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The Nonlinear Input–Output Model*

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This paper develops a nonlinear input–output model in which the production functions can exhibit a mixture of returns to scale at the various stages of production. It is shown that the traditional properties of the linear input–output model can be replicated under an extremely plausible assumption, which we call the uniform dominant diagonal condition. On the basis of this assumption it is shown that the model satisfies a contraction property. This open up the possibility of using some quite powerful results from the contraction mapping theory, establishes the existence of solutions, efficient computational procedures, and leads to a rather transparent mathematical theory for the nonlinear input–output model. *Journal of Economic Literature* Classification Numbers: 022, 023.

The Leontief input–output model is most well-known and most often used static model of the structure of a national economy. The basic assumption of the model, viz., the constancy of the coefficients characterizing the transaction matrix, however, rules out nonconstant returns to scale and substitutability of inputs in each sector of the economy. Interesting cases which arise when there are nonconstant returns to scale or when there are substitution possibilities are not amenable to the Leontief input–output analysis.

In reality, a sector may consist of a large number of minor production units of different efficiency with regard to the use of inputs. Expanding the output of such a sector may involve, e.g., the use of less efficient units which may not be used at lower levels of production. The production function for the sector as a whole may then exhibit nonconstant returns to scale even when there are constant returns to scale in each micro-unit. In addition to this, of course, there may be increasing or decreasing returns to scale, associated with higher levels of production, in the micro-units themselves. The total variations in the returns to scale in a sector may be then a

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combination of these two. Though the analysis presented below is general enough to include both these possibilities, for the sake of simpler economic interpretations (but whithout any loss of mathematical generality) we shall assume nonconstant returns to scale in the sectors, but rule out nonconstant returns to scale in the micro-units.

In addition to the nonconstant returns to scale, the micro-units in a sector may generate substitution possibilities with regard to the inputs for the sector as a whole. Since the micro-units may be particularly efficient with regard to the use of some inputs and not so with regard to others, the same *total* amount of output may be produced with different *total* input combinations depending upon the way in which the micro-units are utilized. In this paper, we shall not focus on this aspect, but assume that the micro-units can be ranked in some definite order of efficiency which may depend upon such factors as the relative prices of inputs (and outputs) determined exogeneously.

Motivated by the considerations such as outlined above, this paper develops a nonlinear input–output model in which the production functions can exhibit a mixture of returns to scale at the various stages of production. It is shown that the traditional properties, including the wellknown computational procedure, of the linear input–output model hold under an extremely plausible assumption. This assumption requires that each micro-unit should have a positive value added at some exogeneously determined relative prices of inputs.

We approach the problem on the basis of certain properties of matrix norms and show that our model has the contraction property. This opens up the possibility of using some quite powerful results from the contraction mapping theory and leads to a rather transparent mathematical theory for the nonlinear input–output model: establishes the existence of solutions, gives simple and efficient computational procedures. In addition to this, it enables us to compare and unify the various other models that have been proposed in the literature. Since our assumptions are technically weaker than those of Sandberg [11], and Chein and Chan [4], our main theorem (Theorem 1) implies that each of these models, like our own, has the contraction property.[1] This basic property shows that the models of these authors are quite similar and forms the basis of a general theory for the nonlinear input–output model.

The contents of this paper are as follows. In Section I we state our model and its assumptions. In Section II we establish the analytic properties of the model. We show that our model satisfies the contraction property (Theorem 1), and then prove certain results (Theorem 2 and 3) concerning

[1] This shows that a claim by Lahiri [7, footnote 9] that his viability conditions are quite different and difficult to compare with those of Sandberg [11] is not correct.

the existence, uniqueness, and computation of the solution by exploiting this property. In Section III, we show that the assumptions of our model are weaker than those of Sandberg [11], and Chien and Chang [4]. Theorem 1 then implies that each of these models like our own satisfies the contraction property.

