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A General Principle for Limit Theorems in Finitely Additive Probability: The Dependent Case

RAJEEVA L. KARANDIKAR*

Indian Statistical Institute, Delhi, India

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In this paper we formulate and prove a general principle which enables us to deduce limit theorems for a sequence of random variables on a finitely additive probability space.

1. INTRODUCTION AND PRELIMINARIES

The main result of this paper can be summarized as "Almost all limit theorems that are true in countably additive probability theory are also true in finitely additive probability theory."

More precisely, given a sequence $\{Y_n\}$ of random variables on a finitely additive probability space, we construct a sequence $\{X_n\}$ of random variables on a countably additive probability space such that a convergence in probability or convergence in distribution type limit theorem holds for $\{Y_n\}$ if and only if it holds for $\{X_n\}$ and if, further the sequence $\{Y_n\}$ satisfies an additional condition, the same is true for a.s. convergence type limit theorems. The sequences $\{X_n\}$ and $\{Y_n\}$ are related via

$$Ef(Y_1, Y_2, ..., Y_n) = Ef(X_1, X_2, ..., X_n) \tag{1}$$

for all continuous functions f on $\mathbb{R}^n$, $n \geqslant 1$. This also gives that if $\{Y_n\}$ is an i.i.d./independent/strongly mixing/martingale/stationary sequence, then so is $\{X_n\}$.

From this, we deduce analogs of several well-known results in the finitely additive case.

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The formulation of the main result is similar to Aldous' formulation of the subsequence principle [1]. The proof is on the same lines as in Karandikar [6], where the same result was proved for a sequence of independent random variables in the finitely additive strategic setting.

We begin with definitions and auxiliary results. For a metric space T, let $C(T)$ be the class of real valued continuous functions on T, $C_b(T)$ be the class of real valued bounded continuous functions on T and $\mathscr{B}(T)$ be the Borel σ field on T. Let S be a separable metric space with a metric ρ.

A finitely additive probability space (FAPS) is a triplet $(H, \mathscr{C}, \mu)$, where H is a set, $\mathscr{C}$ is a field of subsets of H, and μ is a finitely additive probability measure (FAPM) on $(H, \mathscr{C})$. We will give a brief description of integration theory on $(H, \mathscr{C}, \mu)$. For details, see Dunford and Schwartz [4].

For $A \subseteq H$, let $\mu^*(A) = \inf\{\mu(C): C \in \mathscr{C}$ and $A \subseteq \mathscr{C}\}$ and $\mu_*(A) = 1 - \mu^*(A^c)$. Without loss of generality, we will assume that $(H, \mathscr{C}, \mu)$ is complete, i.e., $\mu^*(A) = \mu_*(A)$ implies $A \in \mathscr{C}$.

DEFINITION. Let U, U_n be S-valued mappings on H. Say that U_n converges to U in μ-probability, written as $U_n \to_\mu U$ if for all $\varepsilon > 0$, $\mu^*(\rho(U_n, U) > \varepsilon) \to 0$. Let

$$\mathscr{E} = \left\{ U: U = \sum_{i=1}^{k} a_i 1_{A_i}; A_i \in \mathscr{C}, a_i \in \mathbf{R}, k \geqslant 1 \right\}$$

and

$$\mathscr{L}(H, \mathscr{C}, \mu) = \{ U: H \to \mathbf{R} \text{ s.t. } \exists U_n \in \mathscr{E}, U_n \xrightarrow[\mu]{} U \}.$$

Elements of $\mathscr{E}$ will be called simple functions and elements of $\mathscr{L}(H, \mathscr{C}, \mu)$ will be called measurable functions or random variables on $(H, \mathscr{C}, \mu)$.

Remark 1. It is easy to see that if $U \in \mathscr{L}(H, \mathscr{C}, \mu)$, then for all $\varepsilon > 0$, $\exists K \subseteq \mathbf{R}$, K compact such that $\mu^*(U \notin K) < \varepsilon$. Indeed, given $\varepsilon > 0$, get V such that $\mu^*(|U - V| > 1) < \varepsilon$ and take $K = [-a-1, a+1]$, where a is an upper bound of $|V|$.

From its definition, it is clear that $\mathscr{L}(H, \mathscr{C}, \mu)$ is closed under convergence in probability. It is easy to see that it is closed under addition and multiplication. More generally, if $U_1, U_2, \ldots, U_k \in \mathscr{L}(H, \mathscr{C}, \mu)$ and $g \in C(\mathbf{R}^k)$, then

$$g(U_1, U_2, \ldots, U_k) \in \mathscr{L}(H, \mathscr{C}, \mu). \tag{1.1}$$

Integration. For a simple function $U = \sum_{i=1}^{k} a_i 1_{A_i}$, define $\int U \, d\mu = \sum_{i=1}^{k} a_i \mu(A_i)$. Let $\mathscr{L}^1(H, \mathscr{C}, \mu)$ be the class of $U \in \mathscr{L}(H, \mathscr{C}, \mu)$ such that

$$\exists U_n \in \mathscr{E}, \qquad U_n \xrightarrow[\mu]{} U, \qquad \int |U_n - U_m| \, d\mu \to 0. \tag{1.2}$$

For $U \in \mathcal{L}^1(H, \mathcal{C}, \mu)$, define

$$\int U\,d\mu = \lim_n \int U_n\,d\mu, \tag{1.3}$$

where U_n are as in (1.2). It is shown in Dunford and Schwartz [4, p. 111] that (1.3) unambiguously defines $\int U\,d\mu$. It is easy to see that all bounded measurable functions U belong to $\mathcal{L}^1(H, \mathcal{C}, \mu)$. The following dominated convergence theorem is proved in [4, p. 124].

