STUDIA MATHEMATICA, T. LXXXIII. (1986)

Some results on intersection properties of balls in complex Banach spaces

by

Å. LIMA (Aas) and A. K. ROY (Calcutta)

Abstract. Predual real L^1-spaces are characterized by the 4.2. intersection property. The structure of real spaces with the 3.2. intersection property and of real and complex spaces with the 4.3. intersection property is fairly well understood. In this paper we study complex spaces with the *n.k.* intersection property when $n > k \geq 4$. We show that the 5.4. intersection property characterizes complex L^1-preduals, and that the $(2n+1).2n$. intersection property implies the almost $(2n+1).(2n-1)$. intersection property in the complex case.

1. Introduction. Let A be a Banach space over the complex scalars C. $B(a, r)$ denotes the closed ball in A with centre a and radius r. Let n,k be integers with $n > k \geq 2$. We say that A has the *almost n.k.I.P.* (to be read as the almost *n.k.* intersection property) if for every family $\{B(a_j, r_j)\}_{j=1}^n$ of n balls in A such that for any k of them,

$$\bigcap_{m=1}^{k} B(a_{j_m}, r_{j_m}) \neq \emptyset,$$

we have

$$\bigcap_{j=1}^{n} B(a_j, r_j+\varepsilon) \neq \emptyset \quad \text{for all } \varepsilon > 0.$$

(If we can take $\varepsilon = 0$, we say that A has the *n.k.*I.P.) Introducing the space

$$H^n(A^*) = \{(x_1, \ldots, x_n) \in (A^*)^n: \sum_{k=1}^{n} x_k = 0\}$$

with the norm $\|(x_1, \ldots, x_n)\| = \sum_{k=1}^{n} \|x_k\|$, it was proved in [7] that A has the almost *n.k.*I.P. if and only if each extreme point $(x_1, \ldots, x_n)$ in the unit ball of $H^n(A^*)$ has at most k nonzero components. Thus examination of the extreme point structure of the unit ball of $H^n(A^*)$ furnishes a useful analytic device for the study of the intersection properties of balls in Banach spaces, and this has been effectively used in obtaining various characterizations of complex L^1-preduals (see for example [4], [7], [8], [9]). We pursue this approach here by proving that if A has the $(2n+1).2n$.I.P. then it has the almost $2n.(2n-1)$.I.P. $(n \geqslant 2)$, which yields, as a particular consequence, the interesting fact that A has the 5.4.I.P. if and only if A is an L^1-predual. We also prove that if A has the $2n.(2n-1)$.I.P. and if $(x_1, x_2, \ldots, x_{2n-1})$ is an extreme point of the unit ball of $H^{2n-1}(A^*)$ with all its components non-zero, then $n-1$ of the functionals $x_1, x_2, \ldots, x_{2n-1}$ are linearly independent (over C) and the remaining ones are expressible as linear combinations of these functionals. We conclude by describing the analogue in the context of the higher intersection properties of balls considered here, of the *weak intersection property* which has proved useful in characterizing L^1-preduals ([4], [7]).

It should be emphasized that for the validity of the results proved in this paper it is necessary to work over the field of complex numbers.

2. Notations and main results. Let A_1^* and $H^n(A^*)_1$ denote the unit balls of A^* and $H^n(A^*)$ respectively, and let $\partial_e A_1^*$, $\partial_e H^n(A^*)_1$ denote their (respective) sets of extreme points.

If A has the $(n+1).n$.I.P. and $(x_1, \ldots, x_n) \in \partial_e H^n(A^*)_1$ with all $x_k \neq 0$, it is known (see [8], Lemma 3.3 and the remark following it) that $x_k/\|x_k\| \in \partial_e A_1^*$ for all k. We assert that all these functionals are distinct. To prove this, suppose for instance that $x_1 = cx_2$ for some $c > 0$. Writing

$$(x_1, x_2, \ldots, x_n) = \left(0, x_2, \frac{1}{1+c}x_3, \ldots, \frac{1}{1+c}x_n\right) + \left(cx_2, 0, \frac{c}{1+c}x_3, \ldots, \frac{c}{1+c}x_n\right)$$

we get a contradiction with the fact that $(x_1, \ldots, x_n)$ is an extreme point in $H^n(A^*)_1$.

