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CONSTRUCTION OF ORTHOGONAL FACTORIAL DESIGNS CONTROLLING INTERACTION EFFICIENCIES

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ABSTRACT

This paper employs some variants of the usual Kronecker product to construct orthogonal factorial designs controlling the interaction efficiencies. The methods suggested have a fairly wide coverage and the resulting designs involve a small number of replicates.

1. INTRODUCTION

A factorial experiment in a block design is said to have the orthogonal factorial structure (OFS) if the adjusted treatment sum of squares can be split up orthogonally into components due to different factorial effects. A broad sufficient condition for OFS was obtained by John and Smith (1972) and Cotter, John and Smith (1973), while Mukerjee (1979) derived a necessary and sufficient condition. Construction problems for orthogonal factorial experi-

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ments in generalized cyclic designs were considered by John (1973 a,b), Dean and John (1975) and other authors and for a comprehensive list of references one may refer to John and Lewis (1983).

As an alternative approach, Mukerjee (1980, 1981, 1984) and Gupta (1983) employed Kronecker-type products of varietal designs to generate factorial designs with OFS. Although their approach could control the main effect efficiencies in the resulting designs no general results could be obtained on interaction efficiencies (cf. Gupta (1983), section 5). Gupta (1985), however, obtained some results on the interaction-efficiency in a two-factor Kronecker design particularly considering the situation when the two varietal designs involved in the Kronecker product are balanced.

Recently, Lewis and Dean (1985) established the 'efficiency-consistency' of designs with OFS. Although their result is theoretically elegant, problems still remain in so far as practical applications are concerned. In particular, if an m-factor design D is generated from varietal designs $D_1,\ldots,D_m$, then under OFS of D it follows from Lewis and Dean (1985) that the efficiency of any interaction in D equals that in the subdesign D_{sub} obtained from D by deleting from the treatment combinations all digits except those corresponding to the factors involved in that interaction. It, however, remains uncertain what the interaction efficiencies (either in D or in the appropriate D_{sub}) are in terms of the efficiencies of $D_1,\ldots,D_m$. This problem is important since in practice the experimenter may have $D_1,\ldots,D_m$ at his/her disposal and may wish to choose them suitably to control the interaction-efficiencies in D.

In an attempt to settle this issue, this paper suggests construction procedures for factorial designs with OFS starting from varietal designs and controlling the efficiencies of interactions up to some suitable order in terms of those of the component designs. Such a method of construction has been termed 'faithful' (see section 2). It may be remarked that the notion of faithfulness is different from that of efficiency-consistency defined by

Lewis and Dean (1985) in the sense that (with notations as above) the latter ensures that the efficiency of any interaction in D equals that in the appropriate D_{sub} while the former goes a step further to set a lower bound for this interaction efficiency in terms of the efficiencies of $D_1,\ldots,D_m$ (see (2.4)).

2. PRELIMINARIES

Throughout this paper, whether the design considered is varietal or factorial, the fixed effect intrablock model with independent homoscedastic errors and no block-treatment interaction is assumed. Consider first an equireplicate varietal design D_o in s treatments and r_o replicates. If $\lambda_0 = 0, \lambda_1,\ldots, \lambda_{s-1}$ be the eigenvalues of the usual C-matrix (cf. Raghavarao (1971), Ch. 4) of D_o, then the ϕ_p-efficiency, say $H_o^{(p)}$, of D_o may be defined (cf. Kiefer (1975)) as follows. If $\lambda_1,\ldots, \lambda_{s-1}$ are all positive then

$$
\begin{aligned}
H_o^{(p)} &= \{\prod_{t=1}^{s-1} (\lambda_t/r_o)\}^{1/(s-1)} && \text{when } p = 0,\\
&= \{(s-1)^{-1} \sum_{t=1}^{s-1} (\lambda_t/r_o)^{-p}\}^{-1/p} && \text{when } 0 < p < \infty, \qquad (2.1)\\
&= \min_{1\le t\le s-1} (\lambda_t/r_o) && \text{when } p = \infty,
\end{aligned}
$$

