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SYMMETRY ANALYSIS BY COMPUTER

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Abstract—Approximate locations of axes of symmetry of a 2-dimensional region are detected on the basis of its border. The border, described in terms of certain directional codes, is treated as a regular polygon in a hierarchical manner where a lower level means a greater number of sides. At each level of the hierarchy, the best axis of symmetry is found which for a lower level gives a more accurate position of the unknown axis of symmetry than for a higher level. Along with an axis of symmetry, a certain error is found on the basis of which the degree of symmetry of a 2-dimensional region is defined. Programs are written in FORTRAN IV and are implemented on an EC-1033 computer.

2-Dimensional region Border detection Directional codes Regular polygons Axis of symmetry Degree of symmetry

INTRODUCTION

The perception of shape plays an important role in the human visual learning process, as well as in pattern recognition and scene analysis by computer. For the recognition of shapes, various feature measurements like moments[1,2] and Fourier coefficients[3,4] have been used. However, in many cases these feature measurements have been found to be inadequate for the purpose. A comparatively new technique in this regard is the syntactic technique, which has drawn a lot of interest in recent years. In this, shapes can be represented in a hierarchical manner such that the top levels of the hierarchy contain only the coarse and global aspects of a shape while the lower levels describe the finer details. For example, in one paper[5] a closed curve is represented hierarchically as a chain of certain numbers. The length of the chain indicates the precision with which the chain describes the shape of the curve. Similarly, in another paper[6] a hierarchical representation of the sides and angles of a closed curve is proposed.

An important aspect of shape understanding is symmetry, which can be useful in pattern analysis. For a symmetric pattern, one need store only a half of the pattern along with the axis of symmetry. Again, if a part of a pattern is missing or noisy, with the help of symmetry one can complete the pattern or rid the pattern of noise. The axis of symmetry is also useful in defining the orientation of a shape.

Methods for detection of the axis of symmetry of a simple closed curve have been reported where the closed curve is approximated as a simple polygon. In one approach[7] the global axis of symmetry is found through clustering of the local axes of symmetry, while in another[8] it is detected on the basis of the curvature at the vertices of the polygon.

In the present paper also, the closed curve is hierarchically described as a polygon. The vertices of the polygon lie on the curve such that the distance along the curve between any two consecutive vertices is the same. That is, the shortest or linear distance between consecutive vertices need not always be equal. If the common distance is d and the total perimeter of the closed curve is P then the number of sides or vertices of the polygon is P/d. With a suitable value of d we can get a polygon with any desirable number of sides to describe the input figure. Now, any line dividing the closed area such that the perimeter on both sides is the same can be approximated (depending on how small the value of d is) as a line joining some pair of opposite vertices of the polygon. Thus, for a polygon with $2n$ sides there are n such lines which are possible axes of symmetry. For each of them we find the degree to which the line serves as an axis of symmetry. The line with the maximum degree is finally chosen.

HIERARCHICAL REPRESENTATION OF A CLOSED CURVE AS POLYGONS

The input in the present method is a 2-dimensional region with no holes inside. For symmetry analysis we use only the closed curve indicating the border of the region. The input region is first digitized into two gray levels. The border of the binary picture is extracted in a clockwise manner using an eight point neighbourhood method.[9] Before going into the details of our method we shall define the directional codes.

The directional codes 1, 2, …, 8 are shown in Fig. 1. Each pair of adjacent directions has between then an angle of 45°. Now this definition of directional codes can be extended. For every $d \in (0, 8] = S_8$, a corresponding direction can be defined in the following way: Suppose $i - 1 < d \leqslant i$ for some $i \in \{1, 2, \ldots, 8\} = I_8$; then the directional code d defines a direction that makes an angle of $(i - d)\,45°$ with direction i on the anticlockwise side (Fig. 2). Thus, every direction in the 2-dimensional plane can be associated uniquely with a number in S_8. It can be seen that each pair of opposite directions has directional codes whose difference is 4. Each such pair may sometimes have to be treated as just a line. The directional code of such a line is taken to be the smaller of the two directional codes. This we call the absolute directional code. In other words, the absolute directional code of a line with directional code d is defined as

$$d_0 = d \text{ if } d \leqslant 4$$
$$= d - 4 \text{ if } d > 4.$$
$$\text{Thus} \quad d_0 \in (0, 4].$$