THEOREM 1.1. *Let $U, U_n \in \mathcal{L}(H, \mathcal{C}, \mu)$ be such that $U_n \to_\mu U$. Suppose that $|U_n(h)| \leqslant V(h)$ for all $h \in H$ and $V \in \mathcal{L}^1(H, \mathcal{C}, \mu)$. Then*

$$\int |U_n - U|\,d\mu \to 0.$$

The following inequalities are easy to prove using definition of the integral. Let $U \in \mathcal{L}^1(H, \mathcal{C}, \mu)$, $U \geqslant 0$. Then for all $a > 0$,

$$\mu^*(U \geqslant a) \leqslant \frac{1}{a} \int U\,d\mu \tag{1.4}$$

and if $0 \leqslant U \leqslant 1$, then

$$\mu^*(U > 0) \geqslant \int U\,d\mu. \tag{1.5}$$

We will denote $\int U\,d\mu$ by EU or $E_\mu U$. If $U \in \mathcal{L}(H, \mathcal{C}, \mu)$, $U \geqslant 0$ and $U \notin \mathcal{L}^1(H, \mathcal{C}, \mu)$, we define $EU = \infty$. With this convention, it is easy to check that for all positive measurable functions U,

$$EU = \lim_{n \to \infty} E(U \wedge n). \tag{1.6}$$

For $U \in \mathcal{L}(H, \mathcal{C}, \mu)$ the set $\{U \leqslant a\}$ may not belong to $\mathcal{C}$ for all $a \in \mathbf{R}$ and thus we cannot talk of its distribution function. However, the set $\{U \leqslant a\}$ does belong to $\mathcal{C}$ for all but countably many points $a \in \mathbf{R}$. This is a consequence of the next result.

THEOREM 1.2. *Let $U_1, ..., U_n \in \mathcal{L}(H, \mathcal{C}, \mu)$.*

(a) *There exists a unique countably additive probability measure (CAPM) λ on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$ such that for all $g \in C_b(\mathbf{R}^n)$,*

$$Eg(U_1, U_2, ..., U_n) = \int g\,d\lambda. \tag{1.7}$$

(b) *Let $B \in \mathscr{B}(\mathbf{R}^n)$ be such that $\lambda(\partial B) = 0$ (∂B = boundary of B). Then*

$$\{(U_1, U_2, \ldots, U_n) \in B\} \in \mathscr{C}$$

and

$$\mu((U_1, U_2, \ldots, U_n) \in B) = \lambda(B). \tag{1.8}$$

In fact for all $g \in C_b(\mathbf{R}^n)$, we have

$$Eg(U_1, U_2, \ldots, U_n)\, 1_{((U_1, U_2, \ldots, U_n) \in B)} = \int_B g\, d\lambda. \tag{1.9}$$

Proof. Let $L(g) = Eg(U_1, U_2, \ldots, U_n)$, $g \in C_b(\mathbf{R}^n)$. L is well defined as $g(U_1, U_2, \ldots, U_n) \in \mathscr{L}^1(H, \mathscr{C}, \mu)$ and is bounded. Let $g_k \in C_b(\mathbf{R}^n)$ be such that $g_k \downarrow 0$. Using Dini's theorem and Remark 1, it can be proved that $L(g_k) \to 0$ and hence Daniell's theorem [7, p. 60] implies part (a).

For (b), given $B \in \mathscr{B}(\mathbf{R}^n)$, with $\lambda(\partial B) = 0$, get $g_k \in C_b(\mathbf{R}^n)$, $1_B \leq g_k \leq 1$ such that $g_k(x) \to 1_{\bar{B}}(x)$ pointwise. (Here $\bar{B}$ is the closure of B.) Then

$$\int g_k\, d\lambda \to \lambda(\bar{B}) = \lambda(B). \tag{1.10}$$

Further

$$\begin{aligned} \mu^*((U_1, U_2, \ldots, U_n) \in B) &\leq \mu^*(g_k(U_1, U_2, \ldots, U_k) \geq 1) \\ &\leq \int g_k(U_1, U_2, U_k)\, d\mu \\ &= \int g_k\, d\lambda \end{aligned}$$

by (1.4) and choice of λ. From (1.9) and (1.10) we can conclude that

$$\mu^*((U_1, U_2, \ldots, U_n) \in B) \leq \lambda(B). \tag{1.11}$$

Since $\partial(B^c) = \partial B$, using (1.11) for B^c and remembering that $\mu_*(A) = 1 - \mu^*(A^c)$, $\lambda(B) + \lambda(B^c) = 1$, we get

$$\mu_*((U_1, \ldots, U_n) \in B) \geq \lambda(B). \tag{1.12}$$

Completeness of $\mathscr{C}$ and (1.11), (1.12) now give $((U_1, \ldots, U_n) \in B) \in \mathscr{C}$ and that (1.8) holds. It can be shown that $g_k(U_1, \ldots, U_n) \to_\mu 1_{((U_1, \ldots, U_n) \in B)}$ and (1.9) can be deduced from and the dominated convergence theorem. ∎

COROLLARY 1.3 *Let G be the distribution function of λ and let $(a_1, a_2, \ldots, a_n)$ be a continuity point of G. Then (1.8) gives $\{U_i \leq a_i\} \in \mathscr{C}$ and*

$$\mu(U_i \leq a_i : 1 \leq i \leq n) = G(a_1, a_2, \ldots, a_n). \tag{1.13}$$

Conditional Expectation. Let $U \in \mathcal{L}^1(H, \mathscr{C}, \mu)$ and $\mathscr{F}$ be a subfield of $\mathscr{C}$. In analogy with the usual notion of conditional expectation, we make the following definition.

DEFINITION. If there exists a $V \in \mathcal{L}^1(H, \mathscr{F}, \mu)$ such that for all $F \in \mathscr{F}$, $E(U 1_F) = E(V 1_F)$, then we define V to be the conditional expectation of U given $\mathscr{F}$ and write it as

$$V = E(U | \mathscr{F}).$$

Given U and $\mathscr{F}$, the conditional expectation $E(U | \mathscr{F})$ may not exist, but when it does it has all the properties that the corresponding notion has in the countably additive theory.

Convergence in Distribution. We will define convergence in distribution for random variables on $(H, \mathscr{C}, \mu)$. We first introduce the class of S-valued random variables.

$$\mathcal{L}(H, \mathscr{C}, \mu; S) = \{\xi: H \to S \text{ s.t. } g(\xi) \in \mathcal{L}(H, \mathscr{C}, \mu) \quad \text{for all} \quad g \in C_b(S)\}.$$

DEFINITION. *Let* $\xi_k \in \mathcal{L}(H, \mathscr{C}, \mu; S)$ and λ be a CAPM on $(S, \mathscr{B}(S))$. Say that ξ_k converges in distribution to $\lambda (\xi_k \to^d \lambda)$ if for all $B \in \mathscr{B}(S)$ such that $\lambda(\partial B) = 0$, we have

$$\mu^*(\xi_k \in B) \to \lambda(B) \tag{1.14}$$

and

$$\mu_*(\xi_k \in B) \to \lambda(B). \tag{1.15}$$

Since $\lambda(\partial B) = 0$ implies $\lambda(\partial(B^c)) = 0$ and $\mu_*(A) = 1 - \mu^*(A^c)$, (1.14) for B^c implies (1.15). Thus we can delete (1.15) in the above definition.

