Discrete Mathematics 28 (1979) 291–301

THE NUMBER OF OPEN CHAINS OF LENGTH THREE AND THE PARITY OF THE NUMBER OF OPEN CHAINS OF LENGTH k IN SELF-COMPLEMENTARY GRAPHS

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Received 15 November 1978
Revised 25 May 1979

In this paper it is shown, using the Sieve formula, that the number of open chains of length k, $k \geq 5$, in a self-complementary (s.c.) graph is always even. As a corollary, it follows that the number of hamiltonian chains in a s.c. graph of order $p > 5$ is even, a result proved earlier by Camion. Further, the minimum number and the maximum number of open chains of length 3 in s.c. graphs of order p are determined, and the s.c. graphs of order p which attain these bounds are characterized.

1. Introduction and definitions

All graphs considered in this paper are finite, undirected and have neither loops nor multiple edges. For a graph G, the symbols $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. A graph G is said to be *self-complementary* (abbreviated, s.c.) if G is isomorphic with its complement $\bar{G}$. It is well known that if G is a s.c. graph of order p, then $p \equiv 0$ or $1 \pmod 4$. For most of the known results on s.c. graphs refer to Rao [5]. For a labeled graph G and an unlabeled graph H, let $H(G)$ be the number of nonidentical labeled subgraphs of G isomorphic to H. Denote by C_k, P_k the cycle and open chain, of length k, respectively. *Note that C_k has k vertices and P_k has $k+1$ vertices.* P_k is referred to as an *open k-chain.*

Let $p = 4N$ or $4N+1$; and $P_3^*(p)$ (respectively, $P_3^{**}(p)$) be the minimum (respectively, maximum) value of $P_3(G)$ where G is a s.c. graph of order p.

Clapham [2], (see also Camion [1]) proved that every s.c. graph has a hamiltonian chain.

Camion [1] proved, using the Sieve formula, that if $p > 5$ then the number of hamiltonian chains in a s.c. graph G of order p, that is $P_{p-1}(G)$, is even. In Section 1 using similar methods as in Camion [1], we determine the value of $P_3(G)$ and prove that $P_4(G)$ is even if and only if either $p = 4N$, or $p = 4N+1$ but N is even, and also prove that $P_k(G)$ is even whenever $k \geq 5$. In Section 2, we determine the values of $P_3^*(p)$, and $P_3^{**}(p)$ and characterize the s.c. graphs of order p which attain these bounds.

All sequences considered in this paper have positive terms and are in nonincreasing order. A sequence $\pi=(d_1,\ldots,d_p)$ of length p is said to be *potentially* s.c. *sequence* if there is at least one graph G with degree sequence π, referred to as a *realization* of π, which is a s.c. graph. A sequence π with even sum is *suitable* if *either* $p=4N$, and

(1) $d_i+d_{p+1-i}=p-1$, for $i=1,\ldots,2N$,
(2) $d_i=d_{i-1}$, for $i=2,4,\ldots,2N$.

or $p=4N+1$, for some N, and

(1) $d_i+d_{p+1-i}=p-1$, for $i=1,\ldots,2N+1$,
$d_i=d_{i-1}$, for $i=2,4,\ldots,2N$.

If π is suitable with $p=4N$, then

$$\pi=(a_1,a_1,a_2,a_2,\ldots,a_N,a_N;$$
$$p-1-a_N,p-1-a_N,\ldots,p-1-a_2,p-1-a_2,p-1-a_1,p-1-a_1),$$

and therefore π can be actually specified by the sequence $\pi^*=(a_1,\ldots,a_N)$, called the *reduced sequence* of π. Conversely, given a sequence $\pi^*=(a_1,\ldots,a_N)$ such that $p-1\geqslant a_1\geqslant\cdots\geqslant a_N\geqslant 2N$, then the corresponding *full suitable sequence* π of π^* is determined when $p=4N$. If $p=4N+1$ and π is a suitable sequence, then

$$\pi=(a_1,a_1,\ldots,a_N,a_N,2N,$$
$$p-1-a_N,p-1-a_N,\ldots,p-1-a_1,p-1-a_1)$$

and $\pi^*=(a_1,\ldots,a_N)$ is called the *reduced sequence* of π and π is called the *full suitable sequence* of π^*. A sequence π^* is *reduced-graphic* if the sequence π is graphic.