Now suppose that the border of the digitized region [Fig. 3 (a)] is represented as a chain of directional codes $d_1, d_2, \ldots, d_n$ where $d_i \in I_8$ and n is the total number of border points which depends on the resolution of the picture. The corresponding border points are, say, $v_1, v_2, \ldots, v_n$, such that the directional code of the line $\overline{v_i v_{i+1 \bmod n}}$ is d_i. Now $\{v_i; i = 1, n\}$ can be considered as a polygon with n sides. (Contrary to the common definition of a polygon, three consecutive vertices of a polygon in this paper can be collinear, i.e., change of direction need not occur at every vertex.) Though, strictly speaking, d_i is the directional code of the side $\overline{v_i v_{i+1 \bmod n}}$, from now on d_i will also denote the corresponding side. Clearly, the lengths of the n sides of the polygon are not equal. The sides are, in fact, of two different lengths, x and $\sqrt{2}x$, where x is the size of the square pixel of the input grid. It can be seen that the odd directional codes 1, 3, 5, 7 have length $\sqrt{2}x$ and the even directional codes 2, 4, 6, 8 have length x. To get rid of the inequality in side lengths we modify the chain of directional codes in the following manner: replace d_i by 7 consecutive d_i if d_i is odd, otherwise replace d_i by 5 consecutive d_i for $i = 1, 2, \ldots, n$. The lengths of the new d_i can be considered as the same since 7/5 is very close to $\sqrt{2}$. The number of the new d_i, say N, will be 5 to 7 times n [Fig. 3 (d)]. The new vertices, say, $\{v_i; i = 1, N\}$ describe a polygon P_1 whose sides are of length $x/5$. It is clear that there is no loss of information if the digitized input is represented as the polygon P_1 which indicates the lowest level in the hierarchy. A higher level is defined as a polygon P_r $(1 < r < N)$ whose number of sides is $N_r = N/r$ and whose vertices are $\{v_j^r, j = 1, N_r\}$ where $v_j^r = v_{(j-1)r+1}$. Correspondingly, d_j^r is defined as the directional code of the j-th side $\overline{v_j^r v_{j+1 \bmod N_r}^r}$ of the polygon P_r. Clearly, d_j^r does not always take an integral value. In fact, $d_j^r \in S_8$. It is clear that the sides of the polygon P_r are of equal length. As mentioned before, side length is a distance along the curve. The common value of the polygon P_r is $xr/5$. Higher values of r indicate higher levels of the hierarchy. From the definition of the polygon P_r it is clear that P_r contains only the coarse and global characteristics of the input figure for higher values of r.

DEGREE OF SYMMETRY OF A POLYGON

Let P_r be a polygon as defined above. We consider at this stage only $N_r/2$ lines for possible axes of symmetry. These are lines a_j^r joining the vertices v_j^r and $v_{j+N_r/2}^r$, $j = 1, 2, \ldots, N_r/2$. For each of these lines a cost of symmetry is measured in the following way: Let d_0 be the absolute directional code of a_j^r. Now for a fixed a_j^r, all the N_r sides of the polygon P_r are paired off as $(d_{j-1+k}^r, d_{j-k \bmod N_r}^r)$, $k = 1, 2, \ldots, N_r/2$. For each of these pairs a local axis a_k is found around which the two sides of the pair are symmetric (Fig. 4). The acute angle between this local axis a_k and the global axis a_j^r gives the cost of symmetry due to the k-th pair of sides. The directional code of a_k is $(d_{j-1+k}^r + d_{j-k \bmod N_r}^r)/2 + 2$ mod 8. From this the absolute directional code, d_k, of a_k is found. The cost or error due to a_k is given by $e_k = \min\{|d_0 - d_k|, 4 - |d_0 - d_k|\}$ where $e_k \cdot 45°$ gives the acute angle between a_k and a_j^r. Since $0 < d_0, d_k \leqslant 4$, $0 \leqslant e_k \leqslant 2$ for $k = 1, 2, \ldots, N_r/2$. The overall cost of symmetry for the global axis a_j^r is

$$E_j^r = \sum_{k=1}^{N_r/2} e_k / N_r.$$

If a_j^r is a perfect axis of symmetry at this level of hierarchy, then $e_k = 0$ for all k, so that $E_j^r = 0$. But on the other hand, e_k can not be 2 for all k, since $\{d_j^r, j = 1, N_r\}$ describes a closed curve. Hence $0 \leqslant E_j^r < 1$ for $j = 1, 2, \ldots, N_r/2$.

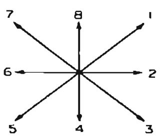


Fig. 1. Directional codes.

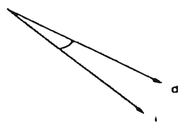
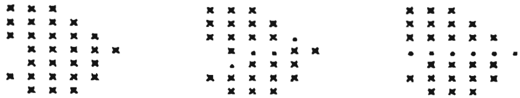


Fig. 2. Angle between d and i is $(i - d)\,45°$.