The following result was suggested by Lemma 3.3 and Theorem 3.6 in [4] and [5].

PROPOSITION 2.1. *Suppose A is a Banach space with the $(n+1).n$.I.P. and let $\mathbf{x} = (x_1, \ldots, x_n) \in \partial_e H^n(A^*)_1$ with $\|\mathbf{x}\| = 1$. The following statements are equivalent*:

(1) $\mathbf{x} \in \partial_e H^n(A^*)_1$ *with all* $x_k \neq 0$.

(2) $\{x_k/\|x_k\|\}_{k=1}^n$ *are affinely independent points of A_1^* over R with each* $x_k/\|x_k\| \in \partial_e A_1^*$.

(3) *The points* $\{\|x_k\|, x_k\}_{k=1}^n \subseteq R \times A^*$ *are linearly independent over R and each* $x_k/\|x_k\| \in \partial_e A_1^*$.

Remark. (2) and (3) are equivalent if we delete the requirement that $x_k/\|x_k\| \in \partial_e A_1^*$.

Proof. (2) $\Leftrightarrow$ (3). $\sum_{k=1}^{n} c_k(\|x_k\|, x_k) = 0$ for some $c_k \in \mathbf{R}$ is equivalent to

$$\sum_{k=1}^{n} c_k \|x_k\| = \sum_{k=1}^{n} c_k x_k = 0 \quad \text{for some } c_k \in \mathbf{R}.$$

Writing $t_k = c_k \|x_k\|$, we see that this in turn is equivalent to

$$\sum_{k=1}^{n} t_k \frac{x_k}{\|x_k\|} = 0 \quad \text{for some } t_k \in \mathbf{R} \text{ with } \sum_{k=1}^{n} t_k = 0.$$

(1) $\Rightarrow$ (2). As remarked above, it follows from (1) that $x_k/\|x_k\| \in \partial_e A_1^*$ for all k and that they are distinct. Let $\mu = \sum_{k=1}^{n} \|x_k\| \varepsilon_k$ where ε_k is the measure with unit mass at $x_k/\|x_k\|$. Clearly $\mu \in Z_0$ where Z_0 denotes the set of probability measures on A_1^* representing 0. By Proposition I.6.10 in [1], (2) follows when we have proved that μ is an extreme point in Z_0.

Thus suppose $\mu_1, \mu_2 \in Z_0$ and $2\mu = \mu_1 + \mu_2$. It is obvious that μ_1 and μ_2 have their support in $\{x_1/\|x_1\|, \ldots, x_n/\|x_n\|\}$. Thus we can write $\mu_1 = \sum_{k=1}^{n} \alpha_k \varepsilon_k$ and $\mu_2 = \sum_{k=1}^{n} \beta_k \varepsilon_k$ where $\alpha_k, \beta_k \geqslant 0$, $\alpha_k + \beta_k = 2\|x_k\|$ for all k and

$$\sum_{k=1}^{n} \alpha_k x_k/\|x_k\| = \sum_{k=1}^{n} \beta_k x_k/\|x_k\| = 0.$$

Writing

$$2(x_1, \ldots, x_n) = (\alpha_1 x_1/\|x_1\|, \ldots, \alpha_n x_n/\|x_n\|) + (\beta_1 x_1/\|x_1\|, \ldots, \beta_n x_n/\|x_n\|)$$

and using the fact that $\mathbf{x} \in \partial_e H^n(A^*)_1$, we see that $\alpha_k = \beta_k = \|x_k\|$ for all k. Thus we get $\mu_1 = \mu_2 = \mu$ and it follows that $\mu \in \partial_e Z_0$.

The proof of (2) $\Rightarrow$ (1) easily follows from Proposition I.6.10 in [1] by an argument similar to that of (1) $\Rightarrow$ (2) above.