while if $\lambda_1,\ldots, \lambda_{s-1}$ are not all positive (in which case D_o is disconnected) then trivially $H_o^{(p)} = 0$ $(0 \le p \le \infty)$. Clearly, if p = 0, 1, ∞, then ϕ_p-efficiency reduces to the standard D-, A-, E- efficiencies respectively. Also if D_o be variance balanced then $\lambda_1 = \lambda_2 = \ldots = \lambda_{s-1} = \lambda$, say, and for each p, $H_o^{(p)} = \lambda/r_o$, which may be termed simply the efficiency of D_o.

Turning to a factorial set-up, consider a factorial design in m factors $F_1,\ldots, F_m$ at $s_1,\ldots, s_m$ (≥ 2) levels respectively. The $v = s_1 s_2 \ldots s_m$ treatment combinations will be assumed to be lexicographically ordered. Let for $1 \le j \le m$, $\underline{1}_j$ be an s_j-component vector with all elements unity, P_j an $(s_j-1)\times s_j$ real matrix such that $(s_j^{-1/2}\underline{1}_j, P_j')'$ is orthogonal and

$$P_j^{x_j} = s_j^{-1/2}\underline{1}_j' \text{ if } x_j = 0; = P_j \text{ if } x_j = 1. \qquad (2.2)$$

Then defining $\underline{\tau}$ as the v×1 vector of treatment effects, J as the set of all non-null m-component vectors with elements 0 or 1 and for any $x = (x_1,\ldots,x_m) \in J$,

$$P^x = \bigotimes_{j=1}^{m} P_j^{x_j} = P_1^{x_1} \otimes\ldots\otimes P_m^{x_m}, \qquad (2.3)$$

where $\otimes$ denotes Kronecker product, it can be seen (cf. Kurkjian and Zelen (1963), Mukerjee (1979))that $P^x\underline{\tau}$ represents a full set of orthonormal contrasts belonging to the factorial effect $F_1^{x_1}\ldots F_m^{x_m}$ ($= I(x)$, say). Let $\alpha(x)$ denote the number of rows of P^x.

Suppose the above factorial experiment is conducted in a block design each treatment combination being replicated r times. Denote by A_x the coefficient matrix of the reduced normal equations for estimating $P^x\underline{\tau}$ (cf. Kiefer (1975)) and by λ_t^x $(1 \le t \le \alpha(x))$ the eigenvalues of A_x. Then the Φ_p-efficiency, say $E^{(p)}(x)$, of the factorial effect $I(x)$ may be defined analogously to (2.1), replacing r_o, s-1 and λ_t in (2.1) by r, $\alpha(x)$ and λ_t^x respectively provided the λ_t^x are all positive. If the λ_t^x are not all positive, then $E^{(p)}(x) = 0$. The effect $I(x)$ will be called balanced (cf. Shah (1958)) if the λ_t^x are all equal in which case $E^{(p)}(x)$ is the same for all p, the common value being termed the efficiency of $I(x)$.

Suppose an $s_1 \times s_2 \times\ldots\times s_m$ factorial design is constructed starting from varietal designs $D_1, D_2,\ldots, D_m$ involving $s_1, s_2,\ldots, s_m$ treatments respectively. A component design D_j will be called relevant for the factorial effect $I(x)$ if $x_j = 1$.

<u>Definition 2.1</u>. A method of construction as above will be called faithful of order g $(1 \le g \le m)$ if for every effect $I(x)$ involving g or less factors and every p $(0 \le p \le \infty)$, the Φ_p-efficiency of $I(x)$ is at least as large as that of each relevant component design i.e. if for every $x \in J_g$ and every p $(0 \le p \le \infty)$,

$$E^{(p)}(x) \ge \max_{1\le j\le m} x_j H_j^{(p)}, \qquad (2.4)$$

where $H_j^{(p)}$ is the Φ_p-efficiency of D_j $(1 \le j \le m)$ and J_g is a subset of J containing vectors with at most g 1's.

