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FACTOR PRICE EQUALIZATION THEOREM IN LINEAR PROGRAMMING *

Pradip MAJTI
Indian Statistical Institute, Calcutta, India

1. Introduction

The celebrated factor price equalization (FPE) theorem in international trade theory states that under certain conditions free trade in goods between countries is a perfect substitute for free international mobility of factors, equalizing factor prices everywhere. [1] The initial as well as the usual proof of this theorem rests on the assumption that a 'smooth' production function of the neoclassical type is available for each country. This assumption is partly, if not solely, responsible for the trouble the trade theorist encounters in generalizing the theorem particularly to the case involving unequal numbers of goods and factors. The trouble would appear unnecessary as McKenzie (1955) has proved the same theorem by means of activity analysis without any restriction on the number of goods and the number of factors and without any assumption that the number of available processes is finite.

The present paper intends mainly to recast some of the theorems of McKenzie in terms of a linear programming model. The merits of this type of approach are manifold. Obviously, the number of goods and the number of factors can be anything finite. Secondly, factor prices will drop out as solutions of the model irrespective of whether they are

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[1] For discussions as well as a complete bibliography on this topic see Chipman (1966).

equalized or not. Finally, some interesting duality results in linear programming can be obtained which, in turn, will help to shorten proofs of various theorems.

Sect. 2 introduces the model. Sect. 3 imposes conditions on the parameters of the model which ensure the existence of competitive equilibrium in a country. This section also rediscovers and eventually uses some results in linear programming theory to derive conditions for FPE, given the world prices of goods. Sect. 4 carries the analysis further to obtain conditions for FPE, first, when the world output is given, and secondly, when the given factor endowments in different countries correspond to an 'efficient' allocation from the point of view of the world as a whole. Sect. 5 concludes the analysis.

2. The model

We make the following assumptions: (1) No factors can move between the countries and all commodities move perfectly freely in international trade, without encountering tariffs or transport costs. (2) Each country possesses an identical set of n linear [2] basic activities which produce g number of (final) goods; both joint production and alternative activities for producing a given good are allowed. (3) Each country possesses a given endowment of m (primary) factors. (4) Perfect competition prevails everywhere.

We introduce some notations and definitions. Let R^f denote the f-space, the set of all real f-vectors. The jth (basic) activity when it is operated at unit level is described by a column vector $a_j \in R^m$ with its ith element representing the amount of the ith factor consumed by the jth activity at unit level. The income from a_j – to be denoted by π_j – is the value of output when the jth activity is being operated at unit level. [3] We further assume that in each country there is available for each factor a free disposal activity which uses up one unit of the corresponding factor only and produces nothing. Let $A = (a_j)$ be an $m \times (n+m)$ input matrix with a_{n+i} being the free disposal activity for the ith factor and hence the ith unit vector ($i \leqq m$). Let a column vector $x = (\xi_j) \in$

[2] Activities show constant returns to scale and no external economies or diseconomies (i.e., are additive).

[3] Free trade in goods yields everywhere the same prices for goods and hence the same incomes from activities.

R^{n+m} denote the intensity vector of activities where ξ_j is the level or intensity at which the jth activity is operated. Let the income vector of activities be denoted by a row vector $p = (\pi_j) \in R^{n+m}$, with $\pi_j = 0$ for $j > n$.

Let us now consider a linear program of finding a nonnegative intensity vector x which maximizes total income subject to the condition that a given factor endowment – a column vector $r \in R^m$ – is fully employed. Thus the problem – henceforth to be referred to as (P) – is to find an x which [4]

$$\text{maximizes} \quad px\,, \tag{1}$$

$$\text{subject to} \quad Ax = r; \;\; x \geqq 0\,. \tag{2}$$

The corresponding dual problem – to be called (D) – is to find prices for factors which minimize the total imputed cost of the given factor endowment subject to the familiar condition that no activity earns positive profit, i.e., to find a row vector $y \in R^m$ which

$$\text{minimizes} \quad yr\,, \tag{3}$$

$$\text{subject to} \quad yA \geqq p\,. \tag{4}$$

Note that since the last m columns in A form an identity matrix and each of the last m elements of p is zero, the last m constraints in (4) require y to be nonnegative. [5]

