PARAMETRIZATIONS OF G_δ-VALUED MULTIFUNCTIONS

BY

H. SARBADHIKARI AND S. M. SRIVASTAVA

ABSTRACT. Let T, X be Polish spaces, $\mathfrak{T}$ a countably generated sub-σ-field of $\mathfrak{B}_T$, the Borel σ-field of T, and $F: T \to X$ a multifunction such that $F(t)$ is a G_δ in X for each $t \in T$. F is $\mathfrak{T}$-measurable and $\mathrm{Gr}(F) \in \mathfrak{T} \otimes \mathfrak{B}_X$, where $\mathrm{Gr}(F)$ denotes the graph of F. We prove the following three results on F.

(I) There is a map $f: T \times \Sigma \to X$ such that for each $t \in T$, $f(t, \cdot)$ is a continuous, open map from Σ onto $F(t)$ and for each $\sigma \in \Sigma$, $f(\cdot, \sigma)$ is $\mathfrak{T}$-measurable, where Σ is the space of irrationals.

(II) The multifunction F is of Souslin type.

(III) If X is uncountable and $F(t)$, $t \in T$, are all dense-in-itself then there is a $\mathfrak{T} \otimes \mathfrak{B}_X$-measurable map $f: T \times X \to X$ such that for each $t \in T$, $f(t, \cdot)$ is a Borel isomorphism of X onto $F(t)$.

1. Introduction. The object of this paper is to study G_δ-valued multifunctions. We take T, X to be Polish spaces, $\mathfrak{T}$ a countably generated sub-σ-field of $\mathfrak{B}_T$, the Borel σ-field of T, and $F: T \to X$ a multifunction such that F is $\mathfrak{T}$-measurable, $\mathrm{Gr}(F) \in \mathfrak{T} \otimes \mathfrak{B}_X$ and $F(t)$ is a G_δ in X for each $t \in T$. Definitions and notation are given in §2. G_δ-valued multifunctions arise in the study of C^*-algebras, group representations, etc. ([5], [12]).

In [15], the existence of a $\mathfrak{T}$-measurable selector for F is established and this article can be viewed as a sequel to [15]. Having proved the existence of a measurable selector for F, several questions arise. Can we express $\mathrm{Gr}(F)$ as a union of the graphs of measurable selectors for F? If yes, can we get these graphs to be, moreover, disjoint? Naturally, for the second problem, $F(t)$, $t \in T$, must all be of the same cardinality.

We approach the first problem in more than one way. In §3, we prove a representation theorem for such multifunctions of the kind recently obtained by Ioffe [4] and Srivastava [14] for closed valued multifunctions. In §4, we prove that these multifunctions are of Souslin type in the sense of Leese [8]. This gives us a very important relationship between F and closed valued multifunctions and enables us to answer our question in the affirmative.

We consider the second problem in §5. By a very old and classical result of Luzin ([9, p. 252], [10]), the answer to this question is "yes" for countable-valued F. In the case, $F(t)$, $t \in T$, are all uncountable, we prove a parametrization theorem, analogous to the one recently obtained by Mauldin [11], for F.

2. Preliminaries. The set of positive integers will be denoted by N. S will denote the set of all finite sequences of positive integers, including the empty sequence e.

For each $k > 0$, we denote by S_k the set of elements of S of length k. For $s \in S$, $|s|$ will denote the length of s and if $i < |s|$ is a positive integer, s_i will denote the ith co-ordinate of s. If $n \in N$, sn will denote the catenation of s and n. We put $\Sigma = N^N$. Endowed with the product of discrete topologies on N, Σ becomes a homeomorph of the space of irrationals. For $\sigma \in \Sigma$ and $k \in N$, σ_k will denote the kth co-ordinate of σ and $\sigma|k = (\sigma_1, \ldots, \sigma_k)$. If $k = 0$, $\sigma|k = e$. If $s \in S$, Σ_s will denote the set $\{ \sigma \in \Sigma : \sigma|k = s \}$.

D will denote the set of all finite sequences of 0's and 1's, including the empty sequence e. C will denote the set $\{0, 1\}^N$. Endowed with the product of discrete topologies on $\{0, 1\}$, it becomes a homeomorph of the Cantor set. For $d \in D$, $k > 0$, $i \in \{0, 1\}$ and $\varepsilon \in C$, d_k, ε_k, $\varepsilon|k$, $|d|$ and di are similarly defined.