We need the following special graph $G_{max}(p)$ in Section 3.

For $p=4N$, let $G_{max}(4N)=G$, be the graph with $V(G)=A\cup B$ where

$$A=\{1,3,\ldots,4N-1\},\qquad B=\{2,4,\ldots,4N\}$$

and A is complete, B is independent in G and for $i\in A$, $j\in B$, $(i,j)\in E(G)$ if and only if one of the following holds: (i) $j=i+1$, (ii) $j=i+3$ and $i\equiv 3 \pmod 4$, (iii) $j>i+4$.

For $p=4N+1$, let $G_{max}(4N+1)=H$, be the graph with $V(H)=A\cup B\cup\{4N+1\}$, and the subgraph induced on $A\cup B$ is identical with $G_{max}(4N)$ and $(4N+1,i)\in E(H)$ if and only if $i\in A$.

Note that the degree sequence of $G_{max}(p)$, denoted by $\pi_{max}(p)$ is the *full suitable sequence* of $(p-2,p-4,\ldots,2N+\varepsilon)$, where $p=4N+\varepsilon$ and $\varepsilon=0$ or 1.

2. The value of $P_3(G)$ and the parity of $P_k(G)$, $k\geqslant 4$

In this section we determine the exact value of $P_3(G)$ and the parity of $P_k(G)$, $k\geqslant 4$ where G is a s.c. graph, by applying the Sieve formula [8, p. 19]. To this end define, for $0<r\le|E(G)|=q(G)$ and subsets U_r of $E(G)$ with $|U_r|=r$; $W_i(U_r)$ to be the number of nonidentical cycles of length i containing the edges of U_r of the labeled K_p, the complete graph of order p, and let

$$W_i(r)=\sum W_i(U_r), \tag{2.0}$$

where the summation is to be taken over all subsets U_r of $E(G)$ with $|U_r|=r$. For $r=0$, define $W_i(0)=C_i(K_p)$. Then by the Sieve formula [8, p. 19], we have

$$C_i(\bar{G})=C_i(K_p)-W_i(1)+W_i(2)+\cdots+(-1)^iW_i(i), \tag{2.1}$$

where p is the order of G and $i\ge 3$. Notice that

$$W_i(i-1)=P_{i-1}(G), \tag{2.2}$$

$$W_i(i)=C_i(G), \tag{2.3}$$

$$C_i(K_p)=\frac{i!}{2i}\binom{p}{i},\quad i\ge 3. \tag{2.4}$$

It is clear from (2.4) that

$$C_i(K_p)=0\ (\mathrm{mod}\ 2),\quad \text{if } i\ge 5 \tag{2.5}$$

Now we are ready to obtain the values of $P_3(G)$ and $C_3(G)$ as functions of p and the degree sequence of G.

Theorem 2.1. *If G is a s.c. graph with degree sequence $\pi=(d_1,\ldots,d_p)$, then $P_3(G)=p_3(\pi)$ and $C_3(G)=c_3(\pi)$ where*

$$p_3(\pi)=2\binom{p(p-1)/4}{2}-3\binom{p}{4}+(p-5)\sum_{i=1}^{p}\binom{d_i}{2}, \tag{2.6}$$

$$2c_3(\pi)=\sum_{i=1}^{p}\binom{d_i}{2}-\frac{p(p-1)(p-2)}{12}, \tag{2.6a}$$

and further

$$P_3(G)-2(p-5)C_3(G)=2\binom{p(p-1)/4}{2}+\frac{p(p-1)(p-2(p-5)}{12}-3\binom{p}{4} \tag{2.6b}$$

In particular, all s.c. graphs with the same degree sequence have the same number of open 3-chains, and have the same number of triangles.