I. The Model

We assume the economy to be divided into n industrial sectors, each of which produces a single kind of good that is traded, consumed, and invested in the economy. The interrelations among the various sectors in such an economy may be described by the system of equations:

$$x_i - \sum_{j=1}^{n} a_{ij}(x_j) = c_i \qquad (i = 1, 2, ..., n), \tag{1}$$

where x_i denotes the quantity of good i produced in the ith sector and $a_{ij}(x_j)$ represents the *total* amount of good i used as input for producing x_j units of good j. Therefore, for each i the total amount of good i available for final consumption, export and investment is $x_i - \sum_{j=1}^{n} a_{ij}(x_j)$, which is the left-hand side of (1). The vector $(c_1, c_2, ..., c_n)$ is called the *final demand vector*. As a matter of convenience (1) may be written in the compact form as

$$x - A(x) = c, \tag{2}$$

where $x = (x_1, x_2, ..., x_n)$ and $[A(x)]_i = \sum_{j=1}^{n} a_{ij}(x_j)$ is the ith component of $A(x)$.

Assumption I. *For each i and j, $a_{ij}(\cdot)$ is defined and continuously differentiable on $[0, \infty)$, $a_{ij}(0) = 0$ and $a'_{ij}(\alpha) \geqslant 0$ for all $\alpha \geqslant 0$, where $a'_{ij}(\alpha)$ denotes the derivative of $a_{ij}(\alpha)$ at α.*

The coefficients $a'_{ij}(\alpha)$, $i = 1, 2, ..., n$, will be referred to as the marginal input coefficients for sector j. Given that each micro-unit has constant returns to scale these must be in fact the constant average input coefficients of some micro-unit in sector j. This assumption thus has the same meaning as the nonnegativity assumption for the input coefficients in the linear case. The remaining part of the assumption that the $a_{ij}(\alpha)$ are continuously differentiable on $[0, \infty)$ is used in this paper to enable us to use certain tools of differential calculus. It will be clear from below that this part of the assumption is quite unnecessary as far as Theorems 1, 2, part of 3, and 4 of our paper are concerned. Similar results can be obtained even when the functions $a_{ij}(\alpha)$, $i, j = 1, 2, \ldots, n$, are piecewise affine

ASSUMPTION II. *There exist $p_i \geqslant 0$, and $v_i > 0$, $i = 1, 2, \ldots, n$, such that $p_j \geqslant \sum_{i=1}^{n} p_i a'_{ij}(\alpha) + v_j$ for $\alpha \in [0, \infty)$ and $j = 1, 2, \ldots, n$.*

This assumption is made meaningful by interpreting p_i as the price of commodity i. The sum $\sum_{i=1}^{n} p_i a'_{ij}(\alpha)$ then represents the marginal raw materials and intermediate goods cost. The assumption thus means that there is a positive value added in each micro-unit for some vector of prices.[2] This is the same kind of assumption as is usually made for the linear case, with the constant average and marginal input coefficients in the linear model now being replaced by variable marginal input coefficients. In fact, Assumption II is a sort of extension of the well-known dominant diagonal condition (cf. [8]).

This completes the statement of our model and its assumptions. We show later in Section III that our assumptions are weaker than those of Sandberg [11], and Chein and Chan [4].

II. PROPERTIES OF THE MODEL

As mentioned earlier, most of the properties of the linear input–output model can be replicated in the nonlinear model. These properties will be summarized in the following theorems.

2.1. *Notation and Definitions*

The following notation and definitions are used throughout the paper. R^n denotes the set of all real n-vectors. For $y \in R^n$, the inequality $y \geqslant 0$ ($y > 0$) means that $y_i \geqslant 0$ ($y_i > 0$), for all i, where y_i denotes the ith component of y. The set R^n_+ denotes $\{y \in R^n \mid y \geqslant 0\}$. I denotes the identity matrix of order n. If $M = (m_{ij})$ is a real square matrix, then $M \geqslant 0$ ($M > 0$) is equivalent to the statement that $m_{ij} \geqslant 0$ ($m_{ij} > 0$) for all i and j. If $A(x)$ is a mapping from R^n into R^n then $M(x) = (m_{ij}(x))$ denotes the Jacobian matrix of $A(x)$ with respect to x at the point $x \in R^n$. Accordingly, $M(0)$ denotes the Jacobian matrix of $A(x)$ with respect to x at the point $x = 0$.