If μ_1 is any extension of μ to $\mathscr{P}(H)$, then $\mu^*(A) \geqslant \mu_1(A) \geqslant \mu_*(A)$ and hence (1.14), (1.15) imply that for $B \in \mathscr{B}(S)$ with $\lambda(\partial B) = 0$, we have

$$\mu_1(\xi_k \in B) \to \lambda(B).$$

The next result is a familiar characterization of convergence in distribution.

THEOREM 1.4. *Let* $\xi_k \in \mathcal{L}(H, \mathscr{C}, \mu; S)$ *and* λ *be a* CAPM *on* $(S, \mathscr{B}(S))$. *Then* $\xi_k \to^d \lambda$ *if and only if for all* $f \in C_b(S)$,

$$Ef(\xi_k) \to \int f \, d\lambda. \tag{1.16}$$

Proof. First observe that $f(\xi_k) \in \mathscr{L}(H, \mathscr{C}, \mu)$ for $f \in C_b(S)$. Suppose (1.16) holds. Given $B \in \mathscr{B}(S)$ with $\lambda(\partial B) = 0$, get $f_n \in C_b(S)$, $1_{\bar{B}} \leqslant f_n \leqslant 1$, $f_n \downarrow 1_{\bar{B}}$. Then

$$\mu^*(\xi_k \in B) \leqslant \mu^*(f_n(\xi_k) \geqslant 1)$$
$$\leqslant \int f_n(\xi_k)\, d\mu = \int f_n\, d\lambda$$

and hence by (1.16)

$$\limsup \mu^*(\xi_k \in B) \leqslant \int f_n\, d\lambda. \tag{1.17}$$

But $\int f_n\, d\lambda \to \lambda(\bar{B}) = \lambda(B)$ and hence (1.17) gives

$$\limsup \mu^*(\xi_k \in B) \leqslant \lambda(B). \tag{1.18}$$

Since $\lambda(\partial B^c) = 0$, using (1.18) for B^c we get

$$\liminf \mu_*(\xi_k \in B) \geqslant \lambda(B). \tag{1.19}$$

Since $\mu^*(\xi_k \in B) \geqslant \mu_*(\xi_k \in B)$, the two relations (1.18) and (1.19) together imply

$$\lim \mu^*(\xi_k \in B) = \lambda(B) \tag{1.20}$$

and

$$\lim \mu_*(\xi_k \in B) = \lambda(B). \tag{1.21}$$

Thus (1.16) implies that $\xi_k \to^d \lambda$. Note that except for the occurence of outer and inner measures μ^*, μ_*, the proof is similar to that in the countably additive theory, as given in [2].

For the other part, let μ_1 be any extension of μ to $\mathscr{P}(H)$. It is easy to see that if $U \in \mathscr{L}^1(H, \mathscr{C}, \mu)$, then $U \in \mathscr{L}^1(H, \mathscr{P}(H), \mu_1)$ and then $\int U\, d\mu = \int U\, d\mu_1$. Thus it suffices to show that $\xi_k \to^d \lambda$ implies $\int f(\xi_k)\, d\mu_1 \to \int f\, d\lambda$ for all $f \in C_b(S)$. This proof is also similar to that in the countably additive theory (see [2]) and since $\mu_1(A)$ is defined for all $A \subseteq H$, outer measures do not appear. We omit the details.

The following observation can be proved easily using the respective definitions. Let $\xi_k \in \mathscr{L}(H, \mathscr{C}, \mu; S)$ and $s \in S$. Let δ_s be the measure on S defined by $\delta_s(A) = 1_A(s)$. Then

$$\xi_k \xrightarrow[\mu]{} s \quad \text{iff} \quad \xi_k \xrightarrow{d} \delta_s. \tag{1.22}$$

2. Main Results

From now on, we fix a complete FAPS $(H, \mathscr{C}, \mu)$ and a sequence $\{Y_n\}$ of real valued random variables on $(H, \mathscr{C}, \mu)$. For each n, let λ_n be a CAPM on $(\mathbf{R}^n, \mathscr{B}(\mathbf{R}^n))$ such that for all $f \in C_b(\mathbf{R}^n)$

$$Ef(Y_1, Y_2, ..., Y_n) = \int f \, d\lambda_n. \tag{2.1}$$

The existence and uniqueness of λ_n has been established in Theorem 1.2. Clearly, $\{\lambda_n\}$ is a consistent sequence and hence we can get a countably additive probability space $(\Omega, \mathscr{A}, P)$ and a sequence $\{X_n\}$ of random variables on $(\Omega, \mathscr{A}, P)$ such that for $B \in \mathscr{B}(\mathbf{R}^n)$, we have

$$P((X_1, X_2, ..., X_n) \in B) = \lambda_n(B). \tag{2.2}$$

Also

$$E_\mu f(Y_1, ..., Y_n) = E_P f(X_1, ..., X_n) \tag{2.3}$$

for all $f \in C_b(\mathbf{R}^n)$. Let $E_n = \{a \in \mathbf{R}: P(X_n = a) = 0\}$. Then it follows from Corollary 1.3 that if $a_i \in E_i$, $1 \leqslant i \leqslant n$, then

$$\mu(Y_i \leqslant a_i: 1 \leqslant i \leqslant n) = P(X_i \leqslant a_i: 1 \leqslant i \leqslant n). \tag{2.4}$$

Note that E_n is dense in $\mathbf{R}$ for each n, indeed E_n^c is atmost countable.

Let $\mathbf{Y} = (Y_1, Y_2, ...)$ and $\mathbf{X} = (X_1, X_2, ...)$ be $\mathbf{R}^\infty$-valued mappings on H and Ω, respectively.

The first result, which has several important implications, admits a very elementary proof.

