm (Parui and Dutta Majumder, 1982) we choose $n+1$ equispaced points $P_r=(x_r, y_r)$, $r=0,1,\dots,n$, on a digital curve with a total length, say, L such that $(x_0,y_0)=(i_0,j_0)$ and $(x_n,y_n)=(i_N,j_N)$.

Computations of D_3 and D_4: Let A and B be two digital open curves with the equispaced points $P_{ir}=(x_{ir}, y_{ir})$, $r=0,1,\dots,n$ and $i=1,2$. Let w_{ir} be the directional code of $\overline{P_{i,r-1}P_{ir}}$. Then w_{ir} gives the slopes of A and B in the finite case (analogous to $w(s)$ in the continuous case). Write

$$D_2(A,B)=\frac{1}{n}\sum_{r=1}^{n}D_1(w_{1r}, w_{2r}).$$

This D_2 is the finite form of D_2 defined in (2). Now, the computation of D_3 is difficult on the basis of the present definition of D_1 in (1) since $D_3=\inf_\alpha D_2(A,B_\alpha)$ and $f(\alpha)=D_2(A,B_\alpha)$ is not always differentiable with respect to α. Note that the slope of B_α is $w_{2r}-4\alpha/\pi$.

Changing D_1 defined in (1) we write

$$D_1(d_1,d_2)=4\sin^2\tfrac{1}{8}\pi(d_1-d_2). \tag{3}$$

Thus

$$f(\alpha)=D_2(A,B_\alpha)=\frac{4}{n}\sum_{r=1}^{n}\sin^2\tfrac{1}{8}\pi(w_{1r}-w_{2r}+4\alpha/\pi).$$

$f(\alpha)$ attains the minimum/maximum at

$$\alpha=\tan^{-1}\left\{\frac{-\sum_{r=1}^{n}\sin\theta_r}{\sum_{r=1}^{n}\cos\theta_r}\right\} \tag{4}$$

where $\theta_r=\frac{1}{4}\pi(w_{1r}-w_{2r})$.

Let α_1 belonging to $[0,\pi)$ satisfy (4). Then $\alpha_2=\alpha_1+\pi$ also satisfies (4). Now, for only one of α_1 and α_2 the second derivative $f''(\alpha)$ is positive. Let that value be $\hat{\alpha}$. So the value of $D_3(A,B)$ is obtained as $f(\hat{\alpha})$. Note that the slope of B^{-1} is $w_{3r} = w_{2,n-r+1} + 4 \pmod 8$. $D_3(A, B^{-1})$ is computed in a similar way. $D_4(A, B)$ is computed as $\min\{D_3(A, B), D_3(A, B^{-1})\}$. Note that $D_4(A, B) \le 2$ (Parui and Dutta Majumder, 1982). The similarity measure μ_1 is taken to be $1 - \frac{1}{2}D_4$.

Computations of D_6 and D_7: Without loss of generality we can assume that (i) the lengths of A and B are the same and (ii) (x_{ir}, y_{ir}) are such that $\sum_{r=0}^{n} x_{ir} = \sum_{r=0}^{n} y_{ir} = 0$ for $i = 1, 2$. (If not, we can always add a constant (a, b) to or multiply a constant c with all (x_{ir}, y_{ir}) to make the above true. That is, a suitable translation or dilation of (x_{ir}, y_{ir}) will serve the purpose). The common length is, say, L. Write

$$D_5(A, B) = \sqrt{\frac{L}{n} \sum_{r=1}^{n} \{(x_{1r} - x_{2r})^2 + (y_{1r} - y_{2r})^2\}}.$$

This D_5 is the finite form of D_5 defined in Section 3 (Definition 3.8). Now, the set of points (x_{2r}, y_{2r}) become $(x_{2r} \cos\alpha - y_{2r} \sin\alpha$, $x_{2r} \sin\alpha + y_{2r} \cos\alpha)$ after rotation by angle α. Let

$$\begin{aligned} f(\alpha) &= \frac{n}{L} D_5^2(A, B_\alpha) \\ &= \sum_{r=1}^{n} \{(x_{1r} - x_{2r}\cos\alpha + y_{2r}\sin\alpha)^2 \\ &\quad + (y_{1r} - x_{2r}\sin\alpha - y_{2r}\cos\alpha)^2\}. \end{aligned}$$

$f(\alpha)$ attains the maximum/minimum at

$$\alpha = \tan^{-1}\left\{\frac{\sum_{r=1}^{n} (y_{1r}x_{2r} - x_{1r}y_{2r})}{\sum_{r=1}^{n} (x_{1r}x_{2r} + y_{1r}y_{2r})}\right\}. \tag{5}$$

Let $\hat{\alpha}$ be such that (i) $\hat{\alpha}$ satisfies (5) and (ii) $f''(\hat{\alpha})$ is positive. Hence $f(\alpha)$ attains the minimum at $\hat{\alpha}$. The value of $D_6(A, B)$ is obtained as $\sqrt{(L/n) f(\hat{\alpha})}$. For B^{-1} we take the points (x_{3r}, y_{3r}) where $(x_{3r}, y_{3r}) = (x_{2,n-r}, y_{2,n-r})$. $D_6(A, B^{-1})$ is computed in the same way as $D_6(A, B)$. $D_7(A, B)$ is computed as $\min\{D_6(A, B), D_6(A, B^{-1})\}$. Note that $D_7(A, B) \le L\sqrt{L}/\sqrt{2}$. The similarity measure μ_2 is taken to be $1 - \sqrt{2}D_7/L\sqrt{L}$. Clearly, $1 \ge \mu_2 \ge 0$.

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References

Parui, S.K. and D. Dutta Majumder (1982). A new definition of shape similarity. *Pattern Recognition Letters* 1, 37–42.

Pavlidis, T. (1980). Algorithms for shape analysis of contours and waveforms. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, PAMI-2, 301-312.