DEFINITION 1. A mapping $A(\cdot)$ of R^n into R^n is said to be *contractive* over a set $D \subset R^n$ if there exists an $\alpha < 1$ and a vector norm $\|\cdot\|$ on R^n such that $\|A(x) - A(y)\| \leqslant \alpha \|x - y\|$ for all $x, y \in D$.

[2] Since the micro-units without positive value added may not participate in the production process, the assumption may be even more reasonable than is being claimed.

DEFINITION 2. Given a vector norm $\|\cdot\|$ on R^n the *norm of an arbitrary* $n \times n$ matrix with respect to $\|\cdot\|$ is

$$\|A\| = \sup_{\|x\|=1} \|Ax\|. \tag{3}$$

2.2. Existence and Properties of the Solution

PROPOSITION 1. *Let $\|\cdot\|$ be an arbitrary vector norm on R^n and P an arbitrary nonsingular $n \times n$ matrix. Then the mapping $\|\cdot\|^*$ defined by $\|x\|^* = \|Px\|$, for all $x \in R^n$, is a vector norm on R^n. Moreover, if A is an arbitrary $n \times n$ (real) matrix, then*

$$\|A\|^* = \|PAP^{-1}\|. \tag{4}$$

Proof. To show that $\|\cdot\|^*$ is a vector norm requires only a simple calculation verifying the axioms of a vector norm.[3] Then (4) results from

$$\|A\|^* = \sup_{\|x\|^*=1} \|Ax\|^* = \sup_{\|Px\|=1} \|PAx\|$$
$$= \sup_{\|y\|=1} \|PAP^{-1}y\| = \|PAP^{-1}\|.$$

THEOREM 1. *Under Assumptions* I *and* II, *the mapping $A(\cdot)$ of R^n into R^n as defined in* (2) *is contractive over R^n_+.*

We prove this theorem by means of a Lemma.[4]

LEMMA 1. *Let A be an arbitrary nonnegative $n \times n$ matrix. If there exist (row) vectors $p \geqslant 0$ and $v > 0$ such that $p \geqslant pA + v$, then there exists a vector norm $\|\cdot\|^*$ on R^n and an $\varepsilon > 0$ such that $\|A\|^* \leqslant 1 - \varepsilon$.*

Proof. Clearly, $p > 0$. Let P denote the (nonsingular) diagonal matrix corresponding to the vector p. Let $\|\cdot\|^*$ be the norm defined as $\|x\|^* = \|Px\|$ for all $x \in R^n$, where $\|\cdot\|$ is the l_1-norm on R^n, i.e., $\|x\| = \sum_{i=1}^n |x_i|$. Then (by Proposition 1) $\|A\|^* = \|PAP^{-1}\| \leqslant 1 - \varepsilon$, where $\varepsilon = \min_{1 \leqslant j \leqslant n} (v_j/p_j)$.

Proof of Theorem 1. Let $x, y \in R^n_+$. Then $x + \gamma(y - x) \in R^n_+$ for all $\gamma \in [0, 1]$. Let $M(x + \gamma(y - x))$ denote the (nonnegative) Jacobian matrix of $A(\cdot)$ at $x + \gamma(y - x)$. By Assumption II, there exist $p \geqslant 0$ and $v > 0$ such that $p \geqslant pM(x + \gamma(y - x)) + v$. Let P be the diagonal matrix corresponding to p

[3] Note that the norm $\|\cdot\|^*$ depends upon the matrix P, although this is not indicated by our notation.