THEOREM 2.1. *Let (S, ρ) be a metric space and let $\{g_k\}$ be a sequence of continuous mappings from $\mathbf{R}^\infty$ into S. Assume that for each k, g_k depends only on finitely many coordinates, i.e., m_k such that*

$$g_k(x_1, x_2, ...) = g_k(x_1', x_2', ...), \qquad x_i = x_i' \quad \textit{for} \quad 1 \leqslant i \leqslant m_k. \tag{2.5}$$

Let $s \in S$ and λ be a CAPM on $(S, \mathscr{B}(S))$. Then

(a) *$g_k(\mathbf{Y}) \to_\mu s$ if and only if $g_k(\mathbf{X}) \to_P s$*

(b) *$g_k(\mathbf{Y}) \to^d \lambda$ if and only if $g_k(\mathbf{X}) \to^d \lambda$.*

Proof. In view of (1.22) and the corresponding result on $(\Omega, \mathscr{A}, P)$, (a) is a special case of (b) when $\lambda(A) = 1_A(s)$. Let $f \in C_b(S)$. Condition (2.5) gives that

$$f(g_k(x_1, x_2, ...)) = h_k(x_1, x_2, ..., x_{m_k})$$

for a certain $h_k \in C_b(\mathbf{R}^{m_k})$. Hence by (2.3), we have

$$\begin{aligned} E_\mu f(g_k(\mathbf{Y})) &= E_\mu h_k(Y_1, Y_2, ..., Y_{m_k}) \\ &= E_p h_k(X_1, X_2, ..., X_{m_k}) \\ &= E_p f(g_k(\mathbf{X})). \end{aligned} \tag{2.6}$$

Therefore,

$$\begin{aligned} g_k(\mathbf{Y}) \xrightarrow{d} \lambda &\Leftrightarrow E_\mu f(g_k(Y)) \to \int f\,d\lambda \qquad \forall f \in C_b(S) \\ &\Leftrightarrow E_p f(g_k(\mathbf{X})) \to \int f\,d\lambda \qquad \forall f \in C_b(S) \\ &\Leftrightarrow g_k(\mathbf{X}) \xrightarrow{d} \lambda. \end{aligned}$$

We have used Theorem 1.3, the relation (2.6), and the definition of convergence in distribution on countably additive probability spaces. ∎

Almost Sure Convergence. The notion of almost sure convergence on $(H, \mathscr{C}, \mu)$ is defined in the obvious manner: $U_n \to U$ a.s. if

$$\mu^*\{h: U_n(h) \nrightarrow U(h)\} = 0.$$

It is well known that the analog of Theorem 2.1 is false for almost sure converge unless one imposes some condition on the infinite dimensional distribution of $\{Y_n\}$, as the following example shows.

EXAMPLE. Let H be the space of all sequences of 0 and 1 and let Y_k be the coordinate mappings on H. Let $\mathscr{C}$ be the field of finite dimensional sets and μ be the FAPM on H given by $\mu(Y_1 = i_1, ..., Y_k = i_k) = 2^{-k}$ for all $i_1, i_2, ..., i_k \in \{0, 1\}$. The associated sequence $\{X_k\}$ is an i.i.d. sequence of Bernoulli random variables and hence by SLLN

$$\frac{X_1 + \cdots + X_n}{n} \to \frac{1}{2} \qquad \text{a.s.}$$

Let $A = \{h \in H: (Y_1(h) + \cdots + Y_n(h))/n \to \frac{1}{2}\}$. Since $\mathscr{C}$ contains only finite dimensional sets, it is easy to see that $\mu^*(A^c) = 1$, and thus SLLN is not valid for $\{Y_n\}$. Here, $\mu_*(A) = 0$ and $\mu^*(A) = 1$. Thus for all $\theta \in [0, 1]$, we can get an extension μ_θ of μ to $\mathscr{P}(H)$ such that $\mu_\theta(A) = \theta$. So SLLN will hold for $\{Y_N\}$ on $(H, \mathscr{P}(H), \mu_\theta)$ only for $\theta = 1$.

Of course, as a consequence of the previous result, it follows that $(Y_1 + \cdots + Y_n)/n \to_\mu \frac{1}{2}$. The condition we impose on $\{Y_n\}$ for the validity of a.s. limit theorems is the following.

For $\{a_{n,i}: 0 \leqslant i \leqslant k_n\} \subseteq E_n, n \geqslant 1$, let $\mathscr{F}_m$ be the field generated by the family

$$\{(Y_n \leqslant a_{n,i}): 0 \leqslant i \leqslant k_n, n \leqslant m\}.$$

Then $\mathscr{F}_m$ is a finite field, $\mathscr{F}_m \subseteq \mathscr{C}$ (as $a_{n,i} \in E_n$) and $\mathscr{F}_m \subseteq \mathscr{F}_{m+1}$. Let $\mathscr{F}_\infty = \bigcup_m \mathscr{F}_m \cdot \mathscr{F}_\infty$ is itself a field.

We will say that $\{Y_n\}$ is *regular* if the smallest σ-field $\sigma(\mathscr{F}_\infty)$ containing $\mathscr{F}_\infty$ is contained in $\mathscr{C}$ and further, the restriction of μ to $\sigma(\mathscr{F}_\infty)$ is countably additive (for all choices of $\{a_{n,i}\} \subseteq E_n$).

It should be noted that *regularity* is a condition on ∞-dimensional joint distribution and not on marginal distributions. A sequence of independent random variables $\{Y_n\}$ in the strategic setting of Dubins and Savage is regular (See [9]).

Given a consistent sequence G_n of "quasi-distribution functions," (i.e., G_n's satisfy the usual properties of distribution functions except right continuity), we can construct a *regular* sequence $\{Y_n\}$ on some FAPS $(H, \mathscr{C}, \mu)$ such that

$$\mu(Y_i \leqslant y_i: 1 \leqslant i \leqslant n) = G_n(y_1, y_2, ..., y_n) \qquad \text{for all} \quad y_i \in \mathbf{R},\ 1 \leqslant i \leqslant n,\ n \geqslant 1. \tag{2.7}$$

Take $H = \mathbf{R}^\infty$ and Y_n be the coordinate mappings. Then (2.7) defines a finitely additive measure μ_0 on the field $\mathscr{C}_0$ of finite dimensional rectangles. μ_0 is easily seen to be countably additive on $\mathscr{F}_\infty = \bigcup_m \mathscr{F}_m$, where $\mathscr{F}_m$ is as described above and thus has an extension to $\sigma(\mathscr{F}_\infty)$ as a CAPM. These extensions are consistent and determine a finitely additive probability measure μ on

$$\mathscr{C} = \bigcup \{\sigma(\mathscr{F}_\infty): \text{all choices of } \mathscr{F}_\infty \text{ as described above}\}.$$

It is easy to check that $\{Y_n\}$ is *regular* on $(H, \mathscr{C}, \mu)$ and satisfies (2.7).