Proof. By (2.1), (2.2) and (2.3) and the fact that $C_i(G)=C_i(\bar{G})$ we have

$$P_3(G)=W_4(3)=C_4(K_p)-W_4(1)+W_4(2). \tag{2.7}$$

Clearly, $W_4(1)=q(G)\cdot(p-2)(p-3)=6\binom{p}{4}$, since for a s.c. graph G or order p, $q(G)=p(p-1)/4$. To calculate $W_4(2)$, divide the set of all unordered pairs of distinct edges of G into two classes, the first (resp. second) class consisting of the pairs of adjacent (resp. nonadjacent) edges of G. The contribution to $W_4(2)$ by the first (resp. second) class is $p-3$ (resp. 2) for each pair and hence is

$$(p-3)\sum_{i=1}^{p}\binom{d_i}{2} \qquad \left(\text{resp. } 2\binom{q(G)}{2}-2\sum_{i=1}^{p}\binom{d_i}{2}\right).$$

Substituting these values of $W_4(1)$, $W_4(2)$ and the value of $C_4(K_p)$ from (1.4) in (2.7) we have that $P_3(G)=p_3(\pi)$. Since $C_3(K_p)=\binom{p}{3}$, $W_3(1)=p(p-1)(p-2)/4$ and $W_3(2)=\sum_{i=1}^{p}\binom{d_i}{2}$, it follows from (2.1) that $C_3(G)=c_3(\pi)$ of (2.6a) and then it is easy to verify that (2.6b) holds.

Corollary 2.2.

$$P_3(G)\equiv[p/4](\text{mod } 2). \tag{2.8}$$

Proof. Since $p=4N$ or $4N+1$, it is enough, by (2.6), to prove that

$$(p-5)\sum_{i=1}^{p}\binom{d_i}{2}\equiv 0\ (\text{mod } 2).$$

This is trivial if $p=4N+1$ and in the case $p=4N$ since, by a result of Ringel [7], Sachs [9] the number of vertices, in a s.c. graph of order $4N$, whose degree is equal to a given value is even, the corollary follows.

We now state and prove the main lemma of this section.

Lemma 2.3. *If G is a* s.c. *graph of order p and $i\geq 5$, then*

$$P_{i-1}(G)\equiv(p-i+1)P_{i-2}(G)\ (\text{mod } 2). \tag{2.9}$$

Proof. From (2.1), (2.2) and (2.3) and the fact that $C_i(\bar{G})=C_i(G)$ it follows that

$$P_{i-1}(G)\equiv[W_i(1)+W_i(2)+\cdots+W_i(i-2)]\ (\text{mod } 2). \tag{2.10}$$

We first prove that

$$W_i(i-2)\equiv(p-i+1)P_{i-2}(G)\ (\text{mod } 2) \tag{2.11}$$

and then show that

$$W_i(j)\equiv 0\ (\text{mod } 2), \quad \text{if } 1\leq j\leq i-2. \tag{2.12}$$

To prove these, consider a subset U_j of $E(G)$ with $|U_j|=j$. We may suppose that $W_i(U_j)>0$. Let s be the number of components in the subgraph of G induced by these j edges of U_j, on the vertices incident to at least one edge of U_j.

First suppose that $j=i-2$, then $s=1$ or 2. If $s=1$, then these $i-2$ edges form an open chain of length $i-2$ and $W_i(U_j)=p-i+1$; whereas if $s=2$, then these two components, which necessarily have to be open chains, together account for i vertices and therefore $W_i(U_j)=2$. Thus (2.11) holds.

Next suppose that $1 \leqslant j < i-2$. We first prove that $W_i(U_j) \equiv 0 \pmod 2$. Let t be the number of vertices of G incident with at least one edge of U_j. Clearly $t \leqslant i$. If $t = i$, then since $j < i-2$, we have that $s > 2$ and in this case it is clear that

$$W_i(U_j) = (s-1)!(2^{s-1}-1) \equiv 0 \pmod 2.$$

If $t = i-1$, then these $s \geqslant 2$ components have together exactly $i-1$ vertices and

$$W_i(U_j) = s!2^{s-1} \cdot (p-i+1) \equiv 0 \pmod 2.$$

Therefore we may assume that $t \leqslant i-2$. Let C be any cycle of length i of K_p containing the edges of U_j. Throughout the following we fix this j and U_j. Label the vertices of C by two marks as follows: label a vertex with a blue mark if it is an end vertex of some edge of U_j, and a red mark otherwise. Call a maximal subchain of C of like terms (either all blue or all red) a *run* of C. For k vertex disjoint open chains, $\mu_1 = (a_1, \ldots, a_{n_1})$, $\mu_2 = (b_1, \ldots, b_{n_2})$, $\ldots$, $\mu_k = (c_1, \ldots, c_{n_k})$ in a complete graph, define $\mu_1 + \mu_2 + \cdots + \mu_k$ to be the cycle