Let $(X, \mathfrak{A})$ and $(Y, \mathfrak{B})$ be measurable spaces. We denote by $\mathfrak{A} \otimes \mathfrak{B}$ the product of the σ-fields $\mathfrak{A}$ and $\mathfrak{B}$. We say that $\mathfrak{A}$ is *countably generated* if there exist subsets A_n, $n \geq 1$, of X such that $\mathfrak{A}$ is generated by $\{A_n : n \geq 1\}$. A nonempty set $A \in \mathfrak{A}$ is called an $\mathfrak{A}$-*atom* if $A \supseteq B \in \mathfrak{A} \Rightarrow B = A$ or $B = \varnothing$. If X is a metric space, $\mathfrak{B}_X$ will denote the *Borel σ-field* of X. If $E \subseteq X \times Y$ and $x \in X$, E^x will denote the set $\{ y \in Y : (x, y) \in E \}$ and will be called the *section* of E at x. The projection maps from $X \times Y$ to X and from $X \times Y$ to Y will be denoted respectively by $\Pi_X^{X \times Y}$ and $\Pi_Y^{X \times Y}$, or simply by Π_X and Π_Y if there is no ambiguity.

A *multifunction* $F \colon T \to X$ is a function whose domain is T and whose values are nonempty subsets of X. A function $f \colon T \to X$ is called a *selector* for F if $f(t) \in F(t)$ for each $t \in T$. The set $\{(t, x) \in T \times X : x \in F(t)\}$ is denoted by $\mathrm{Gr}(F)$ and is called the *graph* of F. If X is a metric space and $\mathfrak{T}$ is a σ-field on T, we say that F is $\mathfrak{T}$-*measurable* if the set $\{t \in T : F(t) \cap V \neq \varnothing\} \in \mathfrak{T}$ for every open set V in X. If M is a subset of $T \times X$, we say that $C \subseteq M$ *uniformizes* M if sections of C are at most a singleton and $\Pi_T(C) = \Pi_T(M)$.

Let X, Y be topological spaces. We say that a function $f \colon X \to Y$ is *open* (resp. *closed*) if for every open (resp. closed) set W in X, $f(W)$ is open (resp. closed) in the range of f.

The rest of our terminology is from [6].

We now state some known results without proof which will be frequently used in the sequel.

LEMMA 2.1 ([2]). *Let T be a Polish space and $\mathfrak{T}$ a countably generated sub-σ-field of $\mathfrak{B}_T$. Let $A \in \mathfrak{B}_T$ be a union of $\mathfrak{T}$-atoms. Then $A \in \mathfrak{T}$.*

LEMMA 2.2. *Let T, X be Polish spaces and $\mathfrak{T}$ a countably generated sub-σ-field of $\mathfrak{B}_T$. Suppose G is a subset of $T \times X$ such that $G \in \mathfrak{T} \otimes \mathfrak{B}_X$ and G^t is a G_δ in X for every $t \in T$. Then for every closed subset A of X the set $\{t \in T : A \subseteq G^t\} \in \mathfrak{T}$.*

PROOF. Let Y be a metric compactification of X. By a well-known result of Alexandrov and Hausdorff, X is a G_δ in Y. Consequently, $G \in \mathfrak{T} \otimes \mathfrak{B}_Y$ and G^t is a G_δ in Y for each $t \in T$. Let $A \subseteq X$ be closed. Then it is easily verified that

$$\{ t \in T : A \subseteq G^t \} = T \setminus \Pi_T((T \times A) \cap ((T \times Y) \setminus G)).$$

By a result of Arsenin and Kunugui [1] (see also [13]) it follows that the set $\{ t \in T :$

$A \subsetneq G'\} \in \mathfrak{B}_T$. Further, this set is a union of $\mathfrak{T}$-atoms. The result now follows from Lemma 2.1.

We now state a very useful result, which is proved in [15], for G_δ-valued multifunctions.

LEMMA 2.3. *Let* T, X *be Polish spaces and* $\mathfrak{T}$ *a countably generated sub-σ-field of* $\mathfrak{B}_T$. *Let* $G \in \mathfrak{T} \otimes \mathfrak{B}_X$ *and* G' *be a* G_δ *in* X *for each* $t \in T$. *Then there exist sets* $G_n \in \mathfrak{T} \otimes \mathfrak{B}_X$ *such that* G_n^t *is open in* X *for* $t \in T$ *and* $n \geqslant 1$ *and* $G = \bigcap_{n=1}^\infty G_n$.