We will now prove our general result on a.s. limit theorems.

THEOREM 2.2. *Suppose that $\{Y_n\}$ is regular. Let $A \in \mathscr{B}(\mathbf{R}^\infty)$ be such that for some p, $0 < p < \infty$, we have*

$$\mathbf{x} \in A \quad \textit{and} \quad \sum_{i=1}^{\infty} |x_i - x_i'|^p < \infty \Leftrightarrow \mathbf{x}' \in A \tag{2.8}$$

for all $\mathbf{x} = (x_1, x_2, ...)$ and $\mathbf{x}' = (x_1', x_2', ...) \in \mathbf{R}^\infty$. Then

$$\mathbf{Y} \in A \text{ a.s. } \mu \qquad \textit{if and only if } \mathbf{X} \in A \text{ a.s. } P. \tag{2.9}$$

Proof. For $n \geq 1$, get $\{a_{n,j}: 0 \leq j \leq k_n\} \subseteq E_n - \{0\}$ such that

$$0 < a_{n,j} - a_{n,j-1} < 2^{-n}, \qquad 1 \leq j \leq k_n \tag{2.9a}$$

and

$$P(a_{n,0} < X_n \leq a_{n,k_n}) \geq 1 - 2^{-n}. \tag{2.10}$$

Let $q_n: \mathbf{R} \to \mathbf{R}$ be defined by

$$q_n(x) = \sum_{j=1}^{k_n} a_{n,j} 1_{\{a_{n,j-1} < x \leq a_{n,j}\}}. \tag{2.11}$$

Let $Z_n = q_n(Y_n)$, $W_n = q_n(X_n)$, $\mathbf{Z} = (Z_1, Z_2, ...)$, $\mathbf{W} = (W_1, W_2, ...)$.

Let $\mathscr{F}_m$ be the finite field generated by the (finite valued) random variables $(Z_1, Z_2, ..., Z_m)$. Since $\{Y_n\}$ is assumed to be regular, we conclude that $\mathscr{G} = \sigma(\bigcup_m \mathscr{F}_m) \subseteq \mathscr{C}$ and restriction μ' of μ to $\mathscr{G}$ is countably additive. Note that $\{Z_n\}$ are measurable w.r.t. $\mathscr{G}$.

Condition (2.4) implies that finite dimensional distribution of $\{Z_n\}$ and $\{W_n\}$ coincide. Since μ' is countably additive on $\mathscr{G}$, this implies

$$\mu'(\mathbf{Z} \in B) = P(\mathbf{W} \in B), \qquad B \in \mathscr{B}(\mathbf{R}^{\infty}). \tag{2.12}$$

We will now prove that

$$\sum_{n=1}^{\infty} |X_n - W_n|^p < \infty \qquad \text{a.s. } P \tag{2.13}$$

and

$$\sum_{n=1}^{\infty} |Y_n - Z_n|^p < \infty \qquad \text{a.s. } \mu. \tag{2.14}$$

By the choice of Z_n's,

$$\{|Z_n - Y_n| \geq 2^{-n}\} \subseteq \{Z_n = 0\}$$

and $\mu'(Z_n = 0) \leq 2^{-n}$. Since μ' is countably additive, the Borel-Cantelli lemma gives

$$\mu'\{Z_n = 0 \text{ i.o.}\} = 0$$

which gives

$$\mu^*\{|Z_n - Y_n| \geq 2^{-n} \text{ i.o.}\} = 0. \tag{2.15}$$

This proves (2.14). The other relation (2.13) can be proved similarly.

To complete the proof, note that

$$\begin{aligned} \mathbf{Y} \in A \text{ a.s. } \mu \Leftrightarrow \mathbf{Z} \in A \text{ a.s. } \mu \quad & \text{(by (2.14))} \\ \Leftrightarrow \mathbf{W} \in A \text{ a.s. } P \quad & \text{(by (2.12))} \\ \Leftrightarrow \mathbf{X} \in A \text{ a.s. } P \quad & \text{(by (2.13)).} \ \blacksquare \end{aligned}$$

Remark 2. Suppose $\{g_k : k \geqslant 1\}$ is a sequence of continuous mappings from $\mathbf{R}^\infty$ into S satisfying, for some $0 < p < \infty$, m_k,

$$\rho(g_k(x), g_k(x')) \leqslant \sum_{i=1}^{m_k} C_{k,i} |x_i - x_i'|^p \tag{2.16}$$

for $\mathbf{x}, \mathbf{x}' \in \mathbf{R}^\infty$, where $C_{k,i}$ are positive constants bounded by C and for each i, $\lim_k C_{k,i} = 0$. Then

$$A = \{x : g_k(x) \text{ converges in } \mathbf{R}\}$$

satisfies (2.8). Thus (2.9) yields

$$g_k(\mathbf{Y}) \text{ converges a.s. } \mu \Leftrightarrow g_k(\mathbf{X}) \text{ converges a.s. } P. \tag{2.17}$$

Let $g_k(\mathbf{Y})$ converge a.s. to U (say). In general, a.s. convergence does not imply convergence is probability on $(H, \mathscr{C}, \mu)$. But in this case, we can first verify that

$$\rho(g_k(\mathbf{Y}), g_k(\mathbf{Z})) \to 0 \qquad \text{a.s. } \mu \text{ and in } \mu\text{-probability.} \tag{2.18}$$

Thus, $g_k(\mathbf{Z}) \to U$ a.s. μ. Since $\mathbf{Z}$ is $\mathscr{D}$-measurable and μ' is countably additive, $g_k(\mathbf{Z}) \to U$ in μ-probability. Then (2.18) gives that $g_k(\mathbf{Y}) \to U$ in μ-probability as well. As a consequence, $U \in \mathscr{L}(H, \mathscr{C}, \mu)$.

3. Consequences

In this section, we define the notions of "*an independent sequence*," "*a strongly mixing sequence*," "*a martingale*," "*a strictly stationary sequence*" on a *finitely additive probability space*. Each of these definitions is a natural one and is equivalent to the usual definition if the underlying probability space is countably additive. Further, if $\{Y_n\}$ has one of the properties listed above, then so does $\{X_n\}$, where $\{X_n\}$ is the sequence associated with $\{Y_n\}$ in the previous section. This enables us to use limit theorems for $\{X_n\}$ and our results in the previous section to deduce analogous results for $\{Y_n\}$. This approach has been illustrated in [6] with full details in the independents case and hence we will be brief in this section. We begin with a lemma. We continue to use notation established in the previous section.