The next sections suggest two construction procedures which are faithful of order g. Clearly, with such a procedure, one can control and remain assured of the efficiencies of the resulting designs for effects involving up to g factors by suitably choosing $D_1,\ldots, D_m$. It may be remarked that the methods in Mukerjee (1981, 1984) and Gupta (1983) are faithful of order 1 and, in general, so is any method of construction yielding designs with OFS (cf. Lewis and Dean(1985)). The following result will be helpful in the subsequent development.

Theorem 2.1 (Mukerjee (1979)). An $s_1 \times s_2 \times \ldots \times s_m$ equireplicate factorial design with constant block size and incidence matrix N will have OFS if and only if NN' has the form

$$NN' = \sum_{\ell=1}^{d} \xi_\ell \{ \bigotimes_{j=1}^{m} V_{\ell j} \},$$

where d is a positive integer, $\xi_1,\ldots, \xi_d$ are some real numbers and for each ℓ, j, the $s_j \times s_j$ matrix $V_{\ell j}$ has all row and column sums equal.

Remark. If the above condition holds, it can be seen (cf. Mukerjee (1979)) that for each $x \in J$, $A_x = P^x C P^{x'}$, C being the C-matrix of the factorial design.

3. COMPONENTWISE KRONECKER PRODUCT OF ORDER g

For $1 \le j \le m$, let D_j be a varietal design in b_j blocks and s_j treatments with common replication number r_j, constant block size k_j and the $s_j \times b_j$ incidence matrix N_j. If $N_1 \otimes \ldots \otimes N_m$ $(= N^{(1)})$ be the usual Kronecker product of $N_1,\ldots, N_m$, then it can be shown (see Theorems 3.1, 4.1, with $g = m$) that the $s_1 \times s_2 \times \ldots \times s_m$ factorial design given by $N^{(1)}$ has OFS and, further, this method of construction is faithful of order m (in fact, as one can check, these observations hold even when each D_j is allowed to have varying block sizes). Thus using the ordinary Kronecker product it is possible to control the efficiencies of all the factorial effects. A difficulty with this method is that the block size and the number of replicates in the design represented by $N^{(1)}$ may become too large. For example, in an m-factor design, based on ordinary Kronecker product, the block size should be at least 2^m in order that all the main effect contrasts are estimable.

As an alternative approach one may, therefore, attempt to achieve a control only over the lower order factorial effects and thereby reduce the block size or the number of replicates. To that effect, some modifications of the ordinary Kronecker product may be employed. With $N_1,\ldots, N_m$ as before, let for $1 \leq j \leq m$,

$$N_j = N_{j0} + N_{j1} + \ldots + N_{ju_j-1}, \tag{3.1}$$

where u_j is a positive integer and the elements of N_{jh_j} $(0 \leq h_j \leq u_j-1)$ are non-negative integers.

Definition 3.1. The componentwise Kronecker product of order g ($\leq$ m) of $N_1,\ldots, N_m$ with respect to the decomposition (3.1) is given by

$$N^{(2)} = \sum_{(h_1,\ldots,h_m)\epsilon T} \left(\bigotimes_{j=1}^{m} N_{jh_j} \right),$$

the sum being taken over only a subset T of the Πu_j possible combinations $(h_1,\ldots,h_m)$ such that the combinations included in T, written as columns, form an orthogonal array (with possibly variable symbols) with m rows, $u_1,\ldots, u_m$ symbols and strength g (cf. Rao (1973)).

The special case g = 1 was considered in Mukerjee (1981). On the other extreme, if g = m then $N^{(2)}$ reduces to the ordinary Kronecker product $N^{(1)}$.

Theorem 3.1. If for each j, h_j $(0 \leq h_j \leq u_j-1,\ 1 \leq j \leq m)$, the design represented by N_{jh_j} be equireplicate with replication number $u_j^{-1}r_j$ and has constant block size $u_j^{-1}k_j$ then the method of componentwise Kronecker product of order g is faithful of order g and the design $D^{(2)}$ represented by $N^{(2)}$ has OFS.