$$(a_1, \ldots, a_{n_1}, b_1, \ldots, b_{n_2}, \ldots, c_1, \ldots, c_{n_k}).$$

Let now B_1, R_1, B_2, $R_2, \ldots, B_k$, R_k be the runs in C in that order so that $C = B_1 + R_1 + B_2 + R_2 + \cdots + B_k + R_k$. Clearly $k = s \geqslant 2$. For two cycles C, C' of length i of K_p each containing U_j, define C to be *semi-equivalent* to C' if the subgraph on $V(C)$ with edges $\bigcup_{l=1}^{s} (E(B_l) \cup E(R_l))$ is identical with the subgraph on $V(C')$ with edges $\bigcup_{l=1}^{s} (E(B'_l) \cup E(R'_l))$, where $B'_1, R'_1, \ldots, B'_s, R'_s$ are the runs of C' so that $C' = B'_1 + R'_1 + \cdots + B'_s + R'_s$; in particular $V(C) = V(C')$. Clearly this is an equivalence relation on the nonidentical cycles of length i containing U_j of the graph K_p. We shall prove that each equivalence class contains an even number of cycles. Any cycle D_1 semi-equivalent to C consists of the edges of the runs B_l, R_l of C, $1 \leqslant l \leqslant s$, and the other edges of D_1 have one end vertex a blue vertex and the other end a red vertex of C. Thus

$$D_1 = B_{j_1} + R^*_{i_1} + B^*_{j_2} + R^*_{i_2} + \cdots + B^*_{j_s} + R^*_{i_s} \tag{2.13}$$

where $B_{j_1} = B_1$ and $j_2, \ldots, j_s$ is a permutation of $2, \ldots, s$; $i_1, \ldots, i_s$ is a permutation of $1, \ldots, s$; and if R_{i_1} is the open chain $(a_1, \ldots, a_n)$, then $R^*_{i_1} = R_{i_1}$ or the chain $(a_n, \ldots, a_1)$ and B^*_k has similar meaning. Note that $R^*_l = R_l$ if $|V(R)| = 1$. Further the chain

$$D_2 = B_1 + R^*_{i'_1} + B^*_{j'_2} + R^*_{i'_2} + \cdots + B^*_{j'_s} + R^*_{i'_s}$$

is semi-equivalent to D_1 if and only if $(i_1, \ldots, i_s) = (i'_1, \ldots, i'_s)$, $(j_2, \ldots, j_s) = (j'_2, \ldots, j'_s)$, $R^*_{i_k} = R_{i_k}$ if and only if $R^*_{i'_k} = R_{i_k}$, and $B^*_{j_k} = B_{j_k}$ if and only if $B^*_{j'_k} = B_{j_k}$. Now we can count the number of elements in the equivalence class containing C. Note that each B_k has at least two vertices, where as R_k may consist of a single vertex for some values of k. Let s_1 be the number of R_k having exactly one vertex and $s_2 = s - s_1$. If $s = 1$, then since $t \leqslant i-2$ it follows that $s_2 = 1$, and the number of cycles of length i semi-equivalent to C is equal to 2. Thus we may assume that $s \geqslant 2$. Then in (2.13) there are $2s_2 + s_1$ choices for $R^*_{i_1}$; and $2(s-1)$ choices for $B^*_{j_2}$, and so on. These imply by what has been said already that the number of cycles semi-equivalent to C is equal to $(2s_2+s_1)\cdot 2(s-1)\cdot t$ (for some integer t) which is congruent to 0 (mod 2). Thus (2.12) holds and then by (2.11) and (2.10), the proof of the lemma is complete.

Corollary 2.4. *If G is a s.c. graph of order p, then $P_4(G)$ is even if and only if $p=4N$; or $p=4N+1$ but N is even.*

Proof. By Lemma 2.3, we have

$$P_4(G) \equiv (p-4)P_3(G)\ (\text{mod } 2)$$

and the corollary follows from Corollary 2.2.

Now we are ready to prove the main theorem of this section which generalizes a result of Camion.

Theorem 2.5. *If G is a s.c. graph and $k \geq 5$, then $P_k(G)$ is even.*

Proof. Since $P_5(G) \equiv (p-5)P_4(G)$ (mod 2) it follows by Corollary 2.4 that $P_5(G)$ is even. Then by Lemma 2.3 we by induction have that $P_k(G)$ is even for every $k \geq 5$, and this completes the proof.