[4] This lemma is, in fact, a restatement of the well-known result in the theory of productive nonnegative input–output matrices that units can be so changed that column sums are less than one, but is given here to familiarise the reader with the matrix-norm notation.

and let $\|\cdot\|^*$ be the norm as defined in Lemma 1. By Lemma 1 then there exists $\varepsilon > 0$ such that $\|M(x + \gamma(y - x))\|^* \leqslant 1 - \varepsilon$. Since γ is arbitrary, $\sup_{0<\gamma<1} \|M(x + \gamma(y - x))\|^* \leqslant 1 - \varepsilon$. Since the mapping $A(\cdot)\colon R^n_+ \to R^n_+$ can be extended to a convex open set containing R^n_+ (see the proof of Theorem 3 below), it follows from [10, Theorem 3.2.2] that

$$\|A(x) - A(y)\|^* \leqslant \sup_{0<\gamma<1} \|M(x + \gamma(y - x))\|^* \, \|x - y\|^* \quad \text{for all } x, y \in R^n_+.$$

Hence the theorem.

THEOREM 2. *Under Assumptions* I *and* II, *there exists a unique* $x \in R^n_+$ *such that* $x - A(x) = c$, *for each* $c \in R^n_+$, *and for any* $x(0) \in R^n_+$ *the sequence* $\{x(t)\}_0^\infty$, *defined by the Jacobi iterates*

$$x(t+1) = A(x(t)) + c, \qquad t \geqslant 0, \tag{5}$$

converges to x.

Proof. Let c be an arbitrary (column) vector belonging to R^n_+ and let $F_c(\cdot)\colon R^n_+ \to R^n_+$ denote the mapping defined as $F_c(x) = A(x) + c$, for all $x \in R^n_+$. Then $F_c(\cdot)$ is contractive over R^n_+. This is so because $F_c(x) - F_c(y) = A(x) - A(y)$ for all $x, y \in R^n_+$, and that, as proved in Theorem 1 above, the mapping $A(\cdot)$ is contractive over R^n_+.

From Assumption I above, $a_{ij}(0) = 0$ and $a'_{ij}(\alpha) \geqslant 0$ for all $\alpha \in [0, \infty)$ and each $i, j = 1, 2,\ldots, n$. Thus, $A(x) \geqslant 0$ for all $x \geqslant 0$ and hence the mapping $F_c(\cdot)$ defined as $F_c(x) = A(x) + c$, for all $x \in R^n_+$ is from R^n_+ into R^n_+, i.e., $F_c(x) \geqslant 0$ for all $x \geqslant 0$.

Proof of our theorem is a direct consequence of the well-known contraction mapping principle (see, e.g., [10, p. 120]) which states

> "Suppose that $F(\cdot)\colon R^n \to R^n$ is contractive over a closed set $D \subset R^n$ and that $F[D] \subset D$. Then $F(\cdot)$ has a unique fixed point in D and for any $x(0) \in D$ the sequence $\{x(t)\}_0^\infty$ defined by the Jacobi iterates $x(t+1) = F(x(t))$, $t \geqslant 0$, converges to the fixed point".

In our case $D = R^n_+$. Thus, if x is a fixed point of $F_c(\cdot)$, then $x \in R^n_+$ and $x = F_c(x)$, i.e., $x = A(x) + c$. Also $x(t+1) = F_c(x(t)) = A(x(t)) + c$ converges to x.

COROLLARY 1. *For* $x(0) = c$ *the sequence* $\{x(t)\}_1^\infty$, *defined by the Jacobi iterates* $x(t+1) = A(x(t)) + c$ *satisfies* $x(t+1) \geqslant x(t) \geqslant c$.

Proof. Assumption I implies that $A(x) \geqslant A(y) \geqslant 0$ whenever $x \geqslant y \geqslant 0$. Hence the corollary.

COROLLARY 2. *If $x = A(x) + c$ and $y = A(y) + d$, in which x, y, c and d belong to R^n_+ such that $c \geqslant d$, then $x \geqslant y$.*

Proof. Let $\{x(t)\}_0^\infty$ and $\{y(t)\}_0^\infty$ be the sequences defined by the Jacobi iterates $x(t+1) = A(x(t)) + c$, $x(0) = c$, and $y(t+1) = A(y(t)) + d$, $y(0) = d$, respectively. Then $x(t) \geqslant y(t)$ for all $t \geqslant 0$. Theorem 2 implies that $x \geqslant y$. Hence the corollary.