A few definitions now: An x (a y) is said to be a *feasible* solution of (P) (of (D)) if it satisfies the constraints (2) (the constraints (4)). Let $\mathfrak{N}$ be the set of indices of columns of A: $\mathfrak{N} = \{1, 2, ..., n+m\}$. Let $\mathfrak{B} \subseteq \mathfrak{N}$, B be the matrix formed by the a_j, $j \in \mathfrak{B}$ and $p(B)$ the row vector of the π_j, $j \in \mathfrak{B}$. A feasible solution $x = (\xi_j)$ of (P) is said to *depend* (*positively*) on the set $\mathfrak{B}$, if $\xi_j = 0$ for $j \notin \mathfrak{B}$ ($\xi_j > 0$ for $j \in \mathfrak{B}$). B is said to be a *feasible basis* of (P), if (i) there is a feasible solution of (P) depending on $\mathfrak{B}$ and (ii) the a_j, $j \in \mathfrak{B}$, are linearly independent and span R^m. (Thus a feasible basis B of (P) is a nonsingular matrix.) A

[4] $x = (\xi_j) \geqq 0$ means $\xi_j \geqq 0$ for all j. $x \geq 0$ means $x \geqq 0$ but $x \neq 0$. $x > 0$ means $\xi_j > 0$ for all j.

[5] It is now obvious that the free disposal activities correspond to the slack vectors in a linear program.

feasible basis B of (P) is said to be *nondegenerate* if the corresponding feasible solution $x = (\xi_j)$ depends positively on $\mathfrak{B}$ and *degenerate* if $\xi_j = 0$ for some $j \in \mathfrak{B}$. A feasible basis B of (P) is said to be an *optimal basis* of it if the feasible solution of (P) depending on $\mathfrak{B}$ is its optimal solution, i.e., maximizes (1) subject to (2). The corresponding maximum value of (P) is called its *optimal value.*

Let the factor endowment in country k be given by a column vector $r_k \in R^m$. Then for a country k the above pair of linear programs will be denoted by $(P)_k$ and $(D)_k$, which are obtained from (P) and (D) by replacing x, r and y by x_k, r_k and y_k, respectively.

We now state a duality theorem of linear programming without proof; for proof see Gale (1960, theorem 4.2, p. 109).

LEMMA 1: If for a feasible basis B of (P) we have

$$p(B)B^{-1}A \geqq p\,, \tag{5}$$

then

(a) B is an optimal basis of (P); and

(b) $y = p(B)B^{-1}$ is an optimal solution of (D).

We shall henceforth call a feasible basis B a *dual optimal basis* if it satisfies (5).

We shall ignore the consumer demand for goods and assume that the prices of goods are determined in the international market outside our system. We now introduce a couple of definitions.

DEFINITION 1: The intensity vector $x = (\xi_j)$ and the factor price vector y yield a competitive equilibrium if they satisfy (2), (4) and

$$ya_j = \pi_j, \text{ whenever } \xi_j > 0\,. \tag{6}$$

The condition (6) gives the so-called 'profit conditions' in competitive equilibrium, namely that whenever an activity is used in competitive equilibrium it must realize zero profits, see McKenzie (1955, p. 241). Note further that whenever a factor is not fully employed in competitive equilibrium by 'legitimate' activities (i.e., the first n activities), its price is zero. This is guaranteed by the nature of our free disposal activities, the constraints (2) and the condition (6).

DEFINITION 2: A country is said to be specialized[6] if in competi-

[6] The usual definition of specialization in international trade theory runs in terms of goods and not activities. The proof of the FPE theorem in the m good–m factor case rests on the

tive equilibrium it uses less than m linearly independent activities out of A.

