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EFFICIENCY OF NON-WALRASIAN EQUILIBRIA

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In this paper, Pareto efficiency properties of a non-Walrasian equilibrium for an exchange economy are analyzed. The equilibrium considered is a generalized version of Drèze's equilibrium with price rigidities and rationing.

1. INTRODUCTION

IT IS GENERALLY recognized that the set of Walras equilibria, the solutions generated by the market mechanism which operates with the help of price signals, may not contain any equitable solution. Pareto efficiency does not ensure equity. So, policy makers frequently resort to quantity rationing which uses prices as well as quantity restrictions as signals as a practical alternative to the market mechanism. Further, a variety of solution concepts involving rationing have been suggested in relation to price rigidities caused by all kinds of market imperfections and institutional factors.

In this paper we define a general non-Walrasian equilibrium that includes many of these suggested solution concepts as special cases. We shall enquire into the efficiency of this equilibrium. Pareto efficiency and its relationship to Walras equilibrium have been discussed extensively in [**1**]. We make use of the same concepts as presented there, except that we confine ourselves to an exchange economy for the sake of simplicity.

In Section 2, we describe the model and the assumptions. In Section 3 we define the general non-Walrasian equilibrium and give some preliminary results. In Section 4 we prove the Pareto inefficiency of our solution concept. In Section 5, we discuss the implications of this result.

2. THE MODEL

Let us consider an exchange economy with N consumers (indexed i) and n commodities (indexed k). The utility maximizing consumer i is characterized by (X_i, U_i, w^i) representing his consumption set, utility function, and initial endowments, in that order.

A feasible allocation is defined as an N-tuple of consumption vectors

$$x^i \in X_i \quad \text{such that} \quad \sum_i x^i \leqslant \sum_i w^i.$$

Define the set of feasible allocations U by

$$U = \{u \in R^n \mid u_i = U_i(x^i) \text{ and } (x^1, x^2, \dots, x^N) \text{ is feasible}\}.$$

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E_1, the set of Pareto efficient utility allocations or the Pareto frontier, is defined by

$$E_1 = \{u \in U \mid \text{there is no } u' \in U \text{ with } u' \geq u \text{ and } u'_i > u_i \text{ for some } i\}.$$

E_2, the set of individually rational Pareto efficient utility allocations is defined by

$$E_2 = \{u \in E_1 \mid u_i \geq U_i(w^i)\}.$$

Conforming to the earlier literature, we define Pareto efficiency using utility allocations and not consumption allocations. This way we avoid the possibility of two distinct Pareto efficient allocations having the same utility allocation.

We make the following assumptions.

ASSUMPTION 1: $X_i = R^n_+$ and $w^i \in$ interior X_i.

ASSUMPTION 2: (Monotonicity): The utility functions U_i are monotone in their arguments.

ASSUMPTION 3 (Smooth indifference curves): The utility functions are differentiable in the interior of R^n_+.

ASSUMPTION 4 (Asymptotic, strictly convex indifference curves): For any strictly positive $x \in X_i$, the set $R(x)$ defined by

$$R(x) = \{y \in X_i \mid U_i(y) \geq U_i(x)\}$$

is strictly convex.

The assumptions have a long history and were used by Hicks [4] among others. They ensure that the preference orderings are strongly convex, i.e. if $U_i(x) = U_i(y)$ and $x \neq y$, then $U_i(\lambda x + (1-\lambda)y) > U_i(x)$ for $0 < \lambda < 1$. One more implication of the assumptions is that the set E_1 coincides with the set

$$\{u \in U \mid \text{there is no } u' \in U \text{ with } u'_i > u_i \ \forall i\}.$$

3. A GENERAL RATIONING EQUILIBRIUM AND PRELIMINARY RESULTS

J. H. Drèze in [2] defines and proves under general assumptions the existence of a rationing equilibrium with stronger requirements than we include here. Hahn's non-Walrasian equilibrium [3] and Younes' p-equilibrium [8] have weaker requirements, so much so that the existence is a trivial issue for the p-equilibrium. Our equilibrium requirements are weaker than the above three, so more allocations are obtained as our equilibrium than any of the above three. We proceed to define it. Consider a rationing scheme $(L^i, l^i) \in R^n_+ \times R^n_-$. Given any price vector $p \in R^n_+$ with $p_1 = 1$, define the budget set of consumer i by

$$\gamma^i(p, L^i, l^i) = \{x \in X_i \mid p(x - w^i) \leq 0, L^i \geq x - w^i \geq l^i\}.$$

DEFINITION: An N-tuple of consumption allocations $\{x^i\}$ is obtained as a general non-Walrasian equilibrium (GNWE) if there are strictly positive $p \in R^n_+$

with $p_1 = 1$ and N rationing schemes (L^i, l^i) with $(L^i, l^i) \in R^n_+ \times R^n_-$ such that these conditions hold:

(i) x^i is maximal for U_i over $\gamma^i(p, L^i, l^i)$.