PROPOSITION 2.2. *Let* $(x_1, \ldots, x_n) \in \partial_e H^n(A^*)_1$ *with all* $x_k \neq 0$. *Then we have* $\operatorname{span}_{\mathbf{R}}\{x_1, \ldots, x_n\} \cong \mathbf{R}^{n-1}$ *where* $\cong$ *means linear isomorphism.*

Proof. Let $E = \operatorname{span}_{\mathbf{R}}\{x_1, \ldots, x_n\}$. Since $\sum_{k=1}^{n} x_k = 0$, we have $\dim_{\mathbf{R}} E \leqslant n-1$. Assume for contradiction that $\dim_{\mathbf{R}} E \leqslant n-2$. By a theorem of Helly every family of n convex sets in E^* such that any $n-1$ intersect has a nonempty intersection. Thus E^* has the $n.(n-1)$.I.P. But then every extreme point in $H^n(E)_1$ has at most $n-1$ nonzero coordinates (see Theorem 2.10 in [7]). Thus $(x_1, \ldots, x_n)$ cannot be an extreme point in $H^n(E)_1$ or in $H^n(A^*)_1$.

Remark. In this paper our main interest is in complex Banach spaces.

Propositions 2.1 and 2.2 are, however, valid for real spaces as well.

In the following we shall need a lemma on complex matrices.

LEMMA 2.3. *Let* $M = B+iI$ *where* B *is a real* $p\times p$ *matrix and* I *is the identity matrix. If* $N(M)$ *denotes the null space of* M, *we have* $\dim_C N(M) \leqslant p/2$.

Proof. Let $x\in C^p$. We have $x\in N(M)$ if and only if $Bx = -ix$, which in turn is equivalent to $B\bar{x} = i\bar{x}$ where $\bar{x}$ is the complex conjugate of x. Since $Bx = -ix$ and $Bx = ix$ if and only if $x = 0$, we get $N(B+iI)\cap N(B-iI) = \{0\}$. Thus $\dim_C N(B+iI) \leqslant p/2$.

LEMMA 2.4. *Assume* A *is a complex Banach space with the almost* $(n+1).n.I.P.$ *and assume* $(x_1, \ldots, x_n)\in \partial_e H^n(A^*)_1$ *with all* $x_k \neq 0$. *Then there exist numbers* a_{kj}, $1 \leqslant k, j \leqslant n-1$, *with* $a_{kj}\in R$ *for* $k\neq j$ *and* $a_{kk}\in C\setminus R$ *for all* k, *such that for* $k = 1, \ldots, n-1$

$$\sum_{j=1}^{n-1} a_{kj}x_j = 0.$$

Proof. Let $E = \operatorname{span}_R\{x_1, \ldots, x_n\}$. By Proposition 2.2 we have $\dim E = n-1$. As noted in [8] (see Lemma 3.3 and the remark following it), $x_k/\|x_k\|\in \partial_e A_1^*$ for all k. Let $\theta = 1+i$. Then $\theta+\bar{\theta} = 2$. By Theorem 3.1 in [8], we can write in $H^{n+1}(A^*)$

$$\begin{array}{rlllll}
(& \theta x_1, & \bar{\theta}x_1, & 2x_2, & 2x_3, \ldots, & 2x_n) \\
=(& 0, & b_{12}\bar{\theta}x_1, & b_{13}x_2, & b_{14}x_3, \ldots, & b_{1,n+1}x_n) \\
+(& b_{21}\theta x_1, & 0, & b_{23}x_2, & b_{24}x_3, \ldots, & b_{2,n+1}x_n) \\
+(& b_{31}\theta x_1, & b_{32}\bar{\theta}x_1, & 0, & b_{34}x_3, \ldots, & b_{3,n+1}x_n) \\
+ & \cdots & \cdots & \cdots & \cdots & \cdots \\
+(& b_{n+1,1}\theta x_1, & b_{n+1,2}\bar{\theta}x_1, & b_{n+1,3}x_2, & b_{n+1,4}x_3, \ldots, & 0)
\end{array}$$

where all $b_{ij}\geqslant 0$ and

$$\begin{array}{llllll}
0 & +\, b_{21} & +\; b_{31} & +\ldots+ & b_{n+1,1} & = 1, \\
b_{12} & +\, 0 & +\; b_{32} & +\ldots+ & b_{n+1,2} & = 1, \\
b_{13} & +\, b_{23} & +\; 0 & +\ldots+ & b_{n+1,3} & = 2, \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
b_{1,n+1} & +\, b_{2,n+1} & +\; b_{3,n+1} & +\ldots+ & 0 & = 2.
\end{array}$$