3. Conditions for FPE

Let us observe that it is possible for a country k to attain competitive equilibrium if and only if $(P)_k$ and $(D)_k$ have optimal solutions. This follows from Definition 1 and the canonical equilibrium theorem of linear programming. [7] Thus "we may paraphrase Adam Smith by saying that there is an 'invisible hand' that leads a competitive system and the economic theorist to the theorems of linear programming", Samuelson (1958, p. 313).

We shall henceforth assume that $r_k > 0$ for all k and that each activity uses at least one factor so that $a_j \geq 0$ for all $j \leqq n$. In this case it is known that $(P)_k$ and $(D)_k$ have optimal solutions, [8] say x_k and y_k, which then yield competitive equilibrium in country k.

Thus an optimal solution of $(D)_k$ gives (equilibrium) factor prices [9] in country k. Of course, if $(D)_k$ has multiple optimal solutions, they together determine a range within which factor prices must lie. (Recall that the consumer demand for goods is ignored here. Hence factor prices, determined after taking the demand side into account, may be unique although technological conditions may specify a range for it. In this connection see Samuelson (1958, p. 316).)

We now turn to our basic question: When do factor prices get equalized among countries? Obviously, a condition is that FPE should be possible among countries. In other words there should exist some y which is an optimal solution of each $(D)_k$. We shall call this condition (CI). However, in order that FPE should necessarily occur, a second

assumption that each country is 'incompletely specialized' – or 'completely diversified' in the words of Chipman (1966, pp. 20–21) – meaning thereby that each should produce all goods. When the number of goods exceeds the number of factors this translates into the condition that no country produces less goods than the number of factors. One of our conditions for FPE will turn out to be the requirement that a competitive equilibrium be possible in each country *without specialization* in the sense of Definition 2.

[7] The theorem states that a feasible solution $x = (\xi_j)$ of (P) and a feasible solution y of (D) are optimal solutions of their respective problems if and only if $\xi_j = 0$, whenever $ya_j > \pi_j$. For proof see Gale (1960, theorem 3.2, p. 82).

[8] For proof see Goldman et al. (1956, lemma 3, p. 47).

[9] That shadow prices in a programming model give competitive prices is not a new idea. See, for example, Dorfman et al. (1958, chs. 3 and 5), Gale (1960, ch. 1) and Samuelson (1958).

condition – to be called (CII) – is needed, namely that the (equilibrium) factor price vector be unique in each country. Our task in this paper would be to translate these conditions in terms of our linear programming model. We take up (CII) first.

LEMMA 2: Given p, let the problem (P) have an optimal solution $\bar{x} = (\bar{\xi}_j)$ depending positively on a set $\mathcal{T} \subseteq \mathcal{N}$ containing t indices, $t \geqq m$, such that some m of the a_j, $j \in \mathcal{T}$, are linearly independent. Then

(a) (P) has an optimal basis B formed from the a_j, $j \in \mathcal{T}$;

(b) the optimal solution of (D) is unique; and

(c) all feasible bases which can be formed from the a_j, $j \in \mathcal{T}$, are dual optimal for (P).

Proof: (a) It follows from the theorem on basic solutions of linear program; see Gale (1960, theorem 3.3, p. 84).

(b) Take any optimal solution $\bar{y}$ of (D). So we have

$$\bar{y}A \geqq p\,. \tag{7}$$

In addition, since by hypothesis $\bar{\xi}_j > 0$ for $j \in \mathcal{T}$, we have by the canonical equilibrium theorem

$$\bar{y}a_j = \pi_j\,, \qquad \text{for } j \in \mathcal{T}\,.$$

Considering only activities in B we therefore get

$$\bar{y}B = p(B)\,,$$

which shows that $\bar{y}$ is unique and is given by $p(B)B^{-1}$.