Corollary 2.6. (Camion [1]) *The number of hamiltonian chains in a s.c. graph of order $p>5$ is even.*

3. Determination of the values of $P_3^*(p)$ and $P_3^{**}(p)$

In this section we determine the values of $P_3^*(p)$, $P_3^{**}(p)$, where $p=4N$ or $4N+1$ and characterize s.c. graphs which attain these bounds. To this end let

$$s_0(\pi)=\sum_{i=1}^{N}\left[2\binom{b_i}{2}+2\binom{p-1-b_i}{2}\right]; \tag{3.1}$$

where π is a potentially s.c. sequence of length p, $N=[p/4]$ and $\pi^*=(b_1,\ldots,b_N)$. Further let

$$s(\pi)=\begin{cases} s_0(\pi), & \text{if } p=4N,\\ s_0(\pi)+\binom{2N}{2}, & \text{if } p=4N+1.\end{cases} \tag{3.2}$$

Then, by Theorem 2.1, for any s.c. graph G of order p, we have

$$P_3(G)=f(p)+(p-5)s(\pi), \tag{3.3}$$

where

$$f(p)=2\binom{p(p-1)/4}{2}-3\binom{p}{5} \quad \text{and} \quad \pi=\pi(G).$$

Therefore, to find the value of $P_3^*(p)$ (respectively, $P_3^{**}(p)$) it is enough to minimize (respectively, maximize) the value of $s_0(\pi)$ over all the degree sequences π of s.c. graphs of order p.

First we determine the value of $P_3^*(p)$ and characterize s.c. graphs G of order p with $P_3(G)=P_3^*(p)$:

Lemma 3.1. *If $p=4N$ or $4N+1$, then*

$$P_3^*(p)=\begin{cases} f(p)+(p-5)(8N^3-8N^2+2N), & \text{if } p=4N,\\ f(p)+(p-5)(8N^3-2N^2-N), & \text{if } p=4N+1. \end{cases} \tag{3.4}$$

Further, if G is a s.c. graph of order p, then $P_3(G)=P_3^*(p)$ if and only if the degree sequence of G is the full suitable sequence of $\pi^*=(2N,\ldots,2N)$.

Proof. By (3.1), (3.2) and (3.3) it is enough to minimize the value of $s_0(\pi)$ over all potentially s.c. sequences π of length p. The ith term of $s_0(\pi)$ is equal to

$$b_i(2b_i-2p+2)+(p-1)(p-2);$$

and since $b_i \geq 2N$, it follows that the minimum value of this term is $2N(4N-2p+2)+(p-1)(p-2)$. Thus the minimum value of $s_0(\pi)$ is $8N^3-8N^2+2N$ if $p=4N$, and is $8N^3-4N^2$ if $p=4N+1$. Also if π is the full suitable sequence of $\pi^*=(2N,\ldots,2N)$ then these minimum values are attained. Thus by (3.2) and (3.3), the equation (3.4) holds. Further, it is clear that if $P_3(G)=P_3^*(p)$, then the degree sequence of G is the full suitable sequence of $\pi^*=(2N,\ldots,2N)$.

To determine the value of $P_3^{**}(p)$ we need the following characterization of potentially s.c. sequences.

Lemma 3.2 (Clapham [3]). *A sequence π of length p is potentially s.c. if and only if π is suitable and its reduced sequence $\pi^*=(b_1,\ldots,b_N)$ satisfies the in-equalities*

$$\sum_{i=1}^{r} b_i \leq r(p-1-r), \quad \textit{for every } r,\ 1\leq r\leq N \tag{3.5}$$

where $N=[p/4]$.

From the above lemma we shall deduce the following

Lemma 3.3. *Let π be a potentially s.c. sequence of length $p=4N$ or $4N+1$, and $\pi^*=(b_1,\ldots,b_N)$. Suppose that equality holds in (3.5) for some r_0. If $1\leq r_0<N$,*

then $b_{r_0} \geq b_{r_0+1}+2$; *if* $r_0=N$ *and strict inequality holds in* (3.5) *for* $r=N-1$, *then* $b_N>2N$.