Proof. Under the conditions of the theorem, for $0 \leq h_j, q_j \leq u_j-1$, $1 \leq j \leq m$, $N_{jh_j}N'_{jq_j}$ has all row and column sums equal. Further, the design $D^{(2)}$ is equireplicate with replication number $n\Pi u_j^{-1}r_j$ (= $r^{(2)}$ say) and has constant block size $n\Pi u_j^{-1}k_j$, n being the cardinality of T. Since by Definition 3.1,

$$N^{(2)}N^{(2)'} = \sum_{\underline{h},\underline{q}\in T} \left(\bigotimes_{j=1}^{m} N_{jh_j} N'_{jq_j} \right), \tag{3.2}$$

where $\underline{h} = (h_1,\ldots,h_m)$, $\underline{q} = (q_1,\ldots,q_m)$, it is clear by Theorem 2.1 that $D^{(2)}$ has OFS.

The C-matrix of $D^{(2)}$ is, say,

$$C = r^{(2)} \left(\bigotimes_{j=1}^{m} I_j \right) - n^{-1} \left(\prod_{j=1}^{m} u_j k_j^{-1} \right) N^{(2)} N^{(2)'}, \tag{3.3}$$

I_j being the identity matrix of order s_j. To show that the method of construction is faithful of order g, take any $\bar{I}(x)$ involving f ($\leq$ g) factors. Let, without loss of generality, $x = \bar{x} = (\bar{x}_1,\ldots,\bar{x}_m)$ with $\bar{x}_j = 1$ ($1 \leq j \leq f$), $= 0$ ($f+1 \leq j \leq m$). Then by (2.2), (2.3), (3.2) and the assumptions regarding N_{jh_j},

$$P^{\bar{x}} N^{(2)} N^{(2)'} P^{\bar{x}'} =$$

$$\left(\prod_{j=f+1}^{m} s_j^{-1} u_j^{-2} k_j^2 b_j \right) \left(\bigotimes_{j=1}^{f} P_j \right) \left\{ \sum_{\underline{h},\underline{q}\in T} \left(\bigotimes_{j=1}^{f} N_{jh_j} N'_{jq_j} \right) \right\} \left(\bigotimes_{j=1}^{f} P'_j \right). \tag{3.4}$$

Since $f \leq g$, (3.1) and the fact that the n members of T form an orthogonal array of strength g yield,

$$\sum_{\underline{h},\underline{q}\in T} \left(\bigotimes_{j=1}^{f} N_{jh_j} N'_{jq_j} \right) = \left\{ n / \left(\prod_{j=1}^{f} u_j \right) \right\}^2 \left(\bigotimes_{j=1}^{f} N_j N'_j \right).$$

Hence if one defines the C-matrix of D_j as $C_j = r_j I_j - k_j^{-1} N_j N'_j$, recalls that $r_j s_j = k_j b_j$ and employs (3.3), (3.4) and the remark following Theorem 2.1, then one obtains after some simplification,

$$A_{\bar{x}} = P^{\bar{x}} C^{(2)} P^{\bar{x}'} = r^{(2)} \left[I - \bigotimes_{j=1}^{f} \{ P_j (I_j - r_j^{-1} C_j) P'_j \} \right],$$

where I is the identity matrix of order $\alpha(\bar{x})$. Denote by $\lambda_{j0} = 0$, $\lambda_{j1},\ldots, \lambda_{js_j-1}$ the eigenvalues of C_j. Then it follows that the eigenvalues of $A_{\bar{x}}$ are say,

$$\lambda^{\bar{x}}_{t_1\ldots t_f} = r^{(2)} \left[1 - \prod_{j=1}^{f} \{ 1 - (\lambda_{jt_j}/r_j) \} \right], \quad 1 \leq t_j \leq s_j - 1, \ 1 \leq j \leq f. \tag{3.5}$$

Clearly for each $t_1,\ldots,t_f$,

$$\lambda^{\bar{x}}_{t_1\ldots t_f} / r^{(2)} \geq \max_{1\leq j\leq f} (\lambda_{jt_j}/r_j). \tag{3.6}$$

From (3.5), (3.6), (2.1) and its analogue for factorial designs, it is clear that for the effect $I(\bar{x})$ under consideration $E^{(p)}(\bar{x}) \geq \max_{1\leq j\leq f} H_j^{(p)}$ $(0 \leq p \leq \infty)$. This proves the theorem.