(c) Take any feasible basis B_0 of (P) which can be formed from the a_j, $j \in \mathcal{T}$. As in case (b) here also we get

$$\bar{y}B_0 = p(B_0)\,,$$

so that the unique optimal solution of (D) is also given by $p(B_0)B_0^{-1}$. This result, together with (7), shows that B_0 is a dual optimal basis of (P).

We shall call all such feasible bases B and B_0 dual optimal bases 'corresponding' to $\bar{x}$.

COROLLARY 1: Given p, if (P) has a nondegenerate optimal basis B, then the optimal solution of (D) is unique, being given by $p(B)B^{-1}$.

By Definition 2 a competitive equilibrium without specialization is possible in country k if and only if $(P)_k$ satisfies the hypothesis [10] in Lemma 2. Thus we can write:

LEMMA 2′: Given p, if a competitive equilibrium is possible in a country without specialization, its factor price vector is unique.

In continuance of the discussion of relating our model to the traditional trade theory it may be pointed out that the possibility of a competitive equilibrium being attained in a country without specialization is analogous to the existence of what Chipman (1966, pp. 20–23) calls a 'diversification cone' which contains the factor endowment of the country in question. In our model a competitive equilibrium is possible in a country k without specialization if, roughly speaking, its factor endowment is sufficiently different in comparison with the differences in factor intensities of the activities, so that its r_k can be expressed as a positive linear combination of at least m linearly independent activities.

Before turning to (CI) we introduce some additional definitions. First we define a 'convex cone' corresponding to a feasible basis B of (P) as follows: [11]

$$R(B) = \{u \in R^m \mid \exists h > 0 \ni Bh = u\}. \tag{8}$$

Note that $u \in R(B)$ implies $u > 0$.

DEFINITION 3: Given p, a pair of countries (k_i, k_j) is said to be *connected* if there exists a chain of countries $\{k_1, k_2, \ldots, k_\nu\}$ joining $k_1 = k_i$ to $k_\nu = k_j$ with the following property: each country k_s in this chain has a dual optimal basis B_{k_s} such that the $R(B_{k_s})$ of successive

[10] The hypothesis in Lemma 2 admits of the possibility of the existence of multiple optimal solutions of (P) as well as of the possibility that some or even *all* of the optimal bases of (P) may be *degenerate*. An example of the second case is the following one. Let

$$A = \begin{bmatrix} 1 & 3 & 1 & 0 & 0 \\ 3 & 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}; \quad r = \begin{bmatrix} 6 \\ 6 \\ 2 \end{bmatrix}; \quad \text{and } p = (2\ 2\ 0\ 0\ 0).$$

It is easily seen that there are two optimal extreme points: $x^1 = (2\ 0\ 4\ 0\ 0)$ and $x^2 = (0\ 2\ 0\ 2\ 0)$, each having only two positive elements. Therefore, all the optimal bases are degenerate. However, any convex combination of x^1 and x^2 is optimal and depends positively on all the first four activities.

[11] Although $R(B)$ does not contain the origin, still it can be called a 'convex cone', following the definitions given by Rockafellar (1970, p. 13).

countries have nonempty intersection.[12] (We shall sometimes call the pair (k_i, k_j) *connected with respect to the* $R(B_{k_s})$.)

We now show that the condition (CI) is realized if every pair of countries is connected.

LEMMA 3: If, given p, a pair of countries (k_i, k_j) is connected,[13] then FPE is possible in this pair.

Proof: By hypothesis there exists a chain of countries $\{k_1, k_2, ..., k_\nu\}$ joining $k_1 = k_i$ to $k_\nu = k_j$ with the following property: country k_s in this chain has a dual optimal basis B_{k_s} such that both $R(B_{k_s})$ and $R(B_{k_{s+1}})$ contain some point $u^s > 0$ (this is true for all $s = 1, ..., \nu-1$).