Proof. If $r_0=1$, then $b_1=p-2$ and $b_1+b_2 \leq 2p-6$. Therefore $b_2 \leq p-4$. Thus we may assume that $1<r_0<N$. From (3.5) for r_0 and r_0-1, we have, after subtraction, that $b_{r_0} \geq p-2r_0$; whereas (3.5) for r_0+1 and r_0 yield, again after subtraction, the inequality $b_{r_0}+1 \leq p-2r_0-2$, and this implies that $b_{r_0} \geq b_{r_0+1}+2$. If $r_0=N$, then from (3.5) for N and $N-1$ and the hypothesis, we have that $b_N>p-2N$ and therefore $b_N>2N$.

To characterize s.c. graphs G of order p with $P_3(G)=P_3^{**}(p)$ we need the following

Lemma 3.4. *The only realization of the sequence* $\pi_{max}(p)$ *which is the full suitable sequence of* $(p-2, p-4, \ldots, 2N+\varepsilon)$ *where* $p=4N+\varepsilon$ *and* $\varepsilon=0$ *or* 1 *is the graph* $G_{max}(p)$.

Proof. By Lemma 3.1, the sequence $\pi_{max}(p)$ is graphic. We prove that $\pi_{max}(p)$ is unigraphic by induction on N. For $N=1$, $\pi_{max}(p)=(2, 2, 1, 1)$ or $(3, 3, 2, 1, 1)$ both of which are unigraphic. Assume that the assertion holds for $N-1$, and $\pi_{max}(p)$ be as in the statement with $N \geq 2$. Let H be any realization of $\pi_{max}(p)=(d_1, \ldots, d_p)$ with $V(H)=\{u_1, \ldots, u_p\}$, and $d_H(u_i)=d_i$, $1 \leq i \leq p$. Then the vertex, u_1(respectively, u_2) is necessarily joined to all the other vertices except one vertex v say, (respectively, w). If $v=w$, then the degree of every vertex of H is at least 2. Thus $v \neq w$. This implies that u_1, u_2 are joined to all the vertices $u_i \neq u, w$. Let $S=\{u_1, u_2, v, w\}$. Then $d_H(v)=d_H(w)=1$, and both u_1, u_2 are joined to every vertex in $V-S$. Now the degree sequence $\pi(H_1)$ where H_1 is the subgraph of H induced on $V-S$ is equal to $\pi_{max}(p-4)$. Therefore, by the inductive hypothesis, $H_1 \simeq G_{max}(p-4)$. Now by the structure of H described above it follows that $H \simeq G_{max}(p)$ completing the proof.

Lemma 3.5. *If* $\pi=\pi_{max}(p)$, *then*

$$s(\pi)=\begin{cases} \dfrac{2N}{3}(16N^2-15N+2), & \text{if } p=4N, \\[2ex] \dfrac{N}{3}(32N^2-6N-5), & \text{if } p=4N+1. \end{cases} \tag{3.6}$$

Proof. If $\pi=\pi_{max}(p)$, then

$$\pi^*=(p-2, p-4, \ldots, 2N+\varepsilon),$$

where $p=4N+\varepsilon$ and $\varepsilon=0$ or 1. Now substituting these values of b_i in $s(\pi)$ of (3.2) and simplifying using values of $\sum_{i=1}^{N} i$ and $\sum_{i=1}^{N} i^2$ it can be seen that $s(\pi)$ is equal to the value mentioned in (3.6) and this completes the proof.

Theorem 3.5. *If $p=4N$ or $4N+1$, then*

$$P_3^{**}(p)=\begin{cases} f(p)+(p-5)\frac{2N}{3}(16N^2-15N+2), & \text{if } p=4N,\\ f(p)+(p-5)\frac{N}{3}(32N^2-6N-5), & \text{if } p=4N+1. \end{cases} \tag{3.7}$$

Further, if G is a s.c. graph of order p with $P_3(G)=P_3^{**}(p)$, then $G\cong G_{max}(p)$.

Proof. Let G be a s.c. graph of order p with $P_3(G)=P_3^{**}(p)$, and let $\pi=\pi(G)$, $\pi^*=(b_1,\ldots,b_N)$. We first prove that $\pi=\pi_{max}(p)$. For this it is enough to show, that equality holds in (3.5) for every r, $1\leqslant r\leqslant N$. Suppose that this is not true for some r_0 and let m be the smallest and M be the largest integers with $m\leqslant r_0\leqslant M$ such that for every r with $m\leqslant r\leqslant M$ strict inequality holds in (3.5). We prove that there is a s.c. graph H of order p with $P_3(H)>P_3(G)$.