Remark. Given $N_1, \ldots, N_m$, relations like (3.5) may be used for exactly determining $E^{(p)}(x)$, $x \in J_g$. Interestingly, as Example 3.1 suggests, the exact values of $E^{(p)}(x)$ for interactions are usually much greater than the lower bound (2.4). Incidentally, Theorem 3.1 strengthens Theorem 3 in Gupta (1985) in the sense that the varietal designs involved are not necessarily balanced and the result holds even in the multifactor case. In particular, if the designs $D_1, \ldots, D_f$ are balanced then $\lambda_{j1} = \ldots = \lambda_{js_j-1} = \lambda_j^*$, say, and by (3.5) in $D^{(2)}$ the effect $I(\bar{x})$ is balanced with efficiency $1 - \prod_{j=1}^{f}(1-H_j)$, where H_j $(= \lambda_j^*/r_j)$ is the efficiency of D_j $(1 \leq j \leq f)$. This holds in general for any $I(x)$, $x \in J_g$, provided the relevant component designs are balanced.

One may follow the line of Mukerjee (1981) to get the matrices N_{jh_j} satisfying the conditions of Theorem 3.1.

Example 3.1. To construct a 3×4×5 design, let D_1, D_2, D_3 be such that the $k_j \times b_j$ arrays Z_j, obtained by writing the blocks of D_j as columns, are as follows:

$$Z_1 = \begin{matrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{matrix}, \quad Z_2 = \begin{matrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \end{matrix}, \quad Z_3 = \begin{matrix} 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 & 2 & 3 & 4 & 0 & 1 \end{matrix}.$$

Then $r_1=r_2=2$, $r_3=4$, $k_1=k_2=k_3=2$. Note that each D_j, written as above, is positionally balanced in the sense that with the blocks written as columns all the treatments occur equally often in each row. Hence if for each j, Z_j be partitioned as $Z_j = [Z_{j0}', Z_{j1}']'$, where Z_{jh_j} is $1\times b_j$ and N_{jh_j} be the incidence matrix of a varietal design with blocks given by the columns of Z_{jh_j} $(h_j = 0,1)$ then the matrices N_{jh_j} satisfy the conditions of Theorem 3.1 with $u_1=u_2=u_3=2$. According to Definition 3.1, now $N^{(2)}$ may be formed taking $T = \{(0,0,0),(0,1,1),(1,0,1),(1,1,0)\}$. Since T represents an orthogonal array of strength 2, the resulting 3×4×5 factorial design has OFS and the method is faithful of order 2. As D_1, D_3 are balanced with efficiencies 0.75 and 0.625, the remark following Theorem 3.1 shows that the effects F_1, F_3, F_1F_3 are balanced with respective efficiencies 0.75, 0.625, 0.9062. The ϕ_p-efficiencies for F_2, F_1F_2, F_2F_3 may be obtained by (3.5). In particular, the A-efficiencies of these effects are 0.6, 0.9130, 0.8667 respectively. The design involves 8 replicates and block size 4, while the corresponding usual Kronecker product requires 16 replicates and block size 8.