In the rest of the paper, T, X *will denote arbitrary Polish spaces and* $\mathfrak{T}$ *a countably generated sub-σ-field of* $\mathfrak{B}_T$. X *will be given a complete metric such that* $\operatorname{diam}(X) <$ 1. $\{V_n: n \geqslant 1\}$ *will be a base for the topology of* X *such that* $V_1 = X$. $F: T \to X$ *will denote a multifunction such that* F *is* $\mathfrak{T}$*-measurable,* $\operatorname{Gr}(F) \in \mathfrak{T} \otimes \mathfrak{B}_X$ *and* $F(t)$ *is a* G_δ *in* X *for each* $t \in T$. G *will denote the graph of* F *and* G_n, $n \geqslant 1$, *will be a nonincreasing sequence of sets in* $\mathfrak{T} \otimes \mathfrak{B}_X$ *such that* G_n^t *is open for* $t \in T$ *and* $n \geqslant 1$ *and* $G = \bigcap_{n=1}^\infty G_n$. *The existence of such a sequence of sets is ensured by Lemma* 2.3.

3. A representation theorem.

LEMMA 3.1. *Let* X *be compact. Then for each* $t \in T$, *there is a system* $\{n_s^t: s \in S\}$ *of positive integers such that for* $s \in S_k$, $k > 0$, *and* $t \in T$,

(i) *the map* $t' \to n_s^{t'}$, *defined on* T, *is* $\mathfrak{T}$*-measurable,*

(ii) $\operatorname{diam}(V_{n_s^t}) < 2^{-k}$,

(iii) $\overline{V}_{n_{sm}^t} \subseteq G_{k+1}^t \cap V_{n_s^t}$, $m \geqslant 1$,

(iv) $G^t \cap V_{n_s^t} \neq \varnothing$,

(v) $G^t \subseteq V_{n_s^t}$,

(vi) $G^t \cap V_{n_s^t} \subseteq \bigcup_{m=1}^\infty V_{n_{sm}^t}$.

PROOF. For each $t \in T$, we define n_s^t, $s \in S$, by induction on $|s|$. We define $n_e^t = 1$, $t \in T$. The above conditions are clearly satisfied for $s = e$. Suppose n_s^t, $t \in T$, are defined satisfying (i)–(vi) for every $s \in \bigcup_{i \leqslant k} S_i$, for some $k > 0$. Fix an $s \in S_k$. We define n_{sm}^t, $t \in T$, $m \in N$, by induction on m. We first make a simple observation. Let $W \subseteq X$ be closed and $t \in T$. Then

$$W \subseteq G_{k+1}^t \cap V_{n_s^t} \Leftrightarrow (\exists \, l \in N) \left(n_s^t = l \text{ and } W \subseteq G_{k+1}^t \cap V_l \right).$$

By the induction hypothesis and Lemma 2.2, it follows that the set

$$\left\{ t \in T: W \subseteq G_{k+1}^t \cap V_{n_s^t} \right\} \in \mathfrak{T}.$$

For $m \geqslant 1$, let

$$T_m^0 = \varnothing \quad \text{if } \operatorname{diam}(V_m) \geqslant 2^{-(k+1)};$$

$$= \Big\{ t \in T: G^t \cap V_m \neq \varnothing, \overline{V}_m \subseteq G_{k+1}^t \cap V_{n_s^t} \text{ and}$$

$$(\forall l < m) \left(\operatorname{diam}(V_l) < 2^{-(k+1)} \Rightarrow \left(G^t \cap V_l = \varnothing \text{ or } \overline{V}_l \nsubseteq G_{k+1}^t \cap V_{n_s^t} \right) \right) \Big\}$$

$$\text{if } \operatorname{diam}(V_m) < 2^{-(k+1)}.$$

As F is $\mathfrak{T}$-measurable, by the above observation, it follows that the sets T_m^0, $m \geqslant 1$,

belong to $\mathcal{T}$. Also, these are pairwise disjoint and $T = \bigcup_{m=1}^{\infty} T_m^0$. We define $n_{s1}^t = m$ if $t \in T_m^0$. Clearly, the map $t \to n_{s1}^t$ is $\mathcal{T}$-measurable. Suppose for some $p \in N$, maps $t \to n_{st}^t$, $i \leqslant p$, have been defined to be $\mathcal{T}$-measurable. For $m > 1$, let

$$T_m^p = \varnothing \quad \text{if diam}(V_m) > 2^{-(k+1)};$$

$$= \Big\{ t \in T : n_{sp}^t < m, \, G^t \cap V_m \neq \varnothing, \, \overline{V}_m \subseteq G_{k+1}^t \cap V_{n_s^t}, \quad \text{and}$$

$$(\forall l < m)\big(\text{diam}(V_l) < 2^{-(k+1)} \Rightarrow \big(n_{sp}^t \geqslant l \text{ or } G^t \cap V_l = \varnothing$$