We consider two cases.

Case 1. $1\leqslant m\leqslant M<N$.

Then by the maximality of M, we have that for $r=M+1$ equality holds in (3.5), and also if $m>1$, then by the minimality of M, we have that for $r=m-1$, equality holds in (3.5). These imply by Lemma 3.3 that $b_{M+1}\geqslant b_{M+2}+2$, if $M\leqslant N-2$, and $b_N>2N$ if $M=N-1$, further if $m>1$, then $b_{m-1}\geqslant b_m+2$.

Define a new sequence $(b_1',\ldots,b_N')$ as follows:

$$b_i'=\begin{cases} b_i & \text{if } i\neq m \text{ and } i\neq M+1,\\ b_i+1 & \text{if } i=m,\\ b_i-1 & \text{if } i=M+1.\end{cases}$$

Now by what had been said above it follows that $b_1'\geqslant\cdots\geqslant b_N'\geqslant 2N$. Further, for any r, $1\leqslant r\leqslant N$, we have that

$$\sum_{i=1}^r b_i'=\begin{cases} \sum_{i=1}^r b_i & \text{if } r<m \text{ or } r>M,\\ \left(\sum_{i=1}^r b_i\right)+1 & \text{if } m\leqslant r\leqslant M.\end{cases}$$

Since π^* satisfies (3.5), it follows by the definition of m and M that $\pi_1^*=(b_1',\ldots,b_N')$ satisfies (3.5) and hence is reduced-graphic. Let π_1 be the full suitable sequence of π_1^*. Then by Lemma 3.2 π_1 is a potentially s.c. sequence of length p. Let H be a s.c. graph with degree sequence π_1. Note that $\pi_1^*=\pi^*+\delta_m-\delta_{M+1}$ and $m\leqslant M$, where δ_m is the vector of length N in which the m-th coordinate is 1 and the rest are zero.

It is easy to check that

$$s(\pi_1)-s(\pi)=2b_m-2b_{M+1}+2\geqslant 2.$$

Therefore, by (3.3), $P_3(H) > P_3(G)$, a contradiction.

Case 2. $1 \leq m \leq M = N$.

Define a new sequence $\pi_1^* = (b_1', \ldots, b_N')$ as follows

$$b_i' = \begin{cases} b_i, & \text{if } i \neq m, \\ b_i + 1, & \text{if } i = m. \end{cases}$$

Then, by Lemma 3.3, it follows that $b_1' \geq \cdots \geq b_N' \geq 2N$ and by the definition of m and M we have that π_1^* is reduced-graphic. Let π_1 be the full suitable sequence of π_1^*. Then π_1 is a potentially s.c. sequence of length p. Let H be a s.c. graph with degree sequence π_1. Note that $\pi_1^* = \pi^* + \delta_m$. It is easy to check that

$$s(\pi_1) - s(\pi) = 2b_m - p + 2 \geq 1.$$

Therefore, by (3.3), $P_3(H) > P_3(G)$, a contradiction.

Thus we have proved that the degree sequence of G is $\pi_{\max}(p)$. Therefore by Lemma 3.5, we have that $P_3^{**}(p) = P_3(G)$ equals the value asserted in (3.7). Further, by Lemma 3.4, we have that $G \simeq G_{\max}(p)$.

From (2.6a), (2.6b) and Theorem 3.5 we have the following:

Theorem 3.6. *If $C_3^{**}(p)$ is the maximum number of triangles in* s.c. *graphs of order p, then*

$$c_3^{**}(p) = \begin{cases} \dfrac{N}{3}(N-1)(8N-1), & \text{if } p = 4N, \\ \dbinom{2N}{2} + \dfrac{N}{3}(N-1)(8N-1), & \text{if } p = 4N+1. \end{cases}$$

Further, if G is a s.c. *graph of order p with $C_3(G) = C_3^{**}(p)$, then $G \simeq G_{\max}(p)$.*

We conclude this paper with the following unsolved problems:

Find the maximum, the minimum number of hamiltonian chains, hamiltonian cycles in s.c. graphs of order p.

For an elegant characterization of s.c. graphs with hamiltonian cycles refer to Rao [6].

Acknowledgement

Many thanks are due to the referee for useful comments and suggestions.

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