On Some Perturbation Inequalities for Operators

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ABSTRACT

An estimate for the norm of the solution to the equation $AX - XB = S$ obtained by R. Bhatia, C. Davis, and A. McIntosh for normal operators A and B is shown to be valid for a larger class. Some other inequalities in the same spirit are obtained, including a "$\sin\theta$ theorem" for singular vectors. Some inequalities concerning the continuity of the map $A \to |A|$ obtained recently by Kittaneh and Kosaki are extended using these ideas.

Let H_1 and H_2 be any two Hilbert spaces, and let $L(H_1, H_2)$ denote the space of bounded linear operators from H_1 to H_2. Let $L(H, H)$ be denoted simply as $L(H)$. For $A \in L(H)$ let $\sigma(A)$ denote the spectrum of A. It has long been known (see [11]) that if A and B are elements of $L(H_1)$ and $L(H_2)$, respectively, such that $\sigma(A)$ and $\sigma(B)$ are disjoint, then for every S in $L(H_2, H_1)$ the equation $AX - XB = S$ has a unique solution $X \in L(H_2, H_1)$. In their study of the subspace perturbation problem [4], R. Bhatia, C. Davis, and A. McIntosh obtained some estimates for the norm of X in terms of that of S and the number $\delta = \operatorname{dist}(\sigma(A), \sigma(B))$. Among other things they proved: If A and B are normal operators such that $\operatorname{dist}(\sigma(A), \sigma(B)) = \delta > 0$ and if S is a Hilbert-Schmidt operator, then the solution X is also a Hilbert-Schmidt

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operator and

$$\delta\|X\|_2 \leqslant \|S\|_2. \tag{1}$$

More generally, if S belongs to a subspace $\mathscr{S}$ of $L(H_2, H_1)$ which is the domain of a symmetric norm $|||\cdot|||$, then X also belongs to $\mathscr{S}$ and

$$\delta|||X||| \leqslant c_2'|||S|||, \tag{2}$$

where c_2' is a universal constant. If A, B are self-adjoint, then the constant c_2' in (2) can be replaced by a (possibly smaller) constant c_1'. (See [4] for details.)

It was also pointed out in [4] that no estimate like (1) or (2) above is possible, in general, for arbitrary operators A and B. It is of some interest, therefore, to know whether estimates like the ones above are possible under conditions weaker than normality. Our first result (Theorem 1 below) replaces normality by subnormality of A and B^*. Further, it turns out that in this broader case the same constant c_2' as in (2) above does the job. This opens up some interesting possibilities which we discuss below.

Recall that an operator A in $L(H)$ is subnormal if it has a normal extension; i.e., there exist a Hilbert space K containing H as a subspace and a normal operator M in $L(K)$ which leaves H invariant and which coincides with A when restricted to H. If there is no reducing subspace of M lying between H and K, then M is called the minimal normal extension of A. Every subnormal operator has a minimal normal extension [6].

THEOREM 1. *Let A, B be operators on H_1, H_2, respectively, such that A and B^* are subnormal and* $\text{dist}(\sigma(A), \sigma(B)) = \delta > 0$. *Let $X \in L(H_2, H_1)$. If $AX - XB$ is a Hilbert-Schmidt operator, then X is also Hilbert-Schmidt and*

$$\delta\|X\|_2 \leqslant \|AX - XB\|_2. \tag{3}$$

If $AX - XB$ lies in a subspace $\mathscr{S}$ of $L(H_2, H_1)$ which is the natural domain of a symmetric norm $|||\cdot|||$ (i.e. $\mathscr{S} = \{S \in L(H_2, H_1) : |||S||| < \infty\}$), then $X \in \mathscr{S}$ and

$$\delta|||X||| \leqslant c_2'|||AX - XB|||, \tag{4}$$

where c_2' is the constant for which the inequality (2) *holds.*

Proof. Let M and N^* be the minimal normal extensions of A and B^*, respectively. If M and N act on K_1 and K_2, respectively, then relative to the decompositions $K_1 = H_1 \oplus H_1^{\perp}$ and $K_2 = H_2 \oplus H_2^{\perp}$ we have the representations

$$M = \begin{pmatrix} A & A_{12} \\ 0 & A_{22} \end{pmatrix}, \qquad N = \begin{pmatrix} B & 0 \\ B_{21} & B_{22} \end{pmatrix}.$$

Let Y be the operator from K_2 to K_1, which in the above decomposition has the representation

$$Y = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}.$$

Then note that

$$MY - YN = \begin{pmatrix} AX - XB & 0 \\ 0 & 0 \end{pmatrix}.$$

Now, by the spectral inclusion Theorem [6, p. 107], $\sigma(M) \subseteq \sigma(A)$ and $\sigma(N^*) \subseteq \sigma(B^*)$, and hence $\sigma(N) \subseteq \sigma(B)$. Hence $\operatorname{dist}(\sigma(M), \sigma(N)) \geqslant \delta > 0$. Now apply the result of Bhatia, Davis, and McIntosh [4] to M and N. All the assertions of the theorem follow.