$$\text{or } \overline{V}_l \not\subseteq G_{k+1}^t \cap V_{n_s^t}\big)\big)\Big\}$$

$$\text{if diam}(V_m) < 2^{-(k+1)}.$$

It is easily checked that the sets T_m^p, $m > 1$, belong to $\mathcal{T}$ and are pairwise disjoint. We define

$$n_{s,p+1}^t = m \quad \text{if } t \in T_m^p,$$

$$= n_{sp}^t \quad \text{if } t \in T \setminus \bigcup_{m=1}^{\infty} T_m^p.$$

The definition of n_s^t, $s \in S$, $t \in T$, is complete. That the conditions (i)–(v) are satisfied follows immediately from the definitions of n_s^t, $s \in S$, $t \in T$. To check (vi) note that $G^t \cap V_{n_s^t} \subseteq G_{k+1}^t \cap V_{n_s^t}$ and $G_{k+1}^t \cap V_{n_s^t}$ is open.

THEOREM 3.2. *There is a map $f: T \times \Sigma \to X$ such that for each $t \in T, f(t, \cdot)$ is a continuous, open map from Σ onto $F(t)$ and for each $\sigma \in \Sigma, f(\cdot, \sigma)$ is $\mathcal{T}$-measurable.*

PROOF. Without loss of generality, we assume that X is a compact metric space. For each $t \in T$, we get a system $\{n_s^t : s \in S\}$ of positive integers satisfying conditions (i)–(vi) of Lemma 3.1.

Let $f(t, \sigma)$ be the unique point of $\bigcap_k \overline{V}_{n_{\sigma|k}^t}$, $t \in T$, $\sigma \in \Sigma$. By conditions (iii)–(vi) of Lemma 3.1, $f(t, \Sigma) = F(t)$, $t \in T$. By standard arguments we show that for each $t \in T, f(t, \cdot)$ is continuous and open. Let $U \subseteq X$ be open, $\sigma \in \Sigma$ and $t \in T$. Then

$$f(t, \sigma) \in U \Leftrightarrow \bigcap_k \overline{V}_{n_{\sigma|k}^t} \subseteq U$$

$$\Leftrightarrow (\exists \, k > 1)(\exists \, l > 1)\left(n_{\sigma|k}^t = l \text{ and } \overline{V}_l \subseteq U\right).$$

Therefore,

$$f(\cdot, \sigma)^{-1}(U) = \bigcup \, \bigcup \left\{ t \in T : n_{\sigma|k}^t = l \right\} \in \mathcal{T},$$

where the inner union is taken over all l such that $\overline{V}_l = U$ and the outer union is over all k. It follows that $f(\cdot, \sigma)$ is $\mathcal{T}$-measurable for each $\sigma \in \Sigma$.

COROLLARY 1. *F admits a $\mathcal{T}$-measurable selector.*

COROLLARY 2. *There exist $\mathcal{T}$-measurable selectors $f_1, f_2, \ldots$ for F such that for each $t \in T$, $\{f_n(t): n > 1\}$ is dense in $F(t)$.*

PROOF. Let $\sigma^1, \sigma^2, \ldots$ be a countable dense set in Σ. Then, for each $t \in T$, $\{f(t, \sigma^n): n > 1\}$ is dense in $F(t)$. Put $f_n = f(\cdot, \sigma^n)$, $n > 1$.

REMARK 1. In [16], it is proved that Theorem 3.2 remains valid if the condition "$f(t, \cdot)$ is open" is replaced by "$f(t, \cdot)$ is closed".

REMARK 2. Let Y be a Polish space and $h: T \times Y \to X$ be a map such that for each $t \in T$, $h(t, \cdot)$ is continuous and open and for each $y \in Y$, $h(\cdot, y)$ is $\mathfrak{T}$-measurable. Define a multifunction $H: T \to X$ by $H(t) = h(t, Y)$, $t \in T$. By a result of Hausdorff [3], $H(t)$ is a G_δ in X for each $t \in T$. Let $\{y_n : n \geqslant 1\}$ be a countable dense set in Y. For $n \geqslant 1$, define $f_n: T \to X$ by $f_n(t) = f(t, y_n)$, $t \in T$. Let $V \subseteq X$ be open. Then

$$\{t \in T: H(t) \cap V \neq \varnothing\} = \bigcup_{n \geqslant 1} f_n^{-1}(V).$$

It follows that the multifunction H is $\mathfrak{T}$-measurable. The question now arises: Is $\text{Gr}(H) \in \mathfrak{T} \otimes \mathcal{B}_X$? We do not know the answer. In [16], it is proved that the answer to this question is 'yes' if the condition "$h(t, \cdot)$ is open" is replaced by "$h(t, \cdot)$ is closed".