(ii) $\Sigma_i x^i - \Sigma_i w^i = 0$.

(iii) $L^i_1 = \infty$, $l^i_1 = -\infty$, $\forall i$.

Now we shall describe how the different equilibrium concepts are obtained as special cases of GNWE. Hahn's non-Walrasian equilibrium requires one additional condition:

(iv) For all k: if for some i, $L^i_k = x^i_k - w^i_k$, then for all i: $x^i_k - w^i_k > l^i_k$. Also, for all k: if for some i, $l^i_k = x^i_k - w^i_k$, then for all i: $L^i_k > x^i_k - w^i_k$.

Drèze's equilibrium requires apart from (i), (ii), (iii), and (iv) that the rationing scheme be equal and requires a further boundary condition depending on a given price set.

In Younes' p-equilibrium, the quantity restrictions are only implicitly present. If we make them explicit, the p-equilibrium satisfies, apart from (i), (ii), and (iii), the following:

(v) Either $L^i_k = 0$ or $l^i_k = 0$ or both are zero, $\forall i, \forall k$.

The proportional rationing equilibrium of [6] is a special case of a p-equilibrium satisfying stronger conditions. Define:

$$G = \{u \in U \mid u_i = U_i(x^i) \text{ and } \{x^i\} \text{ is a GNWE}\},$$

$$W = \{u \in U \mid u_i = U_i(x^i) \text{ and } \{x^i\} \text{ is a Walras equilibrium}\}.$$

For a Drèze equilibrium with any price set, the corresponding utility allocation is seen to lie in G.

The following lemma plays a fundamental role in all that follows:

LEMMA 1: *Let X_i be R^2_+. Let $(\bar{x}_1, \bar{x}_2)$ be the preferred bundle for consumer i at prices $(1, p_2)$ and income $\bar{x}_1 + p_2\bar{x}_2$. Let $\bar{x}_1 + p_2\bar{x}_2 = x_1 + p_2x_2$ and (x_1, x_2) be the preferred bundle at prices $(1, p'_2)$ and income $x_1 + p'_2x_2$. (x_1, x_2), p_2, and p'_2 are positive. Then*

$$x_2 > \bar{x}_2 \Leftrightarrow p'_2 < p_2,$$

$$\bar{x}_2 > x_2 \Leftrightarrow p_2 < p'_2.$$

PROOF: We first prove that $p_2 = p'_2 \Leftrightarrow x_2 = \bar{x}_2$. Let $p_2 = p'_2$. Due to strong convexity of preferences, the preferred bundle is unique in any budget set. So, $x_2 = \bar{x}_2$.Now let $x_2 = \bar{x}_2$. Since $\bar{x}_1 + p_2\bar{x}_2 = x_1 + p_2x_2$, we have $x_1 = \bar{x}_1$. Since $(\bar{x}_1, \bar{x}_2)$ is positive, Slater's constraint qualification for the Kuhn-Tucker theorem is satisfied (see [5]) for the constrained utility maximization problem. So, the marginal rate of substitution (mrs), defined to be the ratio of marginal utilities, equals p_2 at $(\bar{x}_1, \bar{x}_2)$. This also equals p'_2 because $(\bar{x}_1, \bar{x}_2) = (x_1, x_2)$. So $p_2 = p'_2$. Contrary to the hypothesis, suppose $x_2 > \bar{x}_2$ and $p'_2 > p_2$. Now, $\bar{x}_1 + p_2\bar{x}_2 = x_1 + p_2x_2$ (given). Or,

$$(\bar{x}_1 - x_1) = p_2(x_2 - \bar{x}_2)$$
$$< p'_2(x_2 - \bar{x}_2).$$

Rearranging terms,

$$\bar{x}_1 + p_2'\bar{x}_2 < x_1 + p_2' x_2.$$

Since $(\bar{x}_1, \bar{x}_2)$ has been revealed to be preferred to (x_1, x_2) at prices $(1, p_2)$, this contradicts that (x_1, x_2) is preferred at prices $(1, p_2')$ and income $x_1 + p_2' x_2$. A similar contradiction is achieved if $\bar{x}_2 > x_2$ and $p_2 > p_2'$, thus proving the lemma.