LEMMA 3.1. *Let $g \in C(\mathbf{R}^n)$. Then*

$$E_\mu|g(Y_1, Y_2, ..., Y_n)| < \infty \quad \text{if and only if } E_P|g(X_1, X_2, ..., X_n)| < \infty \tag{3.1}$$

and in that case we have

$$E_\mu g(Y_1, Y_2, ..., Y_n) = E_P g(X_1, X_2, ..., X_n) \tag{3.2}$$

and, for all $B \in \mathscr{B}(\mathbf{R}^n)$ with $P((X_1, X_2, ..., X_n) \in \partial B) = 0$,

$$E_\mu g(Y_1, Y_2, ..., Y_n) \cdot 1_{((Y_1, Y_2, ..., Y_n) \in B)} = E_P g(X_1, X_2, ..., X_n) \cdot 1_{((X_1, X_2, ..., X_n) \in B)}. \tag{3.3}$$

Proof. For all $k \geq 1$, we have

$$E_\mu|g(Y_1, Y_2, ..., Y_n)| \wedge k = E_P|g(X_1, X_2, ..., X_n)| \wedge k.$$

Taking the limit as $k \to \infty$, using (1.6) on the left-hand side, and the monotone convergence theorem on the right, we get

$$E_\mu|g(Y_1, Y_2, ..., Y_n)| = E_P|g(X_1, X_2, ..., X_n)|. \tag{3.4}$$

This implies (3.1) and (3.2) follows by using (3.4) for the functions $g^+ = g \vee 0$ and $g^- = -(g \wedge 0)$. Finally, (3.3) can be deduced from (1.9) similarly.

(i) *The Independent Case*

DEFINITION. Say that $\{Y_n\}$ is a sequence of *independent random variables* (on $(H, \mathscr{C}, \mu)$) if for all $n \geq 1$, for all $y_i \in E_i$, $1 \leq i \leq n$, we have

$$\mu(Y_i \leq y_i; 1 \leq i \leq n) = \prod_{i=1}^{n} \mu(Y_i \leq y_i). \tag{3.5}$$

It is easily seen that if $\{Y_n\}$ satisfies (3.5), then the associated sequence $\{X_n\}$ is also a sequence of independent random variables. $\{Y_n\}$ will be said to be identically distributed if

$$\mu(Y_i \leq y) = (Y_1 \leq y) \qquad \text{for all} \quad y \in \bigcap_n E_n, \tag{3.6}$$

and then $\{X_n\}$ will also have the same property. Hence as a consequence of Theorem 2.1, we have that the weak law of large numbers (WLLN), the Lindeberg–Feller central limit theorem, the Donsker invariance principle are valid on finitely additive probability spaces as well, and we do not need to assume *any* aditional condition. Further, if $\{Y_n\}$ is assumed to be regular, then the strong law of large numbers, the law of iterated logarithms, the Kolmogonov 3-series theorem, and the Strassen invariance principle also hold for $\{Y_n\}$. The details are same as those given in [6].

(ii) *The Mixing Case*

For $1 \leqslant m \leqslant n \leqslant \infty$, let $\mathscr{D}_n^m$ be the field on H generated by $\{(Y_i \leqslant y): y \in E_1, m \leqslant i \leqslant n\}$.

DEFINITION. Say that $\{Y_n\}$ is a *strongly mixing sequence* with rate $r(n)$, if $r(n) \downarrow 0$ and

$$|\mu(E_1 \cap E_2) - \mu(E_1)\,\mu(E_2)| \leqslant r(n) \tag{3.7}$$

whenever $E_1 \in \mathscr{D}_k^1$, $E_2 \in \mathscr{D}_\infty^{k+n}$.

DEFINITION. Say that $\{Y_n\}$ is a *mean-zero weakly stationary sequence* if for all $i, j \geqslant 1$,

$$EY_i^2 < \infty, \qquad EY_i = 0, \qquad EY_i Y_{i+j-1} = EY_1 Y_j. \tag{3.8}$$

If $\{Y_n\}$ is a mean-zero weakly stationary strongly mixing sequence, then it is easy to see that so is $\{X_n\}$ with the same rate function $r(n)$. Suppose further that for some $\varepsilon > 0$, $\delta > 0$, $C < \infty$,

$$E|Y_n|^{2+\delta} \leqslant C < \infty, \qquad r(n) = O(n^{-(1+\varepsilon)(1+2/\delta)})$$

then $E|X_n|^{2+\delta} \leqslant C < \infty$ as well and then from results of Kuelbs and Phillips [7, p. 1008] we have that WLLN, SLLN, CLT, and Donsker's and Strassen's invariances principle hold for $\{X_n\}$. Thus, WLLN, CLT, Donsker's invariance principle also hold for $\{Y_n\}$ and if $\{Y_n\}$ is *regular*, SLLN and Strassen's invariance principle are also valid.

Similarly, we can define ϕ-mixing and show that the available results for ϕ-mixing sequences on $(\Omega, \mathscr{A}, P)$ are also valid on the FAPS $(H, \mathscr{C}, \mu)$, with the exception that for a.s. results, we need to add the assumption of regularity.

(iii) *Martingales*

DEFINITION. Say that $\{Y_n\}$ is a martingale if for all n, $E|Y_n| < \infty$ and further for all $m \geqslant n$,

$$E_\mu(Y_m \mid \mathscr{D}_n^1) = Y_n. \tag{3.9}$$

Since $\mathscr{D}_n^1$ is a finite field, it can be shown that (3.9) is equivalent to

$$E_\mu Y_m 1_{\{Y_i \leqslant y_i:\, 1 \leqslant i \leqslant n\}} = EY_n 1_{\{Y_i \leqslant y_i:\, 1 \leqslant i \leqslant n\}} \tag{3.10}$$

for all $y_i \in E_i$. Now (3.3) implies that similar relation holds for $\{X_n\}$ and since E_n is dense for each n, this gives that $\{X_n\}$ is itself a martingale (for the natural σ-fields). Thus the martingale convergence theorem for $\{X_n\}$ implies the following.

THEOREM 3.2. *Suppose that* $\{Y_n\}$ *is a martingale. Suppose that* $\{Y_n\}$ *is also regular. Then we have*

(a) *If* $\sup_n E|Y_n| < \infty$, *then* Y_n *converges* a.s. μ.