Now since B_{k_s} is a dual optimal basis of $(P)_{k_s}$, we know by Lemma 1 that $(D)_{k_s}$ has an optimal solution given by

$$y^0_{k_s} = p(B_{k_s})B^{-1}_{k_s}, \qquad \text{for all } s \leqq \nu . \tag{9}$$

We next show that $y^0_{k_s}$, as defined in (9), must be the same for any two successive countries in this chain and hence for the given pair.

To show this, consider countries k_s and k_{s+1}. Consider now problem (P) and (D) where $r = u^s$. We know u^s can be expressed as a positive linear combination of all activities in B_{k_s} as well as of all activities in $B_{k_{s+1}}$. Therefore, (P) has two nondegenerate optimal bases B_{k_s} and $B_{k_{s+1}}$. Corollary 1 then ensures that the optimal solution of (D) is unique, given by *both* $p(B_{k_s})B^{-1}_{k_s}$ and $p(B_{k_{s+1}})B^{-1}_{k_{s+1}}$. Hence we must have $y^0_{k_s} = y^0_{k_{s+1}}$. This equality is true for any two successive countries in this chain and hence the Lemma.

Note that B_{k_s} may be a *degenerate* optimal basis of $(P)_{k_s}$ – that is r_{k_s} may not lie in $R(B_{k_s})$ – so that the possibility of FPE is no guarantee that the factor prices would necessarily be equalized in this chain of countries.

On the basis of preceding Lemmas we state the FPE theorem as follows:

THEOREM 1: Given p, let every pair of countries be connected and

[12] The parameters of a linear program (P) – r, A and p – would determine which vectors of A would form a dual optimal basis, and how many such bases would exist. Consider now two successive countries in a chain. They *differ* with respect to their factor endowments and hence may have different sets of dual optimal bases (although p and A are identical for all). They are, however, said to be *connected* if it is possible to find an arbitrary factor supply vector u with which *both* countries, using any dual optimal basis out of their respective sets of such bases, can attain competitive equilibrium without specialization.

[13] Here lies the importance of the assumption that $r_k > 0$ for all k. No pair will be connected in our model unless they have positive supplies of the same factors.

further, a competitive equilibrium be possible in every country without specialization. Then FPE is necessary for p.

As has already been demonstrated, the first condition ensures the existence of a common factor price vector compatible with competitive equilibrium in all countries while the second condition guarantees the uniqueness of factor prices in any single country in competitive equilibrium. Together they make FPE necessary.

4. Some more results

We like to prove two more theorems which necessitate a few additional notations. Let $D = (d_j)$ be a $g \times (n+m)$ output matrix where $d_j \in R^g$ gives the amounts of different goods turned out by the jth activity when it is operated at unit level. Obviously $d_j = 0$ for $j > n$. Further given the world prices of goods, $q \in R^g$, the income vector of activities is $p = qD$. Let us consider the following hypothesis, to be called (H):

(H) For some q^* the world output vector in competitive equilibrium is $z \in R^g$ with the following property: $z = \Sigma_k D\bar{x}_k$ where each country k attains competitive equilibrium without specialization with the intensity vector $\bar{x}_k$ which, in turn, has some 'corresponding' dual optimal basis B_k such that every pair of countries is connected with respect to the $R(B_k)$.

It is obvious from Theorem 1 that for q^*, z implies FPE. However, there may be alternative qs at which z is also producible in competitive equilibrium. The interesting result is that FPE is necessary for z for all qs at which it is producible in competitive equilibrium as long as it satisfies the hypothesis (H).