(a) (b) (c)

$n = 16$

Original d_i : 3335456671878822
$(i=1, \ldots n)$

$N = 96$

Modified d_i : 333333333333333333333335555555554444
$(i=1, \ldots N)$ 455555555666666666677777771111111
888887777777088888888882222222222

(d)

$r = 12$ $N_r = 8$

d_j^r : 3.00, 3.41, 4.59, 5.67, 7.24, 0.09, 7.67, 1.75

E_j^r : .302, .280, .412, .300

Absolute directional codes of

a_j^r : 0.21, 1.25, 2.59, 3.41

$E_2^r = 0.280$ is a local minimum. Hence, the best axis of symmetry at this stage is a_2^r which is shown by dots in (b).

(e)

$r = 6$ $N_s = 16$

d_j^s : 3.00, 3.00, 3.00, 4.00, 4.67, 4.50, 5.33, 6.00, 6.67, 8.00, 0.67, 7.50, 7.33, 8.00, 1.41, 2.00

E_j^s : .329, .431, .414, .329, .426, .517, .346, .371

Absolute directional codes of

a_j^r : 0.21, 0.82, 1.25, 2.00, 2.59, 3.00, 3.41, 3.79

Around the axis a_2^r are the axes a_m^s, $m = 2,3,4$ with errors 0.431, 0.414, 0.329. Hence a_4^s with the minimum error 0.329 is chosen as the best axis of symmetry at this stage and is shown by dots in (c).

(f)

Fig. 3.

HIERARCHICAL SEARCH FOR THE BEST GLOBAL AXIS OF SYMMETRY

The search for the axis of symmetry is carried out in a hierarchical manner. In the higher levels of the hierarchy the closed curve is looked upon as a polygon with a small number of sides and hence the choice of the axis of symmetry is from among a small number of lines, while in the lower levels the choice is from among a large number of lines. Clearly, the latter choice is a better one. We start with a high level in the hierarchy, i.e. with a polygon P_r for a suitably high value of r. The local minima among the errors E_j^r are then found, where $E_{j_u}^r$ is said to be a local minimum if $E_{j_u-1 \bmod N_r}^r > E_{j_u}^r < E_{j_u+1 \bmod N_r}^r$. Among the local minima, there is an $E_{j_u}^r$ such that the corresponding axis $a_{j_u}^r$ gives an approximate idea of the position of the best global (unknown) axis of symmetry. The other local minima give global axes of symmetry that are less good. To avoid a futile search for the global axis for a figure that is not symmetric, a threshold t_r is used. If, at this level of hierarchy, all the local minima are greater than t_r, then we call the figure asymmetric and do not go for lower levels of hierarchy. Otherwise, we choose those local minima $E_{j_u}^r$ that are less than t_r and go down to a lower level, say s. We assume $r = ps$, where p is an integer

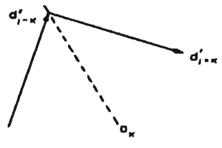


Fig. 4.

greater than 1. At the s-th level, we do not have to check all the lines a^s_m or the errors E^s_m ($m = 1, \ldots, N_s/2$) of the polygon P_s to get a better axis of symmetry. Since at the r-th level, $a^r_{j_0}$ is the closest to the unknown axis of symmetry, the unknown axis must lie between the lines $a^r_{j_1}$ and $a^r_{j_2}$, where $j_1 = j_0 - 1 \bmod N_r$ and $j_2 = j_0 + 1 \bmod N_r$. Note that $a^r_{j_1} = a^s_{w-p \bmod N_s}$ and $a^r_{j_2} = a^s_{w+p \bmod N_s}$, where $w = (j_0 - 1)p + 1 \bmod N_s$. So we have to check only $2p - 1$ lines a^s_m, $m = w - p + 1, \ldots, w + p - 1 \bmod N_s$ and find out their costs, E^s_m, on the basis of t^s_j and d^s_j, $j = 1, 2, \ldots, N_s$. If the minimum among E^s_m is $E^s_{m_0}$, the corresponding line $a^s_{m_0}$ is the best axis of symmetry at the s-th level. In fact, $a^s_{m_0}$ gives a better (or equally good at the worst) approximation of the unknown global axis of symmetry than $a^r_{j_0}$. In a similar fashion, we go down to a still lower level in the hierarchy. If a^s is the best axis at the s-th level, then a^s tends to the unknown global axis of symmetry as s decreases. In practice, we stop at a level where a desired degree of accuracy in the position of the axis of symmetry is achieved. Let the best axis at this lowest level of hierarchy be a and its cost be E. We define the degree of symmetry or the symmetry fuzzy membership of the input figure as $F = 1 - E$. As $0 \leq E < 1$, $0 < F \leq 1$. On the basis of the value of F we can say that a figure is more symmetric or less symmetric than another (Fig. 5). It is clear that the measure F is invariant under rotation of a figure, except that there may be slight variations due to digitization.