Q.E.D.

In what follows, we make use of the following result to be found in [1]. If a Pareto efficient utility allocation corresponds to a consumption allocation consisting of strictly positive vectors, then the consumption allocation can be obtained as a Walras equilibrium with itself as the endowments. We sketch an elementary proof for this here. Let the endowments be given by the consumption allocation. Under our assumptions, there exists a Walras equilibrium and this is Pareto efficient.

Since no one can lose utility through trade, this Walras equilibrium gives the utility allocation we started with. Since under our assumptions the preferred point is unique in any budget set, the given consumption allocation coincides with the Walras equilibrium.[2]

4. INEFFICIENCY OF GNWE

In this section we prove the inefficiency of GNWE for an exchange economy of N consumers and n commodities. We need a few definitions and a lemma to prove this general result.

DEFINITION: Given a strictly positive $x^i \in R_+^n$, define $U_i^h : R_+^2 \to R$ as follows:

$$U_i^h(x_1'^i, x_h'^i) = U_i(x_1'^i, x_2^i, \ldots, x_{h-1}^i, x_h'^i, x_{h+1}^i, \ldots, x_n^i).$$

As U_i^h changes with x^i, instead of using x_i to index U_i^h, we just mention the particular element of X_i determining the U_i^h at any time. This is in order to keep the notation simple. It is easily verified that U_i^h, given any x^i, satisfies assumptions 2, 3, and 4 if U_i does.

DEFINITION: Let $\{x^i\}$ be a GNWE. Let p be an associated price vector. A commodity k is said to have a *binding quantity restriction* on it if for some i_0,

$$(x_1^{i_0}, x_k^{i_0}) \neq (\bar{x}_1^{i_0}, \bar{x}_k^{i_0})$$

where $(\bar{x}_1^{i_0}, \bar{x}_k^{i_0})$ is maximal on R_+^2 for $U_{i_0}^k$ given x^{i_0} with an income $x_1^{i_0} + p_k x_k^{i_0}$ and prices $(1, p_k)$.

To put it in words, freeze the consumption of all the commodities other than 1 and k at the constrained demand level. Respend on 1 and k the money originally

[2] The idea of this simple proof is taken from unpublished lecture notes of Professor V. K. Chetty and it originates there to the best of the author's knowledge.

spent on these two commodities, this time with no constraints on k. If the new demand for k is different from the constrained demand for k a quantity restriction is said to be binding on k.

LEMMA 2: *For any $u \in E_2 \cap G$ with a corresponding GNWE $\{x^i\}$ and an associated vector p, there is no commodity h such that both (i) and (ii) hold: (i) $x_h^{i_0} \neq w_h^{i_0}$ for some consumer i_0. (ii) There is a binding restriction on h.*

PROOF: Suppose there is a $u \in E_2 \cap G$ and a commodity h that satisfies condition (i) and (ii). Let us suppose $\bar{x}_h^{i_0} > x_h^{i_0}$ where $(\bar{x}_1^{i_0}, \bar{x}_h^{i_0})$, maximizes $U_{i_0}^h$ given x^{i_0} on R_+^2 with income $x_1^{i_0} + p_h x_h^{i_0}$ and prices $(1, p_h)$. As $x_h^{i_0} \neq w_h^{i_0}$, there is a consumer g such that $x_h^g < w_h^g$.

Since $\{x^i\}$ is Pareto efficient, there is a price vector p' at which $\{x^i\}$ is a Walras equilibrium for an appropriate distribution of initial endowments, in particular with $\{x^i\}$ itself as the initial endowments. The first components of both p and p' is 1 by convention.

U_{i_0} is maximized on R_+^n at x^{i_0} with income $p'x^{i_0}$ and prices p'. This implies that $U_{i_0}^h$ given x^{i_0} is maximized on R_+^2 at $(x_1^{i_0}, x_h^{i_0})$ with income $x_1^{i_0} + p_h' x_h^{i_0}$ and prices $(1, p_h')$. Also, $\bar{x}_h^{i_0} > x_h^{i_0}$ by supposition. By Lemma 1 it follows that $p_h < p_h'$.

Now consider the consumer g. Suppose that the constraint is not binding on commodity h for consumer g. Then by an application of Lemma 1 it follows that $p_h = p_h'$ which contradicts $p_h < p_h'$.