If $b_{12}\neq 0$, then

$$b_{12}(1-i)x_1+b_{13}x_2+\ldots+b_{1,n+1}x_n = 0.$$

We have

$$b_{1,n+1}x_1+b_{1,n+1}x_2+\ldots+b_{1,n+1}x_n = 0.$$

Subtracting, we get

$$(b_{12}(1-i)-b_{1,n+1})x_1+(b_{13}-b_{1,n+1})x_2+\dots+(b_{1,n}-b_{1,n+1})x_{n-1}=0.$$

Let

$$a_{11}=b_{12}(1-i)-b_{1,n+1},\quad a_{12}=b_{13}-b_{1,n+1},\dots,a_{1,n-1}=b_{1,n}-b_{1,n+1}$$

and we get $\sum_{j=1}^{n-1} a_{1j}x_j=0$ with $a_{11}\in C\setminus R$ and $a_{1j}\in R$ for $j\neq 1$.

If $b_{12}=0$, then $b_{k2}\neq 0$ for some $k\geqslant 3$, say $b_{32}\neq 0$. Thus we have

$$(b_{31}(1+i)+b_{32}(1-i))x_1+b_{34}x_3+\dots+b_{3,n+1}x_n=0.$$

If $b_{31}=b_{32}$, then we have

$$(b_{31}+b_{32})x_1+b_{34}x_3+\dots+b_{3,n+1}x_n=0$$

and this gives $x_1\in \operatorname{span}_R\{x_3,\dots,x_n\}$. But this contradicts $\dim E=n-1$. Hence we must have $b_{31}\neq b_{32}$. We can thus proceed as in the case with $b_{12}\neq 0$ to achieve

$$\sum_{j=1}^{n-1} a_{1j}x_j=0 \quad \text{with } a_{11}\in C\setminus R \text{ and } a_{1j}\in R \text{ for } j\neq 1.$$

Applying the same reasoning to $x_2, x_3,\dots,x_{n-1}$ gives the claimed set of equations. The proof is complete.

Example. Let $A=C^3$ with l_1-norm. If

$$x_1=\begin{bmatrix}1\\1\\1\end{bmatrix},\quad x_2=\begin{bmatrix}i\\-1\\-i\end{bmatrix},\quad x_3=\begin{bmatrix}-1\\-i\\i\end{bmatrix},\quad x_4=\begin{bmatrix}-i\\i\\-1\end{bmatrix}$$

then $\mathbf{x}=(x_1,x_2,x_3,x_4)\in\partial_e H^4(A^*)_4$. In this case, it is not possible to write

$$a_{11}x_1+a_{12}x_2+a_{13}x_3=0$$

with $a_{11}\in C\setminus R$ and $a_{12},a_{13}\in R$. Thus A does not have the 5.4.I.P.

Theorem 2.5. *Assume A is a complex Banach space with the almost $(2n+1).(2n)$.I.P. Then A has the almost $(2n).(2n-1)$.I.P.*

Proof. Suppose $(x_1,\dots,x_{2n})\in\partial_e H^{2n}(A^*)_1$. It suffices to show that some $x_k=0$. Assume for contradiction that all $x_k\neq 0$. By Lemma 2.4 there exist numbers a_{kj}, $1\leqslant k,j\leqslant 2n-1$, with $a_{kj}\in R$ for $k\neq j$ and $a_{kk}\in C\setminus R$ for all k, such that for all $k=1,2,\dots,2n-1$

$$(*)\qquad \sum_{j=1}^{2n-1} a_{kj}x_j=0.$$

Clearly we may assume $\operatorname{Im} a_{kk}=1$ for all k. Let $p=2n-1$ and $M=(a_{kj})$. By Lemma 2.3 we get $\dim_C N(M)\leqslant n-\frac{1}{2}$, so that $\dim_C N(M)\leqslant n-1$. By a standard result in linear algebra (see [3], p. 12), the complex dimension of the solution space of (*) is $\leqslant n-1$. Hence $\operatorname{span}_C\{x_1, \ldots, x_{2n}\} = \operatorname{span}_C\{x_1, \ldots, x_{2n-1}\}$ has complex dimension at most $n-1$, and therefore $\dim_R \operatorname{span}_C\{x_1, \ldots, x_{2n}\} \leqslant 2n-2$. This contradicts Proposition 2.2 which says that $\operatorname{span}_R\{x_1, \ldots, x_{2n}\} \cong R^{2n-1}$. This completes the proof.