As a *consequence of the contraction property* the computational procedure as suggested in Theorem 2 has a *fast convergence*.[5] It also has computational simplicity. All that is necessary by way of the computational facility is the ability to evaluate $A(x)$ for a given value of x. The nature of convergence follows from the following inequalities which follow from the contraction property.

$$\|x(t) - x\|^* \leqslant \frac{\alpha}{1-\alpha} \|x(t) - x(t-1)\|^*,$$

$$\|x(t) - x\|^* \leqslant \alpha^t \|x(0) - x\|^*,$$

$$\|c(t) - c\|^* \leqslant \alpha^t (1-\alpha) \|x(0) - x\|^*.$$

It is clear that the rate of convergence of the Jacobi iterates depends upon the contraction constant α. The first of these inequalities provides a computable error estimate, i.e., if the contraction constant α is known, the actual error $\|x(t) - x\|^*$ (measured in terms of the norm $\|\cdot\|^*$) after the tth iteration can be bounded in terms of the last step $\|x(t) - x(t-1)\|^*$. The second inequality provides an estimate of the number of iterations required for a given tolerable margin of error. For example, if the initial error is 50% and the contraction constant $\alpha = 9/10$, the number of iterations required for a 5% tolerable margin of error is then approximately 25. The last inequality implies that the target errors diminish at a geometric rate.

Let $B(\cdot)$ denote the mapping defined as $B(x) = x - A(x)$ for all $x \in R^n_+$. Then $B(\cdot)$ is from R^n_+ into R^n. It was shown in Theorem 2 that for each $c \in R^n_+$, there exists a unique $x \in R^n_+$ such that $B(x) = c$. We now prove a "smoothness" property of the inverse mapping $B^{-1}: R^n_+ \to R^n_+$.

THEOREM 3. *Under Assumptions* I *and* II, *there exists a constant $n \times n$ matrix K and a continuous mapping $\Delta(\cdot)$ of R^n_+ into R^n such that $\|\Delta(c)\|/\|c\| \to 0$ as $\|c\| \to 0$ with the property that $B^{-1}(c) = Kc + \Delta(c)$ for each $c \in R^n_+$.*

[5] This procedure originated in antiquity, appearing, e.g., in the writings of Heron of Alexandria [2] in the second century B.C. in connection with the extraction of roots. An abstract formalization of this procedure as a property of the contraction mapping was achieved by Banach [1] and further elaborated by Weissinger [12].

Proof. Let $S = \{x \in R^n : x_i > -\eta$ for all $i\}$, where $\eta > 0$ is an arbitrary constant. Let $G(\cdot)$ denote the extension of the mapping $A(\cdot)$ to the domain S, with $G(\cdot)$ defined by the condition that $[G(x)]_i = \sum_{j=1}^{n} g_{ij}(x_j)$ for each $i = 1, 2, \ldots, n$, and all $x \in S$, where $g_{ij}(\alpha) = a_{ij}(\alpha)$ for all $i, j = 1, 2, \ldots, n$, and $\alpha \geqslant 0$, $g_{ij}(\cdot)$ is continuous at $\alpha = 0$, and $g'_{ij}(\alpha) = a'_{ij}(0)$ for all $\alpha \in (-\eta, 0)$. Then $G(\cdot)$ is continuously differentiable over the open set S and for $x \geqslant 0$, the Jacobi matrix of $G(x)$ is same as that of $A(x)$. By essentially the same argument as in Theorem 1, the extended mapping $G(\cdot)$ from S into R^n is contractive over S.

