ON STOCHASTIC MATRICES AND KERNELS

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1. Introduction. It is well-known that every stochastic matrix has 1 as an eigen-value with $(1, \ldots, 1)$ as a corresponding eigen-vector. Further, a stochastic matrix, as a linear transformation, leaves the positive orthant invariant, of which $(1, \ldots, 1)$ is an interior point. Motivated by this, we show in Section 2, that any linear transformation in a finite-dimensional real vector space, positive with respect to a lattice cone and having a fixed vector in its interior is representable by a stochastic matrix with respect to a suitable basis. This gives a coordinate-free characterization of stochastic matrices. The following extension of this result to the infinite-dimensional case is obtained: If T is a positive operator with respect to a normal lattice cone K with interior in a real Banach space having a fixed vector in the interior of K, then T is essentially a stochastic operator. (For a precise statement, see Section 3.)

2. Stochastic matrices. Let V be a real normed linear space. A closed subset $K \subseteq V$, is called a *cone* if

(i) $\quad x, y \in K \Rightarrow \lambda x + \mu y \in K \quad$ for $\lambda, \mu \geqslant 0$,

(ii) $\quad K \cap -K = \theta \quad$ (the zero element of V).

Given a cone K, we can define a partial order $\leqslant$ on V by writing $x \leqslant y$ if and only if $y - x \in K$. K is a *lattice cone* if K is a lattice with respect to the partial order $\leqslant$. A linear transformation T of V into itself is *positive* (with respect to the cone K) if $TK \subseteq K$.

Theorem 1. *Let K be a lattice cone with interior in a real finite (n-) dimensional vector space V. Let A be a positive transformation (with respect to K) with $Az = z$ for some z in the interior of K. Then there exists a basis $\{e_1, e_2, \ldots, e_n\}$ for V with respect to which A is represented by a stochastic matrix.*

Proof. Since K is a lattice cone with interior, by a well-known theorem of Yudin (see [3], p. 22), there exists a basis $\{e_1', e_2', \ldots, e_n'\}$ such that

$$K = \left\{ y : y = \sum_i y_i e_i';\ y_i \geqslant 0, \quad i = 1, 2, \ldots, n \right\}.$$

Since $x = \sum_i x_i e_i'$ is an interior point of K, $x_i > 0$ for each i. Further $e_i' \in K$ for each i and $AK \subseteq K$. Thus if we write

$$Ae_j' = \sum_i a_{ij}' e_i', \qquad j = 1, 2, \ldots, n, \tag{1}$$

then $a_{ij}' \geqslant 0$ for $i, j = 1, 2, \ldots, n$.

Consider the new basis $\{e_1, e_2, \ldots, e_n\}$ defined by

$$e_j = x_j e_j', \qquad j = 1, 2, \ldots, n.$$

Let

$$Ae_j = \sum_i a_{ij} e_i, \qquad j = 1, 2, \ldots, n,$$

i. e.,

$$Ax_j e_j' = \sum_i a_{ij} x_i e_i'.$$

Therefore

$$Ae_j' = \sum_i a_{ij} \frac{x_i}{x_j} e_i' \quad \text{for } i, j = 1, 2, \ldots, n.$$

Using (1), we get $a_{ij} = a_{ij}'(x_j/x_i)$ for $i, j = 1, 2, \ldots, n$. Thus $a_{ij} \geqslant 0$. Since $Ax = x$, $\sum_j a_{ij}' x_j = x_i$, so that

$$\sum_j a_{ij} = \frac{1}{x_i} \sum_j a_{ij}' x_j = 1 \text{ for } i = 1, 2, \ldots, n.$$

Thus the transformation A corresponds to the stochastic matrix (a_{ij}) in the basis $\{e_1, e_2, \ldots, e_n\}$.

Similarly for the doubly stochastic case we have the following

Theorem 2. *A necessary and sufficient condition that the matrix of A of theorem 1 be doubly stochastic is that $A^*\sigma(x) = \sigma(x)$ where σ is the canonical isomorphism of V on to V^* corresponding to the basis $\{e_1, e_2, \ldots, e_n\}$.*

Proof. Sufficiency. Since $x = \sum_i e_i$, $\sigma(x) = \sum_i f_i$ where $\{f_1, f_2, \ldots, f_n\}$ is the basis in V^* dual to $\{e_1, e_2', \ldots, e_n\}$. Also

$$A^* f_j = \sum_i a_{ji} f_i.$$

Thus $A^*\sigma(x) = \sigma(x)$ *gives* $\sum_j a_{ji} = 1$ for $i = 1, 2, \ldots, n$, i. e., the matrix (a_{ij}) *is* doubly stochastic.

The necessary part follows trivially.

3. Stochastic kernels. A stochastic matrix may be thought of as associating a probability measure on a finite set with every point of the set. This leads us to the following definition of a stochastic kernel. We restrict ourselves to compact Hausdorff spaces.

Let S be a compact Hausdorff space, $\mathfrak{B}$ the class of Borel sets of S, $C(S)$ the space of real-valued continuous functions on S and $C^+(S)$ the cone of non-negative functions in $C(S)$. Recall that $C^+(S)$ is a normal lattice cone with interior. (A cone K is *normal* if there exists a $\delta > 0$ such that $\|x + y\| \geqslant \delta \max\{\|x\|, \|y\|\}$ for all $x, y \in K$.)