The above argument gives some additional information when A has the $(2n).(2n-1)$.I.P. as well.

THEOREM 2.6. *Assume A is a complex Banach space with the almost $(2n).(2n-1)$.I.P. and assume $(x_1, \ldots, x_{2n-1}) \in \partial_e H^{2n-1}(A^*)_1$ with all $x_k \neq 0$. Then there exist $n-1$ (complex) linearly independent functionals among $\{x_1, x_2, \ldots, x_{2n-1}\}$ and the remaining n functionals x_k are complex linear combinations of these $n-1$ functionals.*

Proof. As in the proof of Theorem 2.5 we get

$$\sum_{j=1}^{2n-2} a_{kj} x_j = 0$$

but now with $1 \leqslant k, j \leqslant 2n-2$. We also get, with $M = (a_{kj})$, $\dim_C N(M) \leqslant n-1$. By using Proposition 2.2, it follows that $\dim_C N(M) = n-1$ and that $\operatorname{span}_C\{x_1, x_2, \ldots, x_{2n-1}\} \cong C^{n-1}$.

COROLLARY 2.7. *If a complex Banach space A has the almost 5.4.I.P. then A^* is isometric to an $L^1(\mu)$-space.*

Proof. Let $n = 2$ in Theorem 2.5. Then we deduce that A has the almost 4.3.I.P. By Theorem 4.1 in [8], it follows that A^* is isometric to an $L^1(\mu)$-space.

COROLLARY 2.8. *If a complex Banach space has the almost 5.4.I.P., then it has the almost n.3.I.P. for all n.*

Proof. It is known (see e. g. [8]) that predual $L^1(\mu)$-spaces have the almost n.3.I.P. for all n.

If $\dim_C A = k$, it follows from Helly's theorem that A has the $(2k+2)\cdot(2k+1)$.I.P. Hustad [5] has proved that $l_1^k(C)$ does not have the $(2k+1).(2k)$.I.P.

3. Examples. We shall give some examples of spaces with the $(2n).(2n-1)$.I.P. Let $L(X, Y)$ denote the Banach space of bounded linear operators from the Banach space X to the Banach space Y. Since $L(l_1^n, A) \cong (A \otimes \ldots \otimes A)_{l_\infty^n}$ and also $L(l_1^n, A) \cong L(A^*, l_\infty^n)$ by the map $T \to T^*$, we conclude that if $\dim_C A = k$, then A, $L(l_1^n, A)$ and $L(A^*, l_\infty^n)$ have the $(2k+2).(2k+1)$.I.P. by Helly's theorem.

PROPOSITION 3.1. *Assume $\dim_C A = k < \infty$. If $X \cong L^1(\mu)$ for some measure μ, then $L(X, A)$ has the almost $(2k+2).(2k+1)$.I.P.*

The proof is similar to the proof of the next result.

PROPOSITION 3.2. *Assume* $\dim_C A = k < \infty$. *If* $X^* \cong L^1(\mu)$ *for some measure* μ, *then* $L(A, X)$ *has the almost* $(2k+2).(2k+1)$.*I.P.*

Proof. Let $T_1, \ldots, T_{2k+2} \in L(A, X)$ and let $r_1, \ldots, r_{2k+2} > 0$. Assume that any $2k+1$ of $\{B(T_n, r_n)\}_{n=1}^{2k+2}$ intersect. Let $\varepsilon > 0$. Since $Y = \sum_{n=1}^{2k+2} T_n(A)$ is a finite-dimensional subspace of X, there is a subspace Z of X such that $Z \cong l_\infty^m(C)$ for some m and $d(x, Z) \leqslant \varepsilon\|x\|$ for all $x \in Y$ (see [6]). There is a norm-one projection Q in X with $Q(X) = Z$. We get $\|T_n - QT_n\| \leqslant \varepsilon$ for all n. Since $QT_n \in L(A, Z) \cong L(A, l_\infty^m)$, it easily follows from the remark preceding Proposition 3.1 that

$$\bigcap_{n=1}^{2k+2} B(T_n, r_n+\varepsilon) \neq \emptyset.$$

The third example shows that some spaces of continuous functions have the $(2k+2).(2k+1)$.I.P.