Let $F(\cdot)$ from S into R^n be the mapping defined as $F(x) = x - G(x)$ for all $x \in S$. Then $(I - M(0))$ is the Jacobi matrix of $F(\cdot)$ at $x = 0$ and $\det(I - M(0)) \neq 0$ (cf. [9]). By the *inverse function theorem*, S contains an open neighbourhood X of 0 such that $F(\cdot)$ is a homeomorphism of X onto an open neighborhood Y of 0 and there is a continuous mapping $\delta(\cdot)$ of R^n_+ onto R^n such that $\|\delta(y)\|/\|y\| \to 0$ as $\|y\| \to 0$ and such that $F(x) = y$ is satisfied by

$$x = (I - M(0))^{-1}y + \delta(y), \qquad y \in X, \text{ and } x \in X.$$

To complete the proof we must show that $x \geqslant 0$, i.e., $F(x) = B(x)$, for every $c \in Y \cap R^n_+$. Suppose that $F(\bar{x}) = c$ and $\bar{x} \ngeqslant 0$ for some $c \in Y \cap R^n_+$. Since $c \in Y \cap R^n_+$, there exists an $\tilde{x} \geqslant 0$ such that $B(\tilde{x}) = c$, by Theorem 2. Since $F(x) = B(x)$ for all $x \geqslant 0$, this implies that $F(\bar{x}) - F(\tilde{x}) = 0$. This means $\bar{x} - \tilde{x} = G(\bar{x}) - G(\tilde{x})$ and $G(\cdot)$ contractive over $S \supset X$. However, this can be true only if $\bar{x} = \tilde{x} \geqslant 0$, in contradiction to the definition of $\bar{x}$. This proves that $F^{-1}(c) = B^{-1}(c)$ for all $c \in Y \cap R^n_+$ and thus

$$B^{-1}(c) = (I - M(o))^{-1}c + \delta(c) \qquad \text{for each } c \in Y \cap R^n_+.$$

Hence the theorem.

THEOREM 4. *Under Assumptions* I *and* II, *if* $x - A(x) = c$, *and* $y - A(y) = d$, *in which* x, y, c *and* d *are elements of* R^n_+ *such that* $c \geqslant d$ *and* $c - d \neq 0$, *and if* $(I - M(y))$ *is indecomposable, where* $M(y)$ *is the Jacobian matrix, then* $x - y > 0$.

Proof. Corollary 2 above implies that

$$B^{-1}(c) \geqslant B^{-1}(d + \alpha(c - d)) \geqslant B^{-1}(d) \qquad \text{for all } \alpha \in (0, 1). \tag{6}$$

As in Theorem 3 there is a continuous mapping $\Delta(\cdot)$ of R^n_+ into R^n such that $\|\Delta(z)\|/\|z\| \to 0$ as $\|z\| \to 0$, and

$$B^{-1}(d + \alpha(c - d)) = B^{-1}(d) + \alpha(I - M(y))^{-1}(c - d) + \Delta(\alpha(c - d)). \tag{7}$$

Since $(I-M(y))$ is indecomposable. Assumption II implies that $(I-M(y))^{-1} > 0$. Thus, there exists an $\bar{a} \in (0, 1)$ such that $B^{-1}(d+\bar{a}(c-d)) > B^{-1}(d)$. Inequalities (6) and (7) together imply that $B^{-1}(c) > B^{-1}(d)$. Hence the theorem.

Theorems 1–4 demonstrate that our model has similar properties as the linear input–output model. It may be of further interest to observe that any subsystem of (1) has the same properties as (1) itself and that a linear input–output model has the contraction property *if and only if* it satisfies Assumptions I and II. These can be proved easily be utilizing the properties of the contraction mapping.

Finally, note that we have not anywhere utilized the assumption that $[A(x)]_i = \sum_{j=1}^{n} a_{ij}(x_j)$ which rules out intersectoral externalities. All the results above would hold even if we were to assume more generally that $[A(x)]_i = a_i(x)$, a real-valued function of the vector x.[6]

III. Notes on the Literature

Theorem 1 above shows that our model satisfies the contraction property. This enables us to use some quite powerful results from the contraction mapping theory and leads to Theorems 2–4 in a straightforward manner. We now show that some other nonlinear input-output models that have been proposed in the literature [4, 11], also have this property.[7]

To prove that each of the models due to Sandberg [11] and Chein and Chan [4] has the contraction property, we show that their assumptions are technically stronger than ours. Since both these models have an Assumption same as our Assumption I, we need to show this only with regard to our Assumption II. We shall require the following definition and result.

