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ON A PROPERTY OF STRONGLY REPRODUCTIVE EXPONENTIAL FAMILIES ON $\mathbb{R}$

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Abstract: Strongly reproductive exponential models with affine dual foliations are known to allow of a decomposition analogous to the standard decomposition theorem for Chi-squared distributed quadratic forms in normal variates. It is shown that when the components are identically distributed, then necessarily each component follows the gamma law

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1. Introduction

Consider a positive measure μ on $\mathbb{R}^2$ not concentrated on a line such that its Laplace transform

$$L_\mu(\theta) = \int_{\mathbb{R}^2} \exp\langle\theta, t\rangle \mu(\mathrm{d}t)$$

exists on a subset of $\mathbb{R}^2$ with a non-empty interior $\Theta(\mu)$. It is well-known that $\Theta(\mu)$ is convex. Now for $\theta \in \Theta(\mu)$ we write the cumulant transform

$$k_\mu(\theta) = \log L_\mu(\theta),$$

and

$$P_\theta(\mathrm{d}t) = \exp\{\langle\theta, t\rangle - k_\mu(\theta)\}\mu(\mathrm{d}t).$$

The family of probability measures

$$F = F(\mu) = \{P_\theta;\ \theta \in \Theta(\mu)\}$$

is known as the natural exponential family (NEF) generated by μ. In this paper we shall consider the two-parameter exponential family on $\mathbb{R}$ given by

$$\mathrm{d}P_\theta(x) = a(\theta)b(x)\exp\{\theta_1 u(x) + \theta_2 x\}\,\mathrm{d}x,$$
$$x \in \mathbb{R}. \tag{1}$$

Therefore the NEF associated with it in $\mathbb{R}^2$ is generated by the image μ in $\mathbb{R}^2$ of the measure $b(x)\,\mathrm{d}x$ on $\mathbb{R}$ by the map $x \to [u(x), x]$. Here $k_\mu(\theta) = -\log a(\theta)$. If I_F denotes the interior of the closed convex hull of the support of μ in $\mathbb{R}^2$ and T_F the image of $\theta(\mu)$ by $k'_\mu(\theta)$ in $\mathbb{R}^2$, the family is said to be steep if $I_F = T_F$. Let

$$(\tau_1, \tau_2) = \left(\frac{\partial k_\mu(\theta)}{\partial\theta_1}, \frac{\partial k_\mu(\theta)}{\partial\theta_2}\right)$$
$$= (\mathbf{E}_\theta[u(X)], \mathbf{E}_\theta(X)),$$

where X is the real random variable with distribution (1).

If Θ_i is the projection of $\Theta(\mu)$ by the mapping $(\theta_1, \theta_2) \to \theta_i$ and T_j the projection of T_F by the map $(\tau_1, \tau_2) \to \tau$ ($i = 1, 2$ and $j = 1, 2$), then by Lemma 3.1 of Barndorff-Nielsen and Blaesild (BNB) (1983), for any (θ_1, τ_2) in $\Theta_1 \times T_2$, there exists a unique θ_2 and a unique τ_1 such that

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$(\theta_1, \theta_2) \in \Theta(\mu)$, $(\tau_1, \tau_2) \in T_F$ and

$$\tau(\theta_1, \tau_2) = \frac{\partial k_\mu(\theta_1, \theta_2(\theta_1, \tau_2))}{\partial \theta_1},$$

and

$$\tau_2 \doteq \frac{\partial k_\mu(\theta_1, \theta_2(\theta_1, \tau_2))}{\partial \theta_2}.$$

We shall from now on assume that (1) is both steep and satisfies the above properties.

2. A decomposition

Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from (1), we define

(i) $\bar{x}_n = \left(\frac{1}{n}\right) \sum_{i=1}^{n} X_i$,

(ii) $\bar{u}_n = \left(\frac{1}{n}\right) \sum_{i=1}^{n} u(X_i)$,

(iii) $\bar{t}_n = (\bar{u}_n, \bar{x}_n)$, and

(iv) $v_n = \bar{u}_n - \bar{x}_n$.

The following proposition was established in BNB (1983, Corollary 5.4).