4. Multifunctions of Souslin type.

DEFINITION. Let $(L, \mathcal{L})$ be a measurable space and Z a metric space. A multifunction $H: L \to Z$ is said to be of *Souslin type* if there is a Polish space P, a continuous map $\beta: P \to Z$ and a $\mathcal{L}$-measurable, closed-valued multifunction $W: L \to P$ such that $H(t) = \beta(W(t))$, for each $t \in L$.

REMARK 1. Our definition of multifunctions of Souslin type is slightly different from the one given in [8].

REMARK 2. By a representation theorem for closed valued multifunctions proved in [14], we get the following. *If $(L, \mathcal{L})$ is a measurable space, Z a metric space and $H: L \to Z$ a multifunction of Souslin type then there is a map $h: L \times \Sigma \to Z$ such that for each $t \in L$, $h(t, \cdot)$ is a continuous map from Σ onto $H(t)$ and for each $\sigma \in \Sigma$, $h(\cdot, \sigma)$ is $\mathcal{L}$-measurable.*

Now we prove the main result of this section.

THEOREM 4.1. *The multifunction F is of Souslin type.*

PROOF. We first assume that X is a compact, zero-dimensional metric space and basic open sets $V_1, V_2, \ldots$ are clopen. By Lemma 2.2, the sets $T_{nm} = \{t \in T: V_m \subseteq G_n'\}$, $m \geqslant 1$, $n \geqslant 1$, belong to $\mathfrak{T}$. As G_n' is open for $t \in T$ and $n \geqslant 1$, $G_n = \bigcup_{m=1}^{\infty} (T_{nm} \times V_m)$. Put $P = X \times \Sigma$ and $\beta = \Pi_X$. Let

$$B = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} (T_{nm} \times V_m \times \Sigma_m^n),$$

where $\Sigma_m^n = \{\sigma \in \Sigma: \sigma_n = m\}$, $n \geqslant 1$, $m \geqslant 1$. Define $W: T \to P$ by $W(t) = B^t$, $t \in T$, so that $W(t) = \bigcap \bigcup (V_m \times \Sigma_m^n)$, where the inner union is taken over all $m \geqslant 1$ such that $t \in T_{nm}$ and the outer intersection is over all n. For each $n \geqslant 1$, $\{V_m \times \Sigma_m^n: m \geqslant 1\}$ is a discrete family of closed sets in P. It follows that the inner union is closed in P for each n. Therefore, $W(t)$ is closed in P for each $t \in T$. Also, it is easily checked that $\beta(W(t)) = F(t)$, $t \in T$.

We now check that W is $\mathfrak{T}$-measurable. Let $i \in N$ and $s \in S_k$, $k > 0$. It is enough to show that

$$\Pi_T^{T \times P}((T \times V_i \times \Sigma_s) \cap B) \in \mathfrak{T}.$$

Now,

$$(T \times V_i \times \Sigma_s) \cap B = \bigcap_{n \geq 1} \bigcup_{m \geq 1} (T_{nm} \times (V_m \cap V_i) \times (\Sigma_m^n \cap \Sigma_s))$$

$$= \left(\bigcap_{j=1}^{k} (T_{j_j} \times (V_{s_j} \cap V_i) \times \Sigma_s) \right)$$

$$\cap \left(\bigcap_{n > k} \bigcup_{m \geq 1} (T_{nm} \times (V_m \cap V_i) \times (\Sigma_m^n \cap \Sigma_s)) \right)$$

$$= \left(\bigcap_{j=1}^{k} (T_{j_j} \times P) \right)$$

$$\cap \left(\bigcap_{n > k} \bigcup_{m \geq 1} \left(T_{nm} \times \left(V_m \cap V_i \cap \bigcap_{j=1}^{k} V_{s_j} \right) \times (\Sigma_m^n \cap \Sigma_s) \right) \right).$$