So, let the constraint on commodity h be binding for consumer g. Let $(\bar{x}_1^g, \bar{x}_h^g)$ be the bundle that maximizes U_g^h given x^g on R_+^2 with income $(x_1^g + p_h x_h^g)$ and prices $(1, p_h)$. Since U_g^h satisfies Assumption 4, it easily follows that $\bar{x}_h^g < x_h^g$. This implies, by an application of Lemma 1, that $p_h > p_h'$, which again contradicts $p_h < p_h'$.

Similar contradictions are obtained if we start with $x_h^{i_0} > \bar{x}_h^{i_0}$. So, we cannot find u satisfying conditions (i) and (ii) of the Lemma. *Q.E.D.*

THEOREM 1: $E_2 \cap G = W$.

PROOF: Let $u \in E_2 \cap G$. Let $\{x^i\}$ be a corresponding GNWE with associated price $p = (1, p_2, \ldots, p_n)$. Let $p' = (1, p_2', \ldots, p_n')$ be the price vector at which $\{x^i\}$ is a Walras equilibrium with $\{x^i\}$ itself as the initial endowments. Let

$$K_1 = \{k \in \{2, 3, \ldots, n\} | x_k^i \neq w_k^i, \text{ some } i\},$$

$$K_2 = \{k \in \{2, 3, \ldots, n\} | x_k^i = w_k^i, \text{ all } i\},$$

K_1 or K_2 can be null.

Take any $k \in K_1$. Let $(\bar{x}_1^i, \bar{x}_k^i)$ be the bundle that maximizes U_i^k given x^i with an income $x_1^i + p_k x_k^i$ and price $(1, p_k)$. It is clear as explained in the proof of Lemma 2 that (x_1^i, x_k^i) maximizes U_i^k given x^i with an income $x_1^i + p_k' x_k^i$ and prices $(1, p_k')$. From Lemma 1, if $p_k > p_k'$, then $\bar{x}_k^i < x_k^i$, all i, and if $p_k < p_k'$, then $\bar{x}_k^i > x_k^i$, all i.

We obtain from Lemma 2 that there canot be a binding quantity restriction on k. So, $p_k = p'_k$ for all $k \in K_1$.

By monotonicity of the utility functions U_i, we have $pw^i = px^i$ for all i. Since $p_k = p'_k$ for $k \in K_1 \cup \{1\}$ and $w^i_k = x^i_k$ for $k \in K_2$, we have $p'w^i = p'x^i$.

So, x^i is in the budget set for every i with initial endowments w^i and prices p'. We know by the definition of p' that x^i maximizes U_i at income $p'x^i = p'w^i$ and prices p'. So, $\{x^i\}$ is obtained as a Walras equilibrium with initial endowments $\{w^i\}$ and prices p'. So, $u \in W$.

We have just shown that $u \in E_2 \cap G \Rightarrow u \in W$. The reverse inclusion is also true since any Walras equilibrium is Pareto efficient and for a choice of large enough quantity restrictions, a GNWE as well. *Q.E.D.*

The proof of Lemma 2 can be somewhat simplified if we impose Drèze's one way quantity restriction condition on GNWE.

When the number of commodities is only two, then condition (iii) of a GNWE is not necessary for the theorem to be valid. Let

$$G' = \{u \in U \mid u_i = U_i(x^i) \text{ and } \{x^i\} \text{ satisfies conditions (i) and (ii) of a GNWE}\}.$$

COROLLARY 1. *When* $n = 2$, $E_2 \cap G' = W$.

PROOF: It is enough to prove that when $n = 2$, $G = G'$. Let $\{x^i_1, x^i_2\}$ constitute an allocation satisfying conditions (i) and (ii) of a GNWE.

If $x^i_2 = w^i_2$, then redefine $L^i_2 = l^i_2 = 0$. If $x^i_2 \neq w^i_2$, let $(\bar{x}^i_1, \bar{x}^i_2)$ be the demand of consumer i without any constraints. If $x^i_2 > w^i_2$, then as U_i satisfies Assumption 4, $\bar{x}^i_2 \geq x^i_2$. Now redefine $L^i_2 = x_2 - w^i$ and $l^i_2 = -\infty$. Similarly, if $x^i_2 < w^i_2$, let $l^i_2 = x_2 - w^i$ and $L^i_2 = \infty$. In every case, let $L^i_1 = \infty$ and $l^i_1 = -\infty$. It can be checked that $(x^i_1, x^i_2) \in \gamma^i(p, L^i, l^i)$ and condition (iii) of a GNWE is satisfied. So $G = G'$.