Proposition 1. *Suppose that for the exponential model* (1)

(1) $\theta_2(\theta_1, \tau_2) = -\theta_1 h(\tau_2)$ *for some function* h,

(2) $\bar{t}_n \in I_F$ *with probability* 1,

(3) *for every* $c > 1$, c int $\Theta(\mu) \subseteq$ int $\Theta(\mu)$,

(4) u *is continuous*,

then one has

(a) $\bar{x}_n \approx$ *(is distributed as)* $P_{n\theta}$

(b) u' *exists and* $h(\tau_2) = u'(\tau_2)$,

(c) v_n *is independent* ($\perp$) *of* $\bar{x}_n$,

(d) *the Laplace transform of* v_n *defined for all* s *such that* $\theta_1 + s/n \in \Theta_1$ *is*

$$\mathbf{E}_{\theta_1}(\exp sv_n) = \exp - \{M(n\theta_1 + s) - M(n\theta_1)\} + n\{M(\theta_1 + n^{-1}s) - M(\theta_1)\}$$

for $\theta_1 \in \Theta_1$, *for some real valued function* M *on* intΘ_1.

According to Theorem 3.2, BNB (1983, a) models satisfying the above assumptions are said to be strongly reproductive. Now let

$$R_k = k[\bar{u}_k - u(\bar{x}_k)] \quad \text{for } k = 2, 3, \ldots, n.$$

and

$$Q_2 = R_2, \qquad Q_k = R_k - R_{k-1} \quad \text{for } k = 3, \ldots, n. \tag{3}$$

We then have the following lemma.

Lemma. *Let* $X_1, X_2, \ldots, X_n$ *be a random sample from a steep model given by* (1), *satisfying the conditions of Proposition* 1. *Then* $Q_2, Q_3, \ldots, Q_n$ *are independent and, writing* $g_n(\theta_1, s) = M(n\theta_1 + ns) - M(n\theta_1)$,

$$\mathbf{E}_{\theta_1}(\exp sQ_n) = \exp\{-g_n(\theta_1, s) + g_{n-1}(\theta_1, s) + g_1(\theta_1, s)\}$$

for all n.

Proof. From Proposition 1, $Q_k \perp \bar{x}_k$ for all $k = 2, 3, \ldots, n$. Moreover Q_k is a function of $\bar{x}_{k-1}$, and x_k alone. One can then show that $(Q_2, Q_3, \ldots, Q_n, \bar{x}_n)$ are mutually independent. Indeed it is easy to see that $Q_2 \perp Q_3$. Thus if $Y_k = g_k(Q_k)$, where g_k is a bounded function (in our case, we let $Y_k = \exp(-s_k Q_k)$) we can use induction on n to show that $\mathbf{E}(Y_1, Y_2 \cdots Y_n | \bar{x}_n) = \mathbf{E}(Y_1) \cdots \mathbf{E}(Y_n)$. Hence we have

$$\mathbf{E}(\exp sQ_3) = \mathbf{E}(\exp sR_3)/\mathbf{E}(\exp sR_2) = \exp\{-g_3(\theta_1, s) + g_2(\theta_1, s) + g_1(\theta_1, s)\},$$

and in general, since $Q_n = R_n - R_{n-1}$,

$$\mathbf{E}(\exp sQ_n) = \exp\{-g_n(\theta_1, s) + g_{n-1}(\theta_1, s) + g_1(\theta_1, s)\}.$$

3. The main result

In a personal communication, Blaesild has shown that, if for all $c > 1$, $Q_2, Q_3, \ldots, Q_n$ are identically distributed under $P_{c\theta_0}$ for some $\theta_0 \in \Theta(\mu)$ and every $n \in \mathbb{Z}^+$, then their common distribution is Gamma. We shall show here that if Q_2

and Q_3 are identically distributed under $P_{c\theta_0}$ for some $\theta_0 \in \Theta(\mu)$ and every $c > 1$, then their common distribution is gamma. The same proof goes through essentially for any Q_i and Q_j ($i \neq j$), if Q_i and Q_j are assumed to have identical distribution. Our proof is based on Proposition 2 a generalized version of the Choquet–Deny theorem which can be derived from a general result due to Deny (1961). Elementary real analysis proofs of the result can be found in Ramachandran and Prakasa Rao (1984), and Ramachandran (1987) and a proof using the Krein–Milman theorem in Lau and Rao (1984).