REMARK. It was shown in [4] that the constant c_2' satisfies the inequalities

$$\frac{\pi}{2} \leqslant c_2' \leqslant c_2, \tag{5}$$

where c_2 is a constant associated with an extremal problem for the Fourier transform:

$$c_2 = \inf\left\{ \|f\|_{L_1(\mathbf{R}^2)} : f \in L_1(\mathbf{R}^2),\ \hat{f}(x_1, x_2) = \frac{1}{x_1 + ix_2} \text{ for } x_1^2 + x_2^2 \geqslant 1 \right\}.$$

Subsequently, R. Bhatia, C. Davis, and P. Koosis [3] have shown that

$$\frac{\pi}{2} < c_2 \leqslant \frac{\pi}{2} \operatorname{Si}(\pi) < 2.91, \tag{6}$$

where

$$\operatorname{Si}(x) = \int_0^x \frac{\sin t}{t}\, dt.$$

The lower bound on c_2' in (5) was obtained from an example in which A and B were unitary operators. Since our Theorem 1 shows that the same constant works for a larger class of operators A and B (including, for example, shift operators), it might prove useful in the evaluation of the constants c_2' and c_2.

We now consider the case of arbitrary A and B. Here, no estimate like (1) or (2) is possible, as mentioned earlier. However, another kind of inequality can be obtained. Given a Hilbert space H, let $\tilde{H} = H \oplus H$, and for $A \in L(H)$ let $\tilde{A}$ be the element of $L(\tilde{H})$ having the representation

$$\tilde{A} = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}. \tag{7}$$

If H is finite-dimensional, then $\sigma(\tilde{A})$ is the union of $\sigma(|A|)$ and $-\sigma(|A|)$, i.e., the eigenvalues of $\tilde{A}$ are the singular values of A together with their negatives. If H is infinite-dimensional, then the above statement is true with a small modification:

$$\sigma(\tilde{A}) \setminus \{0\} = \sigma(|A|) \cup [-\sigma(|A|)] \setminus \{0\}.$$

(See [6, p. 39].)

Following the ideas introduced by F. Kittaneh [8], we deduce our next result, which, for simplicity, we state only for the Hilbert-Schmidt norm.

THEOREM 2. *For operators* A, B *on* H_1, H_2, *respectively, define* $\tilde{A}$ *and* $\tilde{B}$ *via* (7). *Suppose* $\operatorname{dist}(\sigma(\tilde{A}), \sigma(\tilde{B})) = \tilde{\delta} > 0$. *Let* $X \in L(H_2, H_1)$ *be such that* $AX - XB$ *and* $A^*X - XB^*$ *are both Hilbert-Schmidt operators. Then* X *is also Hilbert-Schmidt and*

$$\tilde{\delta}\|X\|_2 \leqslant \left(\frac{\|AX - XB\|_2^2 + \|A^*X - XB^*\|_2^2}{2} \right)^{1/2} \tag{8}$$

Proof. Define an operator Y from $\tilde{H}_2$ to $\tilde{H}_1$ by putting

$$Y = \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix}.$$

Then note that

$$\tilde{A}Y - Y\tilde{B} = \begin{bmatrix} 0 & A^*X - XB^* \\ AX - XB & 0 \end{bmatrix}. \tag{9}$$

Since $A^*X - XB^*$ and $AX - XB$ are both Hilbert-Schmidt operators, so is $\tilde{A}Y - Y\tilde{B}$. Now apply the result (1) to the self-adjoint operators $\tilde{A}$ and $\tilde{B}$ to get that Y is Hilbert-Schmidt and

$$\tilde{\delta}\|Y\|_2 \leqslant \|\tilde{A}Y - Y\tilde{B}\|_2. \tag{10}$$

Since $\|Y\|_2^2 = 2\|X\|_2^2$ and $\|\tilde{A}Y - Y\tilde{B}\|_2^2 = \|A^*X - XB^*\|_2^2 + \|AX - XB\|_2^2$, the inequality (8) follows from (10).

Remarks.