Alternatively, taking D_1, D_2, D_3 such that

$$Z_1 = \begin{matrix} 0 \\ 1 \\ 2 \end{matrix}, \quad Z_2 = \begin{matrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 0 & 1 \end{matrix}, \quad Z_3 = \begin{matrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \\ 2 & 3 & 4 & 0 & 1 \end{matrix},$$

and with $u_1=u_2=u_3=3$, $T = \{(0,0,0),(0,1,1),(0,2,2),(1,0,1),(1,1,2),(1,2,0),(2,0,2),(2,1,0),(2,2,1)\}$, one may proceed exactly as before to construct a 3×4×5 factorial design in 3 replicates and block size 9. Since T is again an orthogonal array of strength 2, the factorial design obtained has OFS and the method is faithful of order 2. In this design the effects F_1, F_2, F_1F_2 are balanced with respective efficiencies 1, 0.8889, 1, while the A-efficiencies of the effects F_3, F_1F_3, F_2F_3 are 0.8148, 1, 0.9813 respectively. The design, like the earlier one is connected.

The second design in this example requires a smaller number of replicates but a larger block size than the first one. In fact, many other 3×4×5 designs with a fairly wide range of parameter-values and efficiency-levels can be obtained by the above method by choosing D_1, D_2, D_3 suitably. In a practical situation, a choice from amongst the available designs depends on the particular context.

4. KHATRI-RAO PRODUCT OF ORDER g

Khatri and Rao (1968) considered, in a different context, another modification of the ordinary Kronecker product which was used by Mukerjee (1980, 1984) and Gupta (1983) in the construction of factorial designs controlling the main effects alone. This section considers a generalized version of the Khatri-Rao product.

For $1 \leq j \leq m$, let D_j, N_j be as in the first paragraph of section 3 and partition N_j as

$$N_j = \bigcup_{h_j=0}^{u_j-1} N_{jh_j}, \tag{4.1}$$

where N_{jh_j} is of order $s_j \times u_j^{-1} b_j$ $(0 \leq h_j \leq u_j-1)$, u_j is a positive integer and for matrices $M_1, M_2, \ldots, M_a$ of the same order

$$\bigcup_{\ell=1}^{a} M_\ell = (M_1, M_2, \ldots, M_a).$$

Then the Khatri-Rao product of order g ($\leq$ m) of $N_1, \ldots, N_m$ under the decomposition (4.1) is defined as

$$N^{(3)} = \bigcup_{(h_1,\ldots,h_m)\epsilon T} \left(\bigotimes_{j=1}^{m} N_{jh_j} \right), \tag{4.2}$$

as before, T being a subset of the possible combinations $(h_1,\ldots,h_m)$ forming an orthogonal array of strength g. Clearly, $N^{(3)}$ reduces to the ordinary Kronecker product if g = m. The following result may be proved along the line of Theorem 3.1.

<u>Theorem 4.1</u>. If for each j, h_j $(0 \leq h_j \leq u_j-1,\ 1 \leq j \leq m)$, the design represented by N_{jh_j} be equireplicate with replication number $u_j^{-1} r_j$ then the method of Khatri-Rao product of order g is faithful of order g and the factorial design $D^{(3)}$ with incidence matrix $N^{(3)}$ has OFS.

The observations made in the remark following Theorem 3.1 hold in the present set-up as well. Under the conditions of Theorem 4.1, the design $D^{(3)}$ has replication number $n\Pi u_j^{-1} r_j$ and block size Πk_j, n being the cardinality of T. Thus, compared to the ordinary Kronecker product, the Khatri-Rao product can achieve a reduction only in the number of replications but not in the block size and, in this sense, it is inferior to the componentwise Kronecker product which can reduce both.

One may follow Mukerjee (1980, 1984) and Gupta (1983) to get the matrices N_{jh_j} satisfying the conditions of Theorem 4.1. This is fairly simple when each D_j is $(u_j^{-1}r_j)$-resolvable.