Hence,

$$\Pi_T^{T \times P}((T \times V_i \times \Sigma_s) \cap B)$$

$$= \Pi_T^{T \times P}\left[\left(\bigcap_{j=1}^{k} (T_{j_j} \times P) \right) \right.$$

$$\left. \cap \left(\bigcap_{n > k} \bigcup_{m \geq 1} \left(T_{nm} \times \left(V_m \cap V_i \cap \bigcap_{j=1}^{k} V_{s_j} \right) \times (\Sigma_m^n \cap \Sigma_s) \right) \right) \right]$$

$$= \bigcap_{j=1}^{k} T_{j_j} \cap \Pi_T^{T \times X}\left(\bigcap_{n > k} \bigcup_{m \geq 1} \left(T_{nm} \times \left(V_m \cap V_i \cap \bigcap_{j=1}^{k} V_{s_j} \right) \right) \right)$$

$$= \bigcap_{j=1}^{k} T_{j_j} \cap \Pi_T^{T \times X}\left(\left(T \times \left(V_i \cap \bigcap_{j=1}^{k} V_{sj} \right) \right) \cap \left(\bigcap_{n > k} \bigcup_{m \geq 1} (T_{nm} \times V_m) \right) \right)$$

$$= \bigcap_{j=1}^{k} T_{j_j} \cap \Pi_T^{T \times X}\left(\left(T \times \left(V_i \cap \bigcap_{j=1}^{k} V_{sj} \right) \right) \cap G \right).$$

As F is $\mathfrak{T}$-measurable, it follows that W is $\mathfrak{T}$-measurable.

Now, let X be a zero-dimensional Polish space. Let Y be a zero-dimensional compact metric space containing a homeomorph of X. We consider F as a multifunction with values subsets of Y. By the previous case, we get a Polish space Q, a continuous map $g: Q \to Y$ and a $\mathfrak{T}$-measurable, closed set-valued multifunction $H: T \to Q$ such that $F(t) = g(H(t))$, $t \in T$. Put $P = g^{-1}(X)$ and β the restriction of g to P. As X is a G_δ in Y, P is Polish. Note that $H(t) \subseteq P$, $t \in T$. Put $W = H$.

Finally, let X be an arbitrary Polish space. Let $g: \Sigma \to X$ be a continuous, open and onto map. Define a multifunction $H: T \to \Sigma$ by $H(t) = g^{-1}(F(t))$, $t \in T$. Clearly, $H(t)$ is a G_δ in Σ for each $t \in T$. Let $U \subseteq \Sigma$ be open, then

$$\{ t \in T: H(t) \cap U \neq \varnothing \} = \{ t \in T: F(t) \cap g(U) \neq \varnothing \}.$$

As F is $\mathfrak{T}$-measurable and g open, H is $\mathfrak{T}$-measurable. To show $\mathrm{Gr}(H) \in \mathfrak{T} \otimes \mathfrak{B}_\Sigma$, we define $h: T \times \Sigma \to T \times X$ by

$$h(t, \sigma) = (t, g(\sigma)), \qquad t \in T, \sigma \in \Sigma.$$

Then h is continuous and $\mathrm{Gr}(H) = h^{-1}(G)$ so that $\mathrm{Gr}(H)$ is Borel in $T \times \Sigma$. Note that whenever t and t' belong to the same $\mathfrak{T}$-atoms, $G' = G'$ and consequently, $(\mathrm{Gr}(H))^t = (\mathrm{Gr}(H))^{t'}$. This implies that $\mathrm{Gr}(H)$ is a union of $\mathfrak{T} \otimes \mathfrak{B}_\Sigma$-atoms. Therefore, by Lemma 2.1, $\mathrm{Gr}(H) \in \mathfrak{T} \otimes \mathfrak{B}_\Sigma$. By the previous case, we get a Polish space P, a closed set-valued, $\mathfrak{T}$-measurable multifunction $W: T \to P$ and a continuous map $f: P \to \Sigma$ such that $H(t) = f(W(t))$, $t \in T$. Put $\beta = g \circ f$. The desired properties are easy to verify.

REMARK. A close examination of the various cases in the above proof reveals that the map $\beta: P \to X$ is obtained to be continuous, open and onto.

5. Decomposition of $\mathrm{Gr}(F)$ into graphs of measurable selectors. In this section, X and $F(t)$, $t \in T$, are all assumed to be uncountable. We prove

THEOREM 5.1. *If for every $t \in T$, $F(t)$ is dense-in-itself then there is a $\mathfrak{T} \otimes \mathfrak{B}_X$-measurable map $f: T \times X \to X$ such that for each $t \in T$, $f(t, \cdot)$ is a Borel isomorphism of X onto $F(t)$.*

This result is analogous to a result of Mauldin [11] and we follow some of his ideas. We first show by an example that the condition "$F(t)$ is dense in itself" cannot be dropped from Theorem 5.1.