(b) *If* $\sup_n E|Y_n|^p < \infty$ *for some* $p > 1$, *then* $Y_n \to Y_\infty$ a.s. μ *and* $E|Y_n - Y_\infty|^p \to 0$. *Further,*

$$E_\mu(Y_\infty \mid \mathscr{D}_n^1) = Y_n. \tag{3.11}$$

Proof. For (a), take $A = \{\mathbf{x} \in \mathbf{R}^\infty;\ x_n \text{ converges in } \mathbf{R}\}$. Then as noted earlier, $\{X_n\}$ is a martingale and $\sup_n E|X_n| < \infty$. Hence X_n converges a.s., i.e., $\mathbf{X} \in A$ a.s. P and thus $\mathbf{Y} \in A$ a.s. μ by Theorem 2.2.

For (b), we first conclude as in (a) above that $X_n \to X_\infty$ a.s. P and $E|X_\infty - X_n|^p \to 0$ by the martingale convergence theorem for $\{X_n\}$. Now, going back to the proof of Theorem 2.2, it can be seen that $\{a_{n,i}\}$ can be chosen such that $\{Z_n\}$, $\{W_n\}$ also satisfy

$$E_\mu|Y_n - Z_n|^p \leqslant 2^{-n}, \qquad E_P|X_n - W_n|^p \leqslant 2^{-n}. \tag{3.12}$$

Hence $W_n \to X_\infty$ a.s. and $E|W_n - X_\infty|^p \to 0$. Then (2.12) implies that $Z_n \to Y_\infty$ (say) a.s. μ and $E_\mu|Z_n - Y_\infty|^p \to 0$. Now (3.12) implies that $E_\mu|Y_n - Y_\infty|^p \to 0$.

Since $p > 1$, we also have $E_\mu|Y_m - Y_\infty| \to 0$ as $m \to \infty$. Fix $A \in \mathscr{D}_n^1$ then from the martingale property, we have

$$E_\mu Y_m 1_A = E_\mu Y_n 1_A \qquad \text{for} \quad m \geqslant n$$

and hence $E_\mu|Y_m - Y_\infty| \to 0$ implies

$$E_\mu Y_\infty 1_A = E_\mu Y_n 1_A \qquad \text{for all} \quad A \in \mathscr{D}_n^1.$$

Hence (3.11) holds.

The following martingale invariance principle is also a consequence of Theorem 2.1 and for this we *do not* need to assume that $\{Y_n\}$ is regular.

THEOREM 3.3. *Let* $\{Y_n\}$ *be a martingale such that* $EY_n = 0$ *and* $EY_n^2 < \infty$ *for all* n. *Let* $V_n^2 = \sum_{i=1}^n (Y_i - Y_{i-1})^2$, *where* $Y_0 \equiv 0$ *and* $s_n^2 = EV_n^2$. *For each* n, *let* ξ_n *be* $C[0,1]$ *valued map on* H *defined by interpolating between the points*

$$(0,0);\ (V_n^{-2}V_1^2, V_n^{-1}Y_1);\ (V_n^{-2}V_2^2, V_n^{-1}Y_2); \ldots, (1, V_n^{-1}Y_n).$$

Suppose that $\{Y_n\}$ *also satisfies*

$$\mu^*(|s_n^{-2}V_n^2 - s_m^{-2}V_m^2| \geqslant \varepsilon) \to 0 \quad \text{as } n, m \to \infty \quad \text{for all } \varepsilon > 0, \tag{3.13}$$

$$\lim_{\varepsilon \to 0} \limsup_n \mu^*(s_n^{-2}V_n^2 \leqslant \varepsilon) = 0 \tag{3.14}$$

and

$$s_n^{-2} \sum_{j=1}^{n} E[Y_j^2 I_{(|Y_j| > \varepsilon s_n)}] \to 0 \tag{3.15}$$

for all $\varepsilon > 0$ *such that* εs_n *and* $-\varepsilon s_n$, *belong to* $E = \bigcap_n E_n$. *Then* ξ_n *converges in distribution to the Wiener measure on* $C[0, 1]$.

Proof. Let $U_n^2 = \sum_{i=1}^{n} (X_i - X_{i-1})^2$, with $X_0 = 0$. It can be seen that (3.13), (3.14) imply that similar relations hold for $\{U_n\}$ and hence we have

$$s_n^{-2} U_n^2 \xrightarrow[P]{} T \quad \text{with } 0 < T < \infty \text{ a.s. } P. \tag{3.16}$$

Since we can choose $\varepsilon_k \downarrow 0$ such that $\varepsilon_k s_n$ and $-\varepsilon_k s_n$ belong to E, (3.15) and (3.3) imply that the Lindeberg condition holds for $\{X_n\}$. If η_n is a $C[0, 1]$-valued random element on $(\Omega, \mathscr{A}, P)$ obtained by interpolating between the points

$$(0, 0); (U_n^{-2}U_1^2, U_n^{-1}X_1), \ldots, (1, U_n^{-1}X_n)$$

then (3.16) and the Lindeberg condition implies that η_n converges in distribution to the Wiener measure.

It can be seen that ξ_n, η_n can be expressed as $g_n(\mathbf{Y})$ and $g_n(\mathbf{X})$, where g_n are $C[0, 1]$-valued continuous functions on $\mathbf{R}^\infty$ satisfying (2.5). Thus, Theorem 2.1 yields the convergence in distribution of $\xi_n = g_n(\mathbf{Y})$ to the Wiener measure as the same holds for $g_n(\mathbf{X})$.

(iv) *Stationary Sequence*

DEFINITION. Say that $\{Y_n\}$ is a *strictly stationary sequence* if for all $n \geqslant 1$, $y_1, y_2, \ldots, y_n \in E = \bigcap_k E_k$,

$$\mu(Y_1 \leqslant y_1, Y_2 \leqslant y_2, \ldots, Y_n \leqslant y_n) = \mu(Y_2 \leqslant y_1, Y_3 \leqslant y_2, \ldots, Y_{n+1} \leqslant y_n). \tag{3.17}$$

It is easy to see that (3.17) implies that the associated sequence is also a strictly stationary sequence.