THEOREM 2: FPE is necessary for the world output vector z if z satisfies the hypothesis (H). [14]

Proof: Take *any other* goods price vector q' at which z is also producible in competitive equilibrium. This means that for $p' = q'D$, there exist optimal solutions x'_k of the $(P)_k$ such that $z = \Sigma_k Dx'_k$. Now for p', $\bar{x}_k$ is a feasible solution of $(P)_k$ so that we have

[14] As McKenzie (1955, p. 245) puts it, 'Thus far we have been concerned with the question when does a price (income, in our case) vector p imply equalization of factor prices. It is also possible, however, to ask when does an output vector y (z, in our notation) imply equalization of factor prices. This will provide another measure of the prevalence of equalization'.

$$p'\bar{x}_k \leqq p'x'_k \quad (\text{all } k)\,. \tag{10}$$

However,

$$\sum_k p'x'_k = \sum_k q'Dx'_k = q'z = \sum_k q'D\bar{x}_k = \sum_k p'\bar{x}_k\,. \tag{11}$$

(10) and (11) together imply that $p'\bar{x}_k = p'x'_k$ for all k. Thus even for p', $\bar{x}_k$ is an optimal solution of $(P)_k$ so that its 'corresponding' feasible basis B_k still remains a dual optimal basis, in view of (c) of Lemma 2. Thus even for p', competitive equilibrium is possible in every country without specialization, and further, every pair of countries remains connected with respect to the $R(B_k)$. Theorem 1 then ensures the necessity of FPE in all countries.[15]

We shall note one more result which is not in McKenzie (1955). Let us consider a *hypothetical* world program $(P)_w$ which is obtained from (P) by replacing r by $r_w = \Sigma_k r_k$. (As if it were possible to pool together factor endowments of different countries without constraints!) Now the hypothetical world income (in competitive equilibrium) is the optimal value of $(P)_w$. This is obviously greater than or equal to the actual world income which is the sum of incomes of different countries, given their factor endowments. (The income of country k is taken to be the optimal value of $(P)_k$.) A question that arises is: By how much does the actual world income fall short of the hypothetical? The 'loss' would obviously be zero if all countries had the same factor price vector, as will be shown below. However, the converse is also true under an additional assumption, viz., if (CII) holds FPE is necessary for there to be no 'loss'.

THEOREM 3: For a p for which competitive equilibrium is possible in every country without specialization, the actual world income equals the hypothetical world income if and only if each country has the same factor price vector.

Proof: By hypothesis and Lemma 2′ each $(D)_k$ has a unique optimal solution, say $\bar{y}_k$. To prove the 'if' part, let us suppose $\bar{y}_k = \bar{y}$ for all k. Then we have $\Sigma_k \bar{y}_k r_k = \bar{y}\Sigma_k r_k = \bar{y}r_w$. Further, it can be shown that $\bar{y}$ is an optimal solution[16] of $(D)_w$. Therefore, actual world income $\Sigma_k \bar{y}_k r_k$ is equal to the hypothetical world income $\bar{y}r_w$.

[15] Note that the equalized factor price vector for p' may be different from that for p.

[16] Suppose, y^* is an optimal solution of $(D)_w$. However, y^* and $\bar{y}$ are feasible solutions of

To prove the necessity of FPE, let us suppose that

$$\sum_k \bar{y}_k r_k = \bar{y} r_w = \sum_k \bar{y} r_k \,, \tag{12}$$

where $\bar{y}$ is an optimal solution of $(D)_w$. However, since $\bar{y}$ is a feasible solution of each $(D)_k$ we have

$$\bar{y}_k r_k \leqq \bar{y} r_k \quad (\text{all } k)\,. \tag{13}$$

(12) and (13) together imply that $\bar{y} r_k = \bar{y}_k r_k$, i.e., that $\bar{y}$ is an optimal solution of each $(D)_k$. Since the optimal solution of any $(D)_k$ is unique, we must have $\bar{y}_k = \bar{y}$ for all k.