Definition. A function $K(s, E)$ defined on $S \times \mathfrak{B}$ is a *stochastic kernel* if $K(s, \cdot)$ is a regular probability measure on $(S, \mathfrak{B})$ for each $s \in S$, such that for evedy f in $C(S)$, the function g defined by

$$g(s) = \int_S f(t) K(s, dt)$$

is in $C(S)$.

Observe that our definition reduces to the usual definition (see e.g. [3]) if there exists a measure ν on $(S, \mathfrak{B})$ such that $K(s, \cdot) \ll \nu$ for every $s \in S$.

It is easily seen that the linear transformation T on $C(S)$ (to be called a *stochastic operator*) defined by

$$(Tf)(s) = \int_S f(t) K(s, dt)$$

is bounded and has norm 1. In fact, the spectral radius of T is 1. In case $K(s, E)$ is measurable in s for every fixed $E \in \mathfrak{B}$, it can be considered to be the transition function of a Markov process. The corresponding operator on $C(S)$ has been studied in detail by Rosenblatt [4].

It is interesting to note that with our definition, the identity operator is a stochastic operator and corresponds to the stochastic kernel $K(s, \cdot) = \delta_s(\cdot)$ where δ_s is the Dirac measure at s defined by

$$\delta_s(E) = \begin{cases} 1 & \text{if } s \in E \\ 0 & \text{if } s \notin E. \end{cases}$$

Further we may define a large class of stochastic operators which are analogous to the permutation matrices in finite dimensions. The motivation for this definition is the observation that the effect of a permutation matrix on a vector is only to permute the components of the vector.

Definition. An operator P on $C(S)$ is called a *permutation* if there exists a homeomorphism π of S onto S such that

$$(Pf)(s) = f(\pi^{-1}(s)).$$

Clearly for all f in $C(S)$,

$$(Pf)(s) = f(\pi^{-1}(s)) = \int_S f(t) K(s, dt)$$

where $K(s, \cdot) = \delta_{\pi^{-1}(s)}(\cdot)$, i. e., P is a stochastic operator with kernel $\delta_{\pi^{-1}(s)}$.

Every stochastic operator leaves the normal lattice cone $C^+(S)$ invariant and has a fixed vector in its interior, namely the function $f(s) \equiv 1$. The following theorem goes in the opposite direction.

Theorem 3. *Let K be a normal lattice cone with interior in a real Banach space V. If T is a bounded positive operator on V (i. e., $TK \subseteq K$) and $Tx = x$ for some x in the interior of K, then there exists a compact Hausdorff space S and a bicontinuous isomorphism σ of V onto $C(S)$ such that T is carried to a stochastic operator by σ.*

Proof. By a well-known theorem of M. Krein and S. Krein [5] and Kakutani [2], there exists a compact Hausdorff space S and a linear bicontinuous lattice isomorphism τ of V onto $C(S)$ such that $\tau(K) = C^+(S)$. If $T_0 = \tau T \tau^{-1}$, it is clear that T_0 is a bounded operator on $C(S)$ with $T_0 C^+(S) \subseteq C^+(S)$ and $T_0 f_0 = f_0$ where $f_0 (= \tau(x))$ lies in the interior of $C^+(S)$. Hence $f_0 > 0$. If $\Lambda_s(f) = (T_0 f)(s)$, then for fixed s, $\Lambda_s(f)$ is a bounded linear functional on $C(S)$ which is non-negative by the positivity of T_0. Hence we have, by Riesz's theorem,

$$(T_0 f)(s) = \int f(t) K(s, dt), \qquad f \in C(S),$$

where, for fixed s, $K(s, \cdot)$ is a finite positive regular measure on $(S, \mathfrak{B})$.

Put $K_1(s, E) = \dfrac{1}{f_0(s)} \int\limits_E f_0(t) K(s, dt)$. Clearly $K_1(s, E) \geqslant 0$ for all $s \in S$, $E \in \mathfrak{B}$.

Further,

$$K_1(s, S) = \frac{1}{f_0(s)} \int\limits_S f_0(t) K(s, dt) = \frac{1}{f_0(s)} (T_0 f_0)(s) = 1,$$

i. e., $K_1(s, \cdot)$ is a probability measure for fixed s. $K_1(s, \cdot)$ is regular, because $K_1(s, \cdot) \ll \ll K(s, \cdot)$ and $K(s, \cdot)$ is regular.

Also for f in $C(S)$,

$$\int\limits_S f(t) K_1(s, dt) = \int\limits_S f(t) \frac{f_0(t)}{f_0(s)} K(s, dt) = \frac{1}{f_0(s)} (T_0(f \cdot f_0))(s)$$

which is in $C(S)$. Thus $K_1(s, \cdot)$ is a stochastic kernel. Put

$$(T_1 f)(s) = \int\limits_S f(t) K_1(s, dt).$$

The map $\tau' : C(S) \to C(S)$ defined by $\tau'(f) = (1 / f_0) \cdot f$ is a bicontinuous isomorphism such that

$$(T_1 f)(s) = \int\limits_S f(t) K_1(s, dt) = \frac{1}{f_0(s)} \int\limits_S f(t) f_0(t) K(s, dt) =$$

$$= \frac{1}{f_0(s)} \int\limits_S (\tau'^{-1} f)(t) K(s, dt) = (\tau' T_0 \tau'^{-1} f)(s) = (\tau' \tau T \tau^{-1} \tau'^{-1} f)(s).$$

Putting now $\sigma = \tau' \tau$ we see that σ has the required properties and the theorem is proved.

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О СТОХАСТИЧЕСКИХ МАТРИЦАХ И ЯДРАХ

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(Резюме)

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