PROPOSITION 3.3. *Assume that* $\dim_C A = k < \infty$ *and that* S *is a compact Hausdorff space. Then* $C(S, A)$ *has the almost* $(2k+2).(2k+1)$.*I.P.*

Proof. It is well known that we can identify $C(S, A)$ with the injective tensor product $C(S)\check{\otimes}A$ (see [2]). Let $\varepsilon > 0$ and let $\{B(f_i, r_i)\}_{i=1}^{2k+2}$ be balls in $C(S, A)$ such that any $2k+1$ intersect. Arguing as in [2, p. 225], we find an open covering $\{U_j\}_{j=1}^n$ of S, a partition of unity $\{g_j\}_{j=1}^n$ subordinate to this covering and points $\omega_j \in U_j$, $1 \leqslant j \leqslant n$, such that if we put $h_i(\omega) = \sum_{j=1}^n g_j(\omega) f_i(\omega_j)$, then $\|f_i(\omega) - h_i(\omega)\| < \varepsilon$ for each $\omega \in S$ and each i. It follows that any $2k+1$ of the balls $\{B(h_i, r_i+\varepsilon)\}_{i=1}^{2k+2}$ intersect.

Let $x_{ij} = f_i(\omega_j) \in A$. We shall identify h_i with $\sum_{i=1}^n g_j \otimes x_{ij}$ in $C(S)\check{\otimes}A$. Let $E = \operatorname{span}\{g_1, \ldots, g_n\} \subseteq C(S)$. Since E is generated by a partition of unity in $C(S)$, it follows that E is isometric to l_∞^n. Thus E is the range of a norm-one projection P in $C(S)$. $E\check{\otimes}A$ is a closed subspace of $C(S)\check{\otimes}A$ by Proposition 7, p. 225 in [2], and it is the range of the norm-one projection $P\otimes I$ in $C(S)\check{\otimes}A$. Thus it suffices to show that $l_\infty^n\check{\otimes}A$ has the $(2k+2).(2k+1)$.I.P. We have the identifications

$$l_\infty^n\check{\otimes}A \cong K(l_1^n, A) \cong L(l_1^n, A)$$

and it follows from Proposition 3.1 that $l_\infty^n\check{\otimes}A$ has the $(2k+2).(2k+1)$.I.P.

4. The ***n.k.*** **intersection property.** It is known that the n.2.I.P. with $n \geqslant 4$ (n.3.I.P. in the complex case) characterizes L^1-predual spaces. The requirement that any two balls (any three balls in the complex case) intersect is equivalent to the nonempty intersection of their images under any norm-one functional. As shown in Theorem 4.2 below, this can be generalized to be correct also for the $n.k.$I.P.

THEOREM 4.1. *Let $\{B(a_i, 1)\}_{i=1}^k$ be balls in A. The following statements are equivalent:*

(1) $\bigcap_{i=1}^{k} B(a_i, 1+\varepsilon) \neq \emptyset$ *for all* $\varepsilon > 0$.

(2) *If E is a real Banach space with* $\dim E \leqslant k-1$ *and $T: A \to E$ is a real-linear operator with* $\|T\| \leqslant 1$, *then* $\bigcap_{i=1}^{k} B(Ta_i, 1) \neq \emptyset$.