EXAMPLE. Let X be an uncountable Polish space containing a countable, dense, open set U. (Union of the Cantor set and the mid-points of the removed intervals is such a Polish space.) Let $T = \Sigma$, $\mathfrak{T} = \mathfrak{B}_T$ and let $Y = X \setminus U$. Let E be a G_δ set in $\Sigma \times Y$ such that $\Pi_\Sigma(E) = \Sigma$ and E does not admit a Borel uniformization [9, p. 265]. Let $G: Y \to Y$ be a continuous, onto map such that $g^{-1}(y)$ is uncountable for each $y \in Y$. Let $B = \{(t, y) \in T \times Y: (t, g(y)) \in E\}$. Then B is a G_δ set in $\Sigma \times Y$ such that every section of B is uncountable and B does not admit a Borel uniformization. Let $F: T \to X$ be defined by $F(t) = B^t \cup U$, $t \in T$. It is clear that $F(t)$ is a G_δ in X for each $t \in T$ and $\mathrm{Gr}(F) \in \mathfrak{T} \otimes \mathfrak{B}_X$. As U is dense in X, F is $\mathfrak{T}$-measurable. If F satisfies the conclusions of Theorem 5.1, by a result of Mauldin [11], there is a Borel set $M \subseteq \mathrm{Gr}(F)$ such that for each $t \in T$, M^t is nonempty and perfect. Let $H = M \setminus (T \times U) \subseteq B$. The sections of H are nonempty and compact so that H, and therefore, B admits a Borel uniformization.

LEMMA 5.2. *Let X be compact. Then for each $t \in T$, there is a system $\{n_d^t: d \in D\}$ of positive integers such that for $d \in D_k$, $k \geqslant 0$, and $t \in T$,*
(i) *the map $t' \to n_d^{t'}$ is $\mathfrak{T}$-measurable,*
(ii) $\mathrm{diam}(V_{n_d^t}) < 2^{-k}$,
(iii) $d' \in D_k$, $d \neq d' \Rightarrow \bar{V}_{n_d^t} \cap \bar{V}_{n_{d'}^t} = \varnothing$,
(iv) $F(t) \cap V_{n_d^t} \neq \varnothing$,
(v) $\bar{V}_{n_d^t} \subseteq G_{k+1}^t \cap V_{n_d^t}$, $i = 0$ or $i = 1$.

PROOF. We use induction on $|d|$. Let $n_\varnothing^t = 1$ for all t. (i)–(v) are satisfied for

$d = e$. Suppose for some $k > 0$, n_d^t is defined for all $d \in \bigcup_{l < k} D_l$ and for all $t \in T$ satisfying (i)–(v). Fix a $d \in D_k$. Put

$$T^m = \{ t \in T : n_d^t = m \}, \qquad m > 1.$$

By the induction hypothesis, the sets T^m, $m > 1$, belong to $\mathfrak{T}$, are pairwise disjoint and $T = \bigcup_{m > 1} T^m$. Now, for any pair (u, v) of positive integers define T_{uv}^m, $m > 1$, as follows.

If $\operatorname{diam}(V_u) < 2^{-(k+1)}$, $\operatorname{diam}(V_v) < 2^{-(k+1)}$, $\overline{V}_u \cap \overline{V}_v = \varnothing$, $\overline{V}_u \subseteq V_m$, $\overline{V}_v \subseteq V_m$ then

$$T_{uv}^m = \{ t \in T^m : \overline{V}_u \subseteq G_{k+1}^t, \ \overline{V}_v \subseteq G_{k+1}^t, \ V_u \cap F(t) \neq \varnothing \text{ and } V_v \cap F(t) \neq \varnothing \};$$
$$= \varnothing \quad \text{otherwise.}$$

As F is $\mathfrak{T}$-measurable, by Lemma 2.2, $T_{uv}^m \in \mathfrak{T}$. Also $T^m = \bigcup_{(u,v)} T_{uv}^m$.
Let $\alpha: N \to N \times N$ be a one-one, onto function.
Put

$$S_i^m = T_{\alpha(i)}^m \qquad\qquad \text{if } i = 1,$$
$$= T_{\alpha(i)}^m \setminus \bigcup_{j < i} T_{\alpha(j)}^m \quad \text{if } i > 1.$$

The sets S_i^m, $i > 1$, belong to $\mathfrak{T}$, are pairwise disjoint and $T^m = \bigcup_{i > 1} S_i^m$. We define

$$\left.\begin{array}{l} n_{d0}^t = (\alpha(j))_1 \\ n_{d1}^t = (\alpha(j))_2 \end{array}\right\} \quad \text{if } t \in S_j^m \text{ for any } m.$$

This completes the definition of $\{ n_d^t : d \in D_{k+1} \}$, $t \in T$. It is easy to check that (i)–(v) are satisfied.