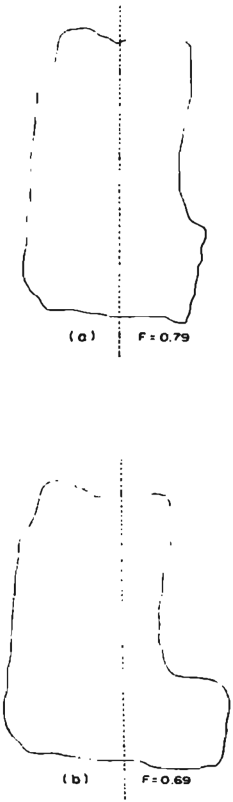


Fig. 5. (a) Is more symmetric than (b).

RESULTS AND DISCUSSION

The input picture is digitized into two gray levels 0 and 1. Gray level 1 indicates the black region, or figure, in question. Hierarchical steps of our algorithm are illustrated in terms of a simple figure [3 (a)]. Original and modified d_i are shown in Fig. 3 (d). We start with a high level of hierarchy, $r = 12$ [Fig. 3 (e)], and the axis a^r_2 [Fig. 3 (b)] is the best axis of symmetry at this level. Next a low level, $s = 6$, is considered [Fig. 3 (f)]. The

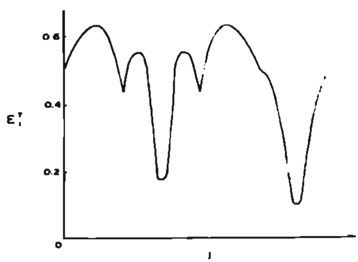


Fig. 6. Errors E^r_j at a medium level of hierarchy for a rectangle with side ratio 2:5. Two local minima in errors with sufficiently small absolute values (0.18 and 0.10) indicate the two axes of symmetry of the rectangle.

best axis of symmetry at this level is the axis a_4^1 [Fig. 3 (c)]. In Fig. 6, the relation between axes of symmetry and local minima in errors is shown where errors E_j^r are given for a rectangle with side ratio 2:5. In the errors there are two local minima which correspond to the two axes of symmetry of the rectangle. It can be seen that if a figure is not symmetric and its outline is concave in some places, then the best axis of symmetry suggested by the present algorithm does not divide the area of the input region equally [Fig. 5 (b)]. However, for outlines that are mainly convex, the axes of symmetry are alright in that respect. The algorithm also gives proper axes of symmetry if the input figure is symmetric or nearly symmetric, irrespective of concavity or convexity. Programs for the hierarchical search for axes of symmetry were written in FORTRAN IV and are implemented on an EC-1033 computer.

SUMMARY

Approximate locations of axes of symmetry of a 2-dimensional region are detected on the basis of the border of the region. The border is described as a chain of certain directional codes and is then regarded as a regular polygon in a hierarchical manner. The vertices of the polygon fall on the border and the distance along the border between two consecutive vertices is taken as the length of the corresponding side. The number of sides of the polygon depends on the level of hierarchy. Lower levels in the hierarchy mean a greater number of sides and less error in describing the original border as a regular polygon. At each level in the hierarchy the best axis of symmetry is found. The best axis at a lower level is always closer to the exact and unknown axis of symmetry of the input figure than the best axis at a higher level. Hence, the best axis at the lowest level is chosen as the ultimate axis of symmetry for the input figure.

The present algorithm starts with a high level in the hierarchy, i.e. with a polygon with, say, n sides. At this level each pair of opposite vertices of the polygon gives a possible axis of symmetry. The best among these $n/2$ axes is to be found. For this a certain kind of error for each of the axes is defined. Less error means the corresponding axis serves better as an axis of symmetry. The axis with the minimum error is taken to be the best axis of symmetry at this level in the hierarchy. Now, to find out the best axis of symmetry at a lower level (i.e. for a polygon with a multiple of n sides) it is not necessary to consider all possible axes of symmetry at that level. Since the best axis at the higher level already gives an approximate location for the exact and unknown axis of symmetry, only a few axes at the lower level, around that best axes, are considered and the axis with the minimum error among those is taken as the best axis of symmetry at this lower level in the hierarchy.

Thus, for a 2-dimensional figure, the axis of symmetry is found, along with a certain error which indicates a sort of deviation from perfect symmetry. On the basis of this error we define the degree of symmetry or symmetry fuzzy membership of a figure, as a decreasing and normalised function of the error. Thus, it is possible to say whether a figure is more or less symmetric than another.

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