Example 4.1. To construct a 4×6×9 factorial design, let D_1, D_2, D_3 be varietal designs such that the corresponding $s_j \times b_j$ arrays Z_j, as in Example 3.1, are

$$Z_1 = \begin{matrix} 0 & 2 & 0 & 1 \\ 1 & 3 & 2 & 3 \end{matrix}, \quad Z_2 = \begin{matrix} 0 & 1 & 2 & 0 & 1 & 2 \\ 4 & 5 & 3 & 5 & 3 & 4 \end{matrix}, \quad Z_3 = \begin{matrix} 0 & 1 & 2 & 0 & 3 & 6 \\ 3 & 4 & 5 & 1 & 4 & 7 \\ 6 & 7 & 8 & 2 & 5 & 8 \end{matrix}.$$

Then $r_1=r_2=r_3=2$, $k_1=k_2=2$, $k_3=3$. Each D_j is 1-resolvable. Hence if Z_j be partitioned as $Z_j = (Z^*_{j0}, Z^*_{j1})$, where $Z^*_{jh_j}$ is $k_j \times (b_j/2)$, then the incidence matrices N_{jh_j} of the varietal designs $Z^*_{jh_j}$ satisfy the conditions of Theorem 4.1 with $u_1=u_2=u_3=2$. As in Example 3.1, the 4×6×9 factorial design with incidence matrix $N^{(3)}$ formed according to (4.2) taking $T = \{(0,0,0),(0,1,1),(1,0,1),(1,1,0)\}$ will have OFS. The method is faithful of order 2 and (3.5) may be applied to obtain the Φ_p-efficiencies of different effects involving at most two factors. In particular, the A-efficiencies of F_1, F_2, F_3, F_1F_2, F_1F_3, F_2F_3 are 0.6, 0.4286, 0.6667, 0.8347, 0.9, 0.8706 respectively. The design involves blocks of size 12 and 4 replications and is connected.

5. CONCLUDING REMARKS

As noted in the remark following Theorem 3.1, the methods of construction presented in this paper lead to designs in which the interaction contrasts are usually estimated with greater precision than the main effect contrasts. The methods may be of practical value in situations where the efficient estimation of the interaction contrasts is considered important. In particular, the procedures that are faithful of order 2 pose no severe combinatorial problem as orthogonal arrays of strength 2 are easily available, and the resulting designs may be useful in many practical situations where emphasis lies on the estimation of the two-factor interaction contrasts.

The methods considered in this paper may be extended in several directions. One may consider a combination of componentwise

Kronecker product and Khatri-Rao product. A similar approach was investigated by Gupta in the case $g = 1$.

Alternatively, instead of starting with varietal designs, one may try to generate more complex factorial designs from simpler factorials. Thus for $1 \leq j \leq w$, let N_j^* be the incidence matrix of an equireplicate factorial design involving factors $F_{j1},\ldots,F_{jm_j}$, and having a constant block size. Suppose N^* represents a factorial design in $m = \Sigma m_j$ factors obtained by suitably combining $N_1^*,\ldots, N_w^*$. For any effect $I(x)$ (in the design N^*), defining $I_j(x)$ as the effect (in the design N_j^*) given by the factors among $F_{j1},\ldots, F_{jm_j}$ that are involved in $I(x)$, one gets the following result:

Theorem 5.1. If for each j, the design given by N_j^* has OFS and N^* be the ordinary Kronecker product of $N_1^*,\ldots, N_w^*$, then the design given by N^* also has OFS. Further, for each x and each p $(0 \leq p \leq \infty)$,

$$\Phi_p\text{-efficiency of } I(x) \text{ in the design given by } N^*$$
$$\geq \max_{1\leq j\leq w} \{\Phi_p\text{-efficiency of } I_j(x) \text{ in the design given by } N_j^*\}.$$

The proof of the above proceeds along the line of Theorem 3.1 and makes use of Theorem 2.1. In fact, in the above setting one may as well introduce the componentwise Kronecker product or Khatri-Rao product of order g to obtain results similar to Theorems 3.1, 4.1. Theorem 5.1 resembles the efficiency-consistency criterion in Lewis and Dean (1985) if the inequality is replaced by an equality. A referee feels that the Kronecker product of incidence matrices of symmetric factorials may belong to the generalized cyclic class of designs.

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