Let us fix a *regular* strictly stationary sequence $\{Y_n\}$. We will now introduce the *invariant field* for $\{Y_n\}$. Let $\{f_n\}$ be a sequence of simple functions of the form

$$f_n(x) = \sum_{j=1}^{k_n} b_{n,j} 1_{(a_{n,j-1} < x \leq a_{n,j})}, \tag{3.18}$$

$a_{n,j} \in E_n$, $0 \leq j \leq k_n$. Let $\mathscr{I}_{\{f_n\}}$ be the invariant σ field on H for the sequence $\{f_n(Y_n)\}$. Since $\{Y_n\}$ is regular, $\mathscr{I}_{\{f_n\}} \subseteq \mathscr{C}$ and μ is countably additive on $\mathscr{I}_{\{f_n\}}$. Let

$$\mathscr{I} = U\mathscr{I}_{\{f_n\}},$$

where union is taken over all sequences (3.18). $\mathscr{I}$ will be called the invariant field for $\{Y_n\}$.

We have the following version of the ergodic theorem.

THEOREM 3.4. *Let $\{Y_n\}$ be a regular strictly stationary sequence. Let $g\colon \mathbb{R}^n \to \mathbb{R}$ be such that for some p, $0 < p < \infty$, for some C,*

$$|g(x_1, x_2, ..., x_n) - g(x_1', x_2', ..., x_n')| \leq C \sum_{i=1}^{n} |x_i - x_i'|^p, \tag{3.19}$$

for all $(x_1, x_2, ..., x_n)$, $(x_1', x_2', ..., x_n') \in \mathbb{R}^n$. Further, suppose that

$$E|g(Y_1, ..., Y_n)| < \infty. \tag{3.20}$$

Then

$$V_m = \frac{1}{m} \sum_{i=0}^{m-1} g(Y_{i+1}, Y_{i+2}, ..., Y_{i+n}) \to E(g(Y_1, Y_2, ..., Y_n) | \mathscr{I})$$

a.s. μ and in $\mathscr{L}^1(H, \mathscr{C}, \mu)$.

Proof. As noted earlier, $\{X_n\}$ is strictly stationary and also (3.20) implies that $E|g(X_1, ..., X_n)| < \infty$. Thus the ergodic theorem [3, p. 118] implies

$$U_m = g_m(\mathbf{X}) \to E(g(X_1, X_2, ..., X_n) | \mathscr{I}') = U_\infty \tag{3.21}$$

a.s. P and in $\mathscr{L}^1(\Omega, \mathscr{A}, P)$, where $\mathscr{I}'$ is the invarient σ field for $\{X_n\}$ and $g_m\colon \mathbb{R}^\infty \to \mathbb{R}$ is given by

$$g_m(x_1, x_2, ...) = \frac{1}{m} \sum_{i=0}^{m-1} g(x_{i-1}, ..., x_{i+n}).$$

It is easy to see that $\{g_m\}$ satisfies (2.16) and that $V_m = g_m(\mathbf{Y})$. Hence by Remark 2, we get that for some V_∞,

$$V_m \to V_\infty \text{ and } g_m(\mathbf{Z}) \to V_\infty \text{ a.s. } \mu \text{ and in } \mu\text{-probability.} \tag{3.22}$$

Continuity of g, Lemma 3.1, and (3.21) imply that

$$E_\mu|V_m - V_k| = E_P|U_m - U_k| \to 0 \quad \text{as} \quad m, k \to \infty \tag{3.23}$$

and hence (3.22) yields

$$E_\mu|V_m - V_\infty| \to 0. \tag{3.24}$$

It remains to show that $V_\infty = E(g(Y_1, ..., Y_n)| \mathscr{I})$. Note that the invariant σ-field for $\{Z_m\}$ is contained in $\mathscr{I}$ and hence in view of (3.22), $V_\infty \in \mathscr{L}(H, \mathscr{I}, \mu)$.

Fix a sequence $\{f_n\}$ satisfying (3.18). Then by Lemma 3.1, for $f \in C(\mathbf{R}^m)$,

$$Ef(Y_1, Y_2, ..., Y_m)\, 1_A = Ef(X_1, X_2, ..., X_m)\, 1_C \tag{3.25}$$

if the integrals are well defined; for $A = \{(f_1(Y_1), ..., f_k(Y_k)) \in B\}$, $C = \{(f_1(X_1), ..., f_k(X_k)) \in B\}$, $B \in \mathscr{B}(\mathbf{R}^k)$. Since distribution of $\{f_n(Y_n)\}$ is countably additive (as $\{Y_n\}$ is regular), the dominated convergence theorem implies that (3.25) holds for any

$$A = (\{f_n(Y_n)\} \in B), \qquad C = (\{f_n(X_n)\} \in B), \qquad B \in \mathscr{B}(\mathbf{R}^\infty). \tag{3.26}$$

Fix a shift invariant set $B \in \mathscr{B}(\mathbf{R}^\infty)$ and let A, C be defined by (3.26). Then as noted above we have

$$Eg(Y_1, Y_2, ..., Y_n)\, 1_A = Eg(X_1, X_2, ..., X_n)\, 1_C \tag{3.27}$$

and

$$EV_m 1_A = EU_m 1_C. \tag{3.28}$$

Since $E|V_m - V_\infty| \to 0$, $E|U_m - U_\infty| \to 0$, we get

$$EV_\infty 1_A = EU_\infty 1_C. \tag{3.29}$$

Since $C \in \mathscr{I}'$ and $U_\infty = E(g(X_1, X_2, ..., X_n)| \mathscr{I}')$, we get that the right-hand sides of (3.27) and (3.29) are equal and hence

$$EV_\infty 1_A = E(Y_1, Y_2, ..., Y_n)\, 1_A. \tag{3.30}$$

Since $\{f_n\}$ satisfying (3.18) and the shift-invariant set B are arbitrary and $V_\infty \in \mathscr{L}(H, \mathscr{I}, \mu)$, we conclude

$$E(g(Y_1, Y_2, ..., Y_n)| \mathscr{I}) = V_\infty.$$

Remark 3. Even in the independent case, results proved in this paper are more general than those in [6]; as for convergence in probability and distribution type theorems, the central limit theorem in particular, we do not need to assume that $\{Y_n\}$ is regular.

Remark 4. Since a Martingale or a Markov chain in the strategic setting may not be regular, the results of Purves and Sudderth [9] on martingales and Ramakrishnan [10] on Markov chains cannot be deduced from our general principle.

Remark 5. We can define an exchangeable sequence in an obvious manner and can obtain an analog of De Fenniti's theorem for regular excangeable sequences. Also, we can prove an analog of Chatterjees subsequence principle (see [1]) for an arbitrary regular sequence $\{Y_n\}$.

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