Proposition 2. *Let f be a continuous non-negative real valued function on $\mathbb{R}$ and μ a sigma-finite measure on the Borel subsets of $\mathbb{R}$ such that*

$$f(x) = \int_{-\infty}^{\infty} f(x+y)\,\mathrm{d}\mu(y) \quad \textit{for all } x \in \mathbb{R},$$

then

$$f(x) = A_1(x)\exp(\lambda_1 x) + A_2(x)\exp(\lambda_2 x) \quad \textit{for all } x \in \mathbb{R},$$

where A_1 and A_2 are continuous, non-negative and periodic with every member of the support of μ as period, and λ_1 and λ_2 are solutions of the equation in λ given by

$$\int_{-\infty}^{\infty} \exp(\lambda y)\,\mathrm{d}\mu(y) = 1.$$

(*At most two such λ's exist.*)

Theorem. *If Q_2, and Q_3 as defined in (3) are identically distributed under $P_{c\theta_0}$ for every $c > 1$ and some $\theta_0 \in \Theta(\mu)$, then their common distribution is gamma.*

Proof. Under the above assumptions and the assumptions of Proposition 1, we may, by reparametrization assume, without loss of generality, that Q_2 and Q_3 are identically distributed under P_θ for every $\theta > 0$. This in turn implies that, for every $\theta_1 > 0$, $s > 0$,

$$M(3\theta_1 + 3s) - 2M(2\theta_1 + 2s) + M(\theta_1 + s) = M(3\theta_1) - 2M(2\theta_1) + M(\theta_1).$$

Recall that $M(2\theta_1 + 2s) - M(2\theta_1)$ is the logarithm of $\mathbf{E}(\exp sQ_2)$; therefore M'' exists and is ≥ 0. Differentiating with respect to s twice and letting $s \to 0$, we see that, for every $\theta_1 > 0$,

$$9M''(3\theta_1) - 8M''(2\theta_1) + M''(\theta_1) = 0.$$

With the substitution $L(x) = M(e^x)$, $x \in \mathbb{R}$, the above equation becomes

$$L(u + \ln 2) = \tfrac{9}{8}L(u + \ln 3) + \tfrac{1}{8}L(u), \quad u \in \mathbb{R},$$

or

$$L(u) = \tfrac{9}{8}L\left(u + \ln\tfrac{3}{2}\right) + \tfrac{1}{8}L(u - \ln 2), \quad u \in \mathbb{R}.$$

Note that $L \geq 0$ on $\mathbb{R}$ (since $M(\theta_1 + s) - M(\theta_1)$ is the logarithm of a Laplace transform, its second derivative with respect to s is ≥ 0 for all $s > 0$, $\theta_1 > 0$, i.e., $M'' \geq 0$ on $(0, \infty)$).

Applying Proposition 2 to equation (4), we see that

$$L(u) = A_1(u)\exp(\lambda_1 u) + A_2(u)\exp(\lambda_2 u)$$

where A_1 and A_2 are continuous and periodic with $\ln\frac{3}{2}$ and $\ln 2$ as periods, that is, with $\ln 3$ and $\ln 2$ as periods. These periods being incommensurable, A_1 and A_2 are necessarily constants, and λ_1 and λ_2 are solutions of the equation

$$1 = \tfrac{9}{8}\left(\tfrac{3}{2}\right)^{\lambda} + \tfrac{1}{8}\left(\tfrac{1}{2}\right)^{\lambda} \tag{5}$$

or $2^{2+\lambda} = 3^{2+\lambda} + 1$.

By inspection it is clear that $\lambda = -2$ and $\lambda = -1$ are solutions. It is easily seen by considering the map $\lambda \to 3^{2+\lambda} - 2^{3+\lambda} + 1$, that they are the only roots of (5). Thus there exist A_1 and $A_2 > 0$ such that

$$M''(s) = \frac{A_1}{s} + \frac{A_2}{s^2} \quad \text{for } s > 0$$

or

$$M(s) = A_1(s \ln s - s) - A_2 \ln s + A_3 s + A_4.$$

Noting that $M(3s) - 2M(2s) + M(s)$ is a constant for all $s > 0$, we see that

$$A_1(3s \ln 3 - 4s \ln 2) + A_2(-\ln 3 + \ln 2)$$

is independent of s and hence $A_1 = 0$. Thus

$$M(s) = A_4 + A_3 - A_2 \ln s.$$

Hence

$$-M(2\theta_1+2s)+M(2\theta_1)+2M(\theta_1+s)-M(\theta_1)$$
$$=-A_2\ \ln\left(1+\frac{s}{\theta_1}\right),$$

so that

$$\mathbf{E}(\exp sQ_2)=\left(1+\frac{s}{\theta_1}\right)^{-A_2}.$$

Since A_2 is positive, it follows that Q_2 is gamma distributed.

Observe that when $u(x)=x^2$ (the normal case) and $u(x)=1/x$ (the inverse-Gaussian case) $A_2=\frac{1}{2}$, and Q_2 and Q_3 are chi-squared distributed. Aside from these two cases we are unaware of other examples of $u(x)$ when gamma distributions arise.

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