EXPERIMENTAL DESIGNS
WITH RESTRICTED RANDOMISATION

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1. — INTRODUCTION

In this paper are discussed some designs for experiments involving several groups of varieties (or treatments), providing precisions of different orders for within group and between group comparisons. A design having this property called the Intra-inter group balanced (IIGB) design was introduced by Nair and Rao (1942) and further developed by Rao (1947). In this design, the association parameters (the number of blocks in which any two varieties occur) for varieties belonging to the same group are different from those for varieties belonging to different groups. No restriction was placed on the randomisation of varieties within a block. It appears that the above object of providing different precisions for different comparisons can be also achieved by imposing the restriction that, within a block, varieties whose differences are to be estimated with a higher precision should be close to one another. A number of situations where such restricted randomisation of varieties within a block is useful and the possible types of experimental arrangements are considered in the following sections.

2. — COMPLETE BLOCK DESIGN WITH RESTRICTED RANDOMISATION

Suppose there are 2v varieties with two distinguishable groups of equal size. The varieties belonging to one group are represented by $\alpha_1, \ldots, \alpha_v$ and those of the other by $\beta_1, \ldots, \beta_v$. It is desired to estimate the differences involving two α or two β-varieties with higher precision than those involving one α and β-variety. We first consider the simple situation where blocks of size 2v (i.e. with 2v plots) are available for experimentation, so that if equal precision was demanded for all comparisons, a complete randomised block design could be used. In the restricted randomised design, we first divide each whole block of 2v plots into two sub-blocks of equal size v and assign the α and β-sets at random to these sub-blocks. Within each sub-block, the varieties belonging to a set are completely randomised. It may be observed that such a design resembles the split plot design used in certain situations, when the treatments have a factorial structure.

Two following notations are made :

r = the number of replications,

$T_i(x)$ = the total of r yields for the variety x_i, $x = \alpha, \beta$,

$B_i(x)$ = the sub-block total for x-varieties in the ith whole block,

$d(x_i - x_j)$ = the estimated difference (per plot) between varieties x_i and x_j.

It is easy to see that

$$d(\alpha_i - \alpha_j) = [T_i(\alpha) - T_j(\alpha)] \div r$$

$$d(\beta_i - \beta_j) = [T_i(\beta) - T_j(\beta)] \div r$$

and $V\{d(\alpha_i - \alpha_j)\} = 2\sigma_w^2/r = V\{d(\beta_i - \beta_j)\}$

where σ_w^2 is between plot variance within sub-blocks.

$$d(\alpha_i - \beta_j) = [T_i(\alpha) - T_j(\beta)] \div r$$

$$V\{d(\alpha_i - \beta_j)\} = \frac{2}{r}\left(\sigma_b^2 + \frac{v-1}{v}\sigma_w^2\right)$$

where σ_b^2 is the per plot variance between sub-blocks.

The estimates of σ_b^2 and σ_w^2 are obtained from appropriate sums of squares in the following analysis of variance tables.

TABLE 1.

ANALYSIS OF VARIANCE FOR α- AND β-VARIETIES SEPARATELY

(as complete randomised blocks with v varieties each)

	d.f.	Sum of squares		Expectation
		β-varieties	α-varieties	
Blocks	$r-1$	$B(\beta)$	$B(\alpha)$	—
Varieties	$v-1$	$V(\beta)$	$V(\alpha)$	—
Error	$(r-1)(v-1)$	$E(\beta)$	$E(\alpha)$	$(r-1)(v-1)\sigma_w^2$

TABLE 2.

ANALYSIS OF VARIANCE FOR α- AND β-VARIETIES TOGETHER
(as a single complete randomised block with 2v varieties)

	d.f.	Sum of squares	Expectation
Blocks	$r-1$	$B(\alpha,\beta)$	—
Varieties	$2v-1$	$V(\alpha,\beta)$	—
Error	$(r-1)(2v-1)$	$E(\alpha,\beta)$	$v(r-1)\sigma_b^2+2(v-1)(r-1)\sigma_w^2$

$$\hat{\sigma}_w^2 = \frac{E(\alpha)+E(\beta)}{2(r-1)(v-1)}$$

$$\hat{\sigma}_b^2 = \frac{E(\alpha,\beta)-E(\alpha)-E(\beta)}{v(r-1)}$$

In this design, where each block contains all the varieties, there is an alternative and a simpler method of estimating σ_b^2 as shown below. The totals for α- and β-varieties in the different sub-blocks are obtained and arranged as in a two-way table.