Q.E.D.

It can be checked that Assumption 4, though strong, is necessary for the theorem to hold. If we were to relax this assumption to strong convexity, then examples exist (see Figure 3.6 in [7]) of non-decentralizable Pareto efficient points with zero consumption of a commodity for some consumer. In the figure mentioned, by taking the initial endowments on a supporting hyperplane at such a point to the indifference curve of the consumer with zero consumption of a commodity, we get a non-decentralizable Pareto efficient point as a GNWE. Existence of a non-decentralizable Pareto efficient allocation is closely related to non-existence of a Walras equilibrium with endowment of one commodity zero for some consumer. However, we can prove a weaker version of Theorem 1 by weakening Assumption 4.

Now we relax Assumption 4 to the following:

ASSUMPTION 4a (Strong convexity): If $U_i(x) = U_i(y)$ and $x \neq y$, then $U_i(\lambda x + (1-\lambda)y) > U_i(x)$ for $0 < \lambda < 1$.

Define:

$$E_3 = \{u \in E_2 | \text{ there is } \{x^i\} \text{ with } \sum_i x^i \leq \sum_i w^i, \ U_i(x_i) = u_i$$
$$\text{and } x^i \text{ strictly positive, all } i\},$$

$$W_1 = \{u \in W | \text{ there is a Walras equilibrium } \{x^i\} \text{ with } U_i(x_i) = u_i$$
$$\text{and } x^i \text{ strictly positive, all } i\}.$$

COROLLARY 2: *Under Assumptions* 1, 2, 3, *and* 4*a*, $E_3 \cap G = W_1$.

PROOF: Lemma 1 continues to be valid and U_i^h given strictly positive x^i satisfies Assumptions 1, 2, 3, and 4a if U_i does. Utility allocations belonging to E_3 can be decentralized, so a proof on the lines of Lemma 2 and Theorem 2 is valid. *Q.E.D.*

5. CONCLUDING REMARKS

It is easy to prove that under the assumptions made, the set G contains an uncountable number of allocations if the initial endowments do not constitute a Walras equilibrium. This is a distinct advantage a policy of rationing has over the market mechanism which is rather restrictive.

Given a welfare function that is monotone in the individual utilities, a Pareto efficient allocation is not necessarily good (equitable) but a Pareto inefficient allocation is necessarily not good since it can be improved upon. The results of Section 4 indicate that rationing equilibria are in general Pareto inefficient except in the trivial case when the allocation is a Walras equilibrium. Since a Pareto inefficient allocation does not belong to the core, there is every incentive for private redistribution under rationing. This may explain why sometimes black markets do come up under rationing. Even if these are restricted, there is no reason for the central authority to remain satisfied with an inferior allocation. The results further indicate that non-Walrasian equilibrium is not as natural a state of the economy as is Walras equilibrium. Permanence of a non-Walrasian equilibrium requires some agency to counter this incentive for private redistribution.

The only policy known that can generate all the Pareto efficient points is a policy of lump sum transfer of initial endowments. This policy violates the property rights of individuals in private ownership economies. Further, since some of these endowments may include types of labor, a lump sum transfer of these amounts to allowing slavery.

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REFERENCES

[1] ARROW, K. J., AND F. H. HAHN: *General Competitive Analysis*. London: Oliver and Boyd, 1971.

[2] DRÈZE, JACQUES H.: "Existence of an Exchange Equilibrium Under Price Rigidities," *International Economic Review*, 14 (1975), 301–320.

[3] HAHN, F. H.: "On Non-Walrasian Equilibria," *Review of Economic Studies*, 45 (1978), 1–17.

[4] HICKS, J. R.: *Value and Capital*. Oxford: Clarendon Press, 1957.

[5] MANGASARIAN, O. L.: *Nonlinear Programming*. New York: McGraw-Hill, 1969.

[6] NAYAK, P. R.: "Existence of Production Equilibrium under Proportional Rationing," Discussion Paper No. 7812, Indian Statistical Institute, New Delhi, 1977.

[7] QUIRK, J., AND R. SAPOSNIK: *Introduction to General Equilibrium Theory and Welfare Economics*. New York: McGraw-Hill, 1968.

[8] YOUNES, Y.: "On the Role of Money in the Process of Exchange and the Existence of a Non-Walrasian Equilibrium," *Review of Economic Studies*, 42 (1975), 489–502.