1. The results of Bhatia, Davis, and McIntosh for self-adjoint A, B [4] are valid for other norms. Using those results, we get under similar conditions, instead of (10), the inequality

$$\tilde{\delta}|||Y||| \leqslant c_1'|||\tilde{A}Y - Y\tilde{B}||| \tag{11}$$

for every symmetric norm, where the constant c_1' is the one for which the inequality (2) holds when A and B are self-adjoint. Norms of the "blown-up" operators are linked to those of their matrix components. Thus, for example, for the operator norm (11) becomes

$$\tilde{\delta}\|X\| \leqslant c_1' \max(\|AX - XB\|, \|A^*X - XB^*\|).$$

In the same way, for the Schatten p-norms we have

$$\|Y\|_p^p = 2\|X\|_p^p \quad \text{and} \quad \|\tilde{A}Y - Y\tilde{B}\|_p^p = \|A^*X - XB^*\|_p^p + \|AX - XB\|_p^p.$$

Thus inequalities akin to the above can be derived from (11) for these norms as well.

2. It was shown in [4] that

$$\sqrt{\tfrac{3}{2}} \leqslant c_1' \leqslant c_1, \tag{12}$$

where c_1 is a constant associated with a Fourier extremal problem on the line:

$$c_1 = \inf\left\{ \|f\|_{L_1(\mathbf{R})} : f \in L_1(\mathbf{R}),\ \hat{f}(t) = \frac{1}{t} \text{ for } |t| \geqslant 1 \right\}.$$

It follows from old work of B. Sz.-Nagy [12] that $c_1 = \pi/2$. The lower bound in (12) was obtained in [4] by constructing a matrix example with self-adjoint A and B. It is conceivable that the inequalities derived above could be useful in the evaluation of the constant c_1'.

3. Let

$$\tilde{A}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \quad\text{and}\quad \tilde{B}=\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}.$$

Then $\tilde{\delta}=0$, whereas $\delta=\operatorname{dist}(\sigma(A),\sigma(B))=2$. On the other hand, if we take

$$A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \quad\text{and}\quad B=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix},$$

then $\delta=0$ but $\tilde{\delta}>0$. So, in general, there is no relation between the distance δ between $\sigma(A)$ and $\sigma(B)$ and the distance $\tilde{\delta}$ between $\sigma(\tilde{A})$ and $\sigma(\tilde{B})$.

4. If the spaces H_1 and H_2 are finite-dimensional, then by the remarks preceding the theorem, $\tilde{\delta}$ is the distance between the singular values of A and those of B.

5. If A and B are normal, then the spectral mapping theorem implies that $\tilde{\delta}\leqslant\delta$. This inequality may be strict, as is demonstrated by the example

$$A=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \quad\text{and}\quad B=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}.$$

However, for nonnormal operators we may have $\delta\leqslant\tilde{\delta}$, as we have seen in remark 3.

6. If A and B are normal, then by the Fuglede-Putnam theorem modulo the Hilbert-Schmidt class [13], we have $\|AX-XB\|_2=\|A^*X-XB^*\|_2$. So, in this case (8) becomes $\tilde{\delta}\|X\|_2\leqslant\|AX-XB\|_2$. In view of Remark 5 above, this is weaker than the inequality (1).

Our next result is one of the kind which are known as "$\sin\theta$ theorems" to numerical analysts [5]. We will obtain such a theorem for singular vectors of arbitrary operators by combining a result of [4] and an extension of the Araki-Yamagami inequality [1] obtained by Kittaneh [7].

Let $A, B\in L(H)$, and let K_A, K_B be two closed subsets of the positive real line $\mathbf{R}_+$. By the spectral theorem we have the representations

$$|A|=\int\lambda\, dP_A(\lambda),\qquad |B|=\int\lambda\, dP_B(\lambda),$$

$$|A^*|=\int\lambda\, dP_A'(\lambda),\qquad |B^*|=\int\lambda\, dP_B'(\lambda),$$

where P_A, P_B, P'_A, P'_B are spectral measures concentrated on $\mathbf{R}_+$. Let $E = P_A(K_A)$, $E' = P'_A(K_A)$, $F = P_B(K_B)$, $F' = P'_B(K_B)$. (In the matrix theorist's language E is the projection operator onto the subspace of H spanned by the right singular vectors of A corresponding to its singular values lying in K_A. In the same way E' is the projection operator corresponding to the left singular vectors of A for its singular values lying in K_A). As explained in [4], the theorem below gives a bound for the "angle" between E and $F^{\perp}$ and that between E' and $F'^{\perp}$.