Proof. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1). Assume $\bigcap_{i=1}^{k} B(a_i, 1+\varepsilon) = \emptyset$ for some $\varepsilon > 0$. Then there exists [7] $(x_1, \ldots, x_k) \in \partial_e H^k(A^*)_1$ with

$$1 < \sum_{i=1}^{k} x_i(a_i).$$

Let A_R denote the real restriction of A. Then we have $(A_R)^* \cong (A^*)_R$. Let $E^* = \operatorname{span}_R\{\operatorname{Re} x_1, \ldots, \operatorname{Re} x_k\}$ in $(A_R)^*$. Then $\dim E^* \leqslant k-1$. Let $T^*: E^* \to (A_R)^*$ be the identity map. Then we may consider $T: A_R \to E$ as a quotient map. Define $z_i \in E^*$ by $T^* z_i = \operatorname{Re} x_i$. Then $(z_1, \ldots, z_k) \in H^k(E^*)$ since $\|\operatorname{Re} x_i\| = \|x_i\|$ for all i. Moreover, since for any $a \in A_R$ and all i, $\operatorname{Re} x_i(a) = T^* z_i(a) = z_i(Ta)$ we get

$$1 < \sum_{i=1}^{k} \operatorname{Re} x_i(a_i) = \sum_{i=1}^{k} z_i(Ta_i).$$

Thus $\bigcap_{i=1}^{k} B(Ta_i, 1) = \emptyset$ in E.

THEOREM 4.2. *Let $n > k \geqslant 2$. The following statements are equivalent:*

(3) *A has the almost $n.k.$I.P.*

(4) *Let $\{B(a_i, 1)\}_{i=1}^n$ be n balls in A. If for every real Banach space E with* $\dim E \leqslant k-1$ *and every real-linear operator $T: A \to E$ with* $\|T\| \leqslant 1$ *we have* $\bigcap_{i=1}^{n} B(Ta_i, 1) \neq \emptyset$, *then* $\bigcap_{i=1}^{n} B(a_i, 1+\varepsilon) \neq \emptyset$ *for all* $\varepsilon > 0$.

Proof. (3) $\Rightarrow$ (4) follows from (2) $\Rightarrow$ (1) above.

(4) $\Rightarrow$ (3). Assume (3) is false. By the proof of Theorem 4.3 in [10], it follows that there exist n balls $\{B(a_i, 1)\}_{i=1}^n$ in A such that any k intersect, but $\bigcap_{i=1}^{n} B(a_i, 1+\varepsilon) = \emptyset$ for some $\varepsilon > 0$. Now use (1) $\Rightarrow$ (2) above to see that (4) is false.

References

[1] E. M. Alfsen, *Compact Convex Sets and Boundary Integrals*, Springer-Verlag, Berlin 1971.
[2] J. Diestel and J. J. Uhl, *Vector Measures*, Math. Surveys 15, Amer. Math. Soc., Providence, R. I. 1977.
[3] K. Hoffman and R. Kunze, *Linear Algebra*, 2nd ed., Prentice-Hall of India, Private Ltd., 1975.
[4] O. Hustad, *Intersection properties of balls in complex Banach spaces whose duals are L^1-spaces*. Acta Math. 132 (1974), 283–313.
[5] —, *Intersection properties of balls in finite dimensional l_1 spaces*, Oslo 1974.
[6] A. Lazar and J. Lindenstrauss, *Banach spaces whose duals are L_1-spaces and their representing matrices*, Acta Math. 126 (1971), 165–193.
[7] Å. Lima, *Intersection properties of balls and subspaces in Banach spaces*, Trans. Amer. Math. Soc. 227 (1977), 1–62.
[8] —, *Complex Banach spaces whose duals are L^1-spaces*. Israel J. Math. 24 (1976), 59–72.
[9] Å. Lima and A. K. Roy, *Characterizations of complex L^1-preduals*, Quart. J. Math. Oxford (2) 35 (1984), 439–453.
[10] J. Lindenstrauss, *Extension of compact operators*, Mem. Amer. Math. Soc. 48 (1964).

DEPARTMENT OF MATHEMATICS AND STATISTICS
AGRICULTURAL UNIVERSITY OF NORWAY
1432 Aas-NLH, Norway

and

STAT/MATH. DIVISION
INDIAN STATISTICAL INSTITUTE
203 B.T. Road, Calcutta 700035, India

Received July 24, 1984 (1990)
Revised version October 15, 1984