LEMMA 5.3. *There is a map* $g: T \times C \to X$ *such that for each* $t \in T$, $g(t, \cdot)$ *is a homeomorphism from* C *into* $F(t)$ *and for each* $\varepsilon \in C$, $g(\cdot, \varepsilon)$ *is* $\mathfrak{T}$-*measurable.*

PROOF. Without loss of generality, we assume that X is a compact metric space. For each $t \in T$, we get a system $\{ n_d^t : d \in D \}$ of positive integers satisfying (i)–(v) of Lemma 5.2. Let $g(t, \varepsilon)$ be the unique point in $\bigcap_k \overline{V}_{n_{\varepsilon|k}^t}$, $t \in T$, $\varepsilon \in C$. By standard arguments, we show that for each $t \in T$, $g(t, \cdot)$ is a homeomorphism defined on C and by (v), it is into $F(t)$. Let $t \in T$, $\varepsilon \in C$ and $U \subseteq X$ be open. Then

$$g(t, \varepsilon) \in U \Leftrightarrow \bigcap_k \overline{V}_{n_{\varepsilon|k}^t} \subseteq U$$
$$\Leftrightarrow (\exists\, k > 1)\left(\overline{V}_{n_{\varepsilon|k}^t} \subseteq U \right)$$
$$\Leftrightarrow (\exists\, k > 1)\,(\exists\, l > 1)\left(\overline{V}_l \subseteq U \right)\left(n_{\varepsilon|k}^t = l \right)$$

so that

$$g(\cdot, \varepsilon)^{-1}(U) = \bigcup\ \bigcup \left(\{ t \in T : n_{\varepsilon|k}^t = l \} \right) \in \mathfrak{T},$$

where the inner union is taken over all l such that $\overline{V}_l \subseteq U$ and the outer union is over all k. It follows that $g(\cdot, \varepsilon)$ is $\mathfrak{T}$-measurable for every $\varepsilon \in C$.

PROOF OF THEOREM 5.1. By Lemma 5.3, we get a map $g: T \times C \to X$ such that for each $t \in T$, $g(t, \cdot)$ is a homeomorphism from C into $F(t)$ and for all $\varepsilon \in C$, $g(\cdot, \varepsilon)$ is $\mathfrak{T}$-measurable. In particular, g is $\mathfrak{T} \otimes \mathfrak{B}_C$-measurable [6, p. 378]. As X and C are Borel isomorphic we get a $\mathfrak{T} \otimes \mathfrak{B}_X$-measurable map $h: T \times X \to X$ such that for each $t \in T$, $h(t, \cdot)$ is a Borel isomorphism from X into $F(t)$. Let k: $T \times X \to T \times X$ be defined by

$$k(t, x) = (t, h(t, x)), \qquad t \in T, x \in X,$$

and let

$$B = \{(t, x) \in T \times X : x \in h(t, X)\}.$$

Then, $B \subseteq G$ and as k is one-one, Borel, B is Borel in $T \times X$. By Lemma 2.1, $B \in \mathfrak{T} \otimes \mathfrak{B}_X$. Also, $k: (T \times X, \mathfrak{T} \otimes \mathfrak{B}_X) \to (T \times X, \mathfrak{T} \otimes \mathfrak{B}_X)$ is a measurable map such that for each $t \in T$, $k(t, \cdot)$ is a Borel isomorphism from X into $\{t\} \times F(t)$. Now, we do a Schroeder-Bernstein type argument as done by Mauldin [11] and get a measurable map $\alpha: (T \times X, \mathfrak{T} \otimes \mathfrak{B}_X) \to (T \times X, \mathfrak{T} \otimes \mathfrak{B}_X)$ such that for each $t \in T$, $\alpha(t, \cdot)$ is a Borel isomorphism from X onto $\{t\} \times F(t)$. Put $f = \Pi_X \circ \alpha$.

COROLLARY 1. *Let $M \subseteq T \times X$ be a Borel set such that for every $t \in \Pi_T(M)$, M^t is dense in itself, and both a K_σ and a G_δ set in X. Then M is a union of $2^{\aleph_0}$ disjoint Borel uniformizations.*

Under the hypothesis of the above corollary, Larman [7] proved that M contains $2^{\aleph_0}$ disjoint Borel uniformizations. The problem of the existence of $2^{\aleph_0}$ disjoint Borel uniformizations of M when its sections are not assumed to be dense in itself remains open.

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STAT-MATHEMATICS DIVISION, INDIAN STATISTICAL INSTITUTE, CALCUTTA 700 035, INDIA