THEOREM 3. *Let $A, B \in L(H)$ be such that their difference $A - B$ is a Hilbert-Schmidt operator. Let K_A, K_B be two subsets of $\mathbf{R}_+$ with $\operatorname{dist}(K_A, K_B) = \delta > 0$. Let E, F, E', F' be the projection operators defined above. Then*

$$\delta^2\left(\|EF\|_2^2 + \|E'F'\|_2^2\right) \leqslant 2\|A - B\|_2^2.$$

Proof. By the results in Sections 2 and 6 of [4] we have

$$\delta^2\|EF\|_2^2 \leqslant \|\,|A| - |B|\,\|_2^2,$$

$$\delta^2\|E'F'\|_2^2 \leqslant \|\,|A^*| - |B^*|\,\|_2^2.$$

On the other hand, Theorem 2 in [7] tells us that

$$\|\,|A| - |B|\,\|_2^2 + \|\,|A^*| - |B^*|\,\|_2^2 \leqslant 2\|A - B\|_2^2.$$

Combining these inequalities leads to the desired result.

Our next result concerns the continuity of the map $A \to |A|$ for Hilbert-space operators. Recently F. Kittaneh and H. Kosaki [9] have proved that

$$\|\,|A| - |B|\,\|_{2p} \leqslant \left(\|A + B\|_{2p}\|A - B\|_{2p}\right)^{1/2} \qquad \text{for} \quad 1 \leqslant p \leqslant \infty,$$

where $\|\cdot\|_p$ denotes the Schatten p-norm. This can be rewritten in the form

$$\left\|(|A| - |B|)^2\right\|_p \leqslant \|A + B\|_{2p}\|A - B\|_{2p} \qquad \text{for} \quad 1 \leqslant p \leqslant \infty. \tag{13}$$

This result was proved by applying an inequality of Kittaneh [8] which extends to the p-norm an earlier result of Powers and Størmer [10]. This result, however, has been further extended to all symmetric norms by Bhatia [2]. Using this latter result and the arguments of [9], we obtain:

THEOREM 4. *Let* $A, B \in L(H)$ *be such that* $A - B$ *belongs to the norm ideal associated with a symmetric norm* $|||\cdot|||$. *Then*

$$|||(|A| - |B|)^2||| \leqslant \|A + B\| \, |||A - B|||.$$

Proof. By [2, Corollary 3] we have

$$|||(|A| - |B|)^2||| \leqslant ||| \, |A|^2 - |B|^2 \, |||. \tag{14}$$

We can write

$$|A|^2 - |B|^2 = \tfrac{1}{2}[(A - B)^*(A + B) + (A + B)^*(A - B)]. \tag{15}$$

For every symmetric norm we have the well-known inequality $|||XTY||| \leqslant \|X\| \, |||T||| \, \|Y\|$, valid for all T in the norm ideal associated with $|||\cdot|||$ and all $X, Y \in L(H)$. (See, e.g. [4].) Use this and the identity (15) to estimate the right-hand side of (14). Recall that $|||T^*||| = |||T|||$ for all T. The theorem follows.

Notice that for the p-norms Theorem 4 says

$$\|(|A| - |B|)^2\|_p \leqslant \|A + B\| \, \|A - B\|_p. \tag{16}$$

The inequality (13) is stronger than (16) for some operators and weaker for some others. Notice that (16) could be obtained by the argument in [9] if instead of using the Holder inequality one used the inequality $\|XY\|_p \leqslant \|X\| \, \|Y\|_p$.

In the same spirit Theorem 2.2 of [9] can be generalized using a result from [4] as follows:

THEOREM 5. *Let* $A, B \in L(H)$ *be such that* $|A| + |B| \geqslant cI \geqslant 0$ *and* $A - B$ *belongs to the norm ideal associated with a symmetric norm* $|||\cdot|||$. *Then*

$$c||| \, |A| - |B| \, ||| \leqslant \|A + B\| \, |||A - B|||.$$

To prove the theorem we need:

PROPOSITION 6. *If A, B are self-adjoint operators and $A + B \geqslant cI \geqslant 0$, then for every symmetric norm we have*

$$c|||A - B||| \leqslant |||A^2 - B^2|||.$$

Proof. Let $S = A - B$, $T = i(A + B)$. The hypothesis implies that $\sigma(T)$ and $\sigma(T^*)$ lie in half planes separated by a distance $2c$. Hence by Theorem 3.3 in [4] applied to the equation $TS - ST^* = 2i(A^2 - B^2)$ we get $c|||S||| \leqslant |||A^2 - B^2|||$.

Proof of Theorem 5. Apply the above proposition to the operators $|A|$ and $|B|$. Then proceed as in the proof of Theorem 4.

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