Definition 3. *An $n \times n$ (real) matrix $M = (m_{ij})$ is said to be a P-matrix if all its principal minors are positive.*

Proposition 2. (Nikaido [9, Theorem 6.11]). *An $n \times n$ matrix M is a P-matrix if, and only if, there exists a (row) vector $p \in R^n_+$ such that $pM > 0$.*

We first state the assumptions of Sandberg [11]. In addition to continuous differentiability,[8] the following assumptions are required.

[6] This generalization was suggested to me by the referee.

[7] Lahiri [7] directly assumes his model to have the contraction property, but considers his assumptions to be quite different and difficult to compare with those of Sandberg [11].

[8] As in our case the assumption of continuous differentiability is not essential for most of Sandberg's results, but is used merely as a matter of convenience.

SANDBERG [11]. *For each i and j, $0 \leqslant a'_{ij}(\alpha) \leqslant a'_{ij}(0)$ for all $\alpha \geqslant 0$, and $(I - M(0))$ is a P-matrix, where $m_{ij}(0) = a'_{ij}(0)$.*

In Chein and Chan the functions $a_{ij}(x_j)$, $i, j = 1, 2, \ldots, n$, are assumed to be piecewise affine and the assumptions are stated accordingly. These assumptions can be stated in the context of differentiable functions as follows.[9]

CHIEN AND CHAN [4]. *There exists a $n \times n$ matrix M such that for each i and j, $m_{ij} \geqslant a'_{ij}(\alpha) \geqslant 0$ for all $\alpha \geqslant 0$, and $(I - M)$ is a P-matrix.*

Note that the assumptions of Chein and Chan as stated above are slightly weaker than those of Sandberg. In that it is assumed in Sandberg that $M = M(0)$. In both cases, however, the assumptions are purely technological. In our terminology it is assumed that *some* micro-units are more efficient than *all others* in the utilization of *every* input. If the micro-units which are efficient in utilizing *some* inputs are not necessarily efficient in utilizing *others*, then these assumptions will not hold. That our assumptions are weaker follows from the following proposition.

PROPOSITION 3. *If there exists a $n \times n$ matrix M such that for each i and j, $m_{ij} \geqslant a'_{ij}(\alpha) \geqslant 0$ for all $\alpha \geqslant 0$, and $(I - M)$ is a P-matrix then there exist $p_i \geqslant 0$ and $v_i > 0$, $i = 1, 2, \ldots, n$, such that $p_j \geqslant \sum_{i=1}^{n} p_i a'_{ij}(\alpha) + v_j$ for $\alpha \in [0, \infty)$, and $j = 1, 2, \ldots, n$.*

Proof. Since $(I - M)$ is a P-matrix, by Proposition 2 there exists a (row) vector $p \in R^n_+$ such that $p > pM$. Clearly, we can find a vector $v > 0$ such that $p \geqslant pM + v$, that is, $p_j \geqslant \sum_{i=1}^{n} p_i m_{ij} + v_j$ for $j = 1, 2, \ldots, n$. This implies that $p_j \geqslant \sum_{i=1}^{n} p_i a'_{ij}(\alpha) + v_j$ for all $\alpha \in [0, \infty)$ and $j = 1, 2, \ldots, n$. Hence the proposition.

IV. CONCLUSION

We have presented above a nonlinear input–output model and shown that the traditional properties of the linear input–output model, including the computational procedure, can be replicated. Theorems 1–4 are of particular interest from this point of view. The attractiveness of these results adds credence to the generalization of the linear model attempted by Sandberg [11], Lahiri [7] and Chien and Chan [4]. This we expect will lead to a wider acceptability and applications of the input–output analysis to problems that have been traditionally considered as not amenable to this type of analysis.

[9] Alternatively, we can state the assumptions of our model in the context of piecewise affine functions and show that similar results hold.

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