The economic rationale behind Theorem 3 is obvious. An optimal solution of $(D)_k$, if unique, evaluates marginal value productivities of different factors in country k in competitive equilibrium. Therefore, a redistribution of factors among countries will not give any increase in the actual world income if and only if the marginal value productivity of a given factor is equalized everywhere.[17]

5. Conclusion

We indicate a few possible extensions of our analysis. Let us suppose that for some income vector p^0 and some factor endowment r_k^0, B_k is a nondegenerate optimal basis of $(P)_k$ and further that every pair of countries is connected with respect to the $R(B_k)$. Then we know from Theorem 1 that FPE is necessary for p^0 and the r_k^0. A question may now be asked: What variations of p and the r_k would still make FPE necessary? The answer is simple: All such variations as would retain B_k as a nondegenerate optimal basis of $(P)_k$.

Consider first only variations in the r_k. All we need is that the variations in factor supplies in country k, Δr_k, should be such that

respectively $(D)_k$ (all k) and $(D)_w$. Therefore, on the one hand, we have $\bar{y}\Sigma_k r_k \geqq y^* \Sigma_k r_k$ while, on the other hand, we get $y^* r_k \geqq \bar{y} r_k$ (all k), or $y^* \Sigma_k r_k \geqq \bar{y} \Sigma_k r_k$. Thus we have in fact $\bar{y}\Sigma_k r_k = y^* \Sigma_k r_k$.

[17] It can be easily shown that our Theorem 3 holds good even when technologies differ between the countries. We further point out that the idea developed here is very similar to the one expressed in Uzawa (1959, theorems 4 and 5).

$r_k = r_k^0 + \Delta r_k$ remains [18] in $R(B_k)$. This happens if

$$B_k^{-1} \Delta r_k > -B_k^{-1} r_k^0 \, . \tag{14}$$

Consider now variations in p as well as variations in factor supplies where the latter satisfy (14). It is possible to find a critical value – let us call it λ_0 – such that FPE is necessary for all p in the range $p^0 \leqq p \leqq p^0 + \lambda_0 c$, where c is some specified, but arbitrary vector. [19] Doing this exercise for every c one would arrive at a set of all ps which ensure FPE.

References

Chipman, J.S., 1966, A survey of the theory of international trade: Part 3, The modern theory, Econometrica 34, 18–76.

Dorfman, R., P.A. Samuelson and R.M. Slow, 1958, Linear programming and economic analysis (McGraw-Hill, New York).

Gale, D., 1960, The theory of linear economic models (McGraw-Hill, New York).

Goldman, A.J. and A.W. Tucker, 1956, Theory of linear programming, in: Kuhn, H.W. and A.W. Tucker, eds., Linear inequalities and related systems (Princeton University Press, Princeton, New Jersey) 53–97.

Hadley, G., 1965, Linear programming (Addison–Wesley, World Student Series, Reading).

McKenzie, L.W., 1955, Equality of factor prices in world trade, Econometrica 23, 239–257.

Rockafellar, R.T., 1970, Convex analysis (Princeton University Press, Princeton, New Jersey).

Samuelson, P.A., 1958, Frank Knight's theorem in linear programming, Zeitschrift für Nationalökonomie 18, 310–317.

Uzawa, H., 1959, Prices of the factors of production in international trade, Econometrica 27, 448–465.

[18] This result is analogous to Theorem 2' of McKenzie (1955, p. 246).

[19] Let $c = (\gamma_j) \in R^{n+m}$, with $\gamma_j = 0$ for $j > n$. Let for $p^0 = (\pi_j^0)$ the equalized factor price vector be $\bar{y}$. Consider country k and write $c(B_k) = (\gamma_j, j \in \mathfrak{B}_k)$ and $w_k = c(B_k) B_k^{-1}$. Define

$$\lambda_k = \begin{cases} \min\limits_j \left\{ -\dfrac{\bar{y} a_j - \pi_j^0}{w_k a_j - \gamma_j}, \quad w_k a_j - \gamma_j < 0 \right\} \\ \infty \quad \text{if } w_k a_j - \gamma_j \geqq 0 \quad \text{for all } j. \end{cases}$$

Note $\lambda_k \geqq 0$. (The above exercise is based on Hadley (1965, pp. 380–381).) Then λ_0 in the text is the minimum value of λ_k over all k.