TABLE 3.

SUB-BLOCK TOTALS

Whole block number	Totals for α-varieties	Totals for β-varieties	Total
1	B_{11}	B_{21}	$B_{.1}$
.	.	.	.
.	.	.	.
.	.	.	.
r	B_{1r}	B_{2r}	$B_{.r}$
Total	$B_{1.}$	$B_{2.}$	$B_{..}$

The analysis of variance for a two-way classification is carried out on the sub-block totals.

TABLE 4.

ANALYSIS OF VARIANCE FOR SUB-BLOCK TOTALS

	d.f.	S.S.	Expectation
Between whole blocks	$r-1$	$\frac{B_{.1}^2 + \dots + B_{.r}^2}{2v} - \frac{B_{..}^2}{2vr}$	—
Between α- and β-varieties	1	$\frac{B_{1.}^2 + B_{2.}^2}{vr} - \frac{B_{..}^2}{2vr}$	—
Error	$r-1$	(by subtraction)	$v(r-1)\sigma_b^2$
Total	$2r-1$	$\Sigma\Sigma\frac{B_{ij}^2}{v} - \frac{B_{..}^2}{2vr}$	—

The estimate of σ_b^2 is obtained from the « error » row.

In the design (1) discussed above, both the α- and β-varieties are kept together. This condition can be relaxed to obtain a wider variety of designs. For instance, only one group of varieties can be kept together, say the α-set. If the number in the α-set is v_1 and that in the β-set, v_2 and the whole block is of size $v_1 + v_2$, we first select v_1 contiguous plots at random and assign the α-set to these plots, randomising the varieties within. The varieties of the β-set are then distributed at random over the rest of the plots. In this case, there are 3 types of errors, one for comparing varieties within the α-set, another for within the β-set and a third for between the sets.

Another variant of this design is to use plots of different sizes for the varieties in the α and β-sets. A contiguous portion of the block, not necessarily equivalent in area to $v_1/(v_1 + v_2)$ of the whole block, is chosen at random and divided into v_1 plots and the rest into v_2 plots.

Recently, a design of this type was suggested to determine the best mixture of seeds of two corps in some areas in India where the method of mixed crops is practised. Instead of devoting entire fields to single crops, the seeds of two crops, say wheat and gram, are mixed in a certain proportion and broadcast over the fields. It is believed that the yield per unit area is increased for both the crops by following this procedure. In an experiment, the following mixtures of wheat and gram seeds are considered:

(1) After this paper was read at the conference my attention was drawn to an earlier publication by J. Taylor on « A valid restriction of randomization for certain field experiments » *Jour. Agr. Sc.* 39, 303, where the use of a such a design was discussed.

$$1:0,\ \tfrac{1}{3}:\tfrac{2}{3},\ \tfrac{1}{2}:\tfrac{1}{2},\ \tfrac{2}{3}:\tfrac{1}{3},\ 0:1$$

which need 5 plots in a block if a complete randomised block design is adopted. With blocks of 4 plots, a design of the type considered in this paper can be used. Three plots are used for actual mixtures and the fourth is split into two halves, one for each of the single crops. Thus the combinations 1 : 0 and 0 : 1 are accommodated on a single plot of the size used for the mixtures. The randomisation is of a restricted nature because the combinations 1 : 0 and 0 : 1 always occur together. The suggested design appears to be efficient for determining the response curve of yield of any variety as a function of the proportion of mixture.

The methods of analysing these designs have been worked out and the full details together with practical illustrations will be published elsewhere.

3. — INCOMPLETE BLOCK DESIGN WITH RESTRICTED RANDOMISATION

As in the case of a complete block design, we divide each whole block of size 2k into sub-blocks of k plots and assign k of the α-varieties in one sub-block and k of the β-varieties in another. The following combinatorial structure is imposed on the arrangement.

The number of varieties in the α-set is denoted by v_1 and that in the β-set by v_2. Each variety of the α-set occurs in r_1 blocks and that of the β-set in r_2 blocks. Any two varieties of the α-set occur together in λ_{11} sub-blocks, and any two of the β-set in λ_{22} sub-blocks and any two one from each of the α and β-sets in λ_{12} whole blocks. This is an intra and inter-group balanced (IIGB) design with restricted randomisation.

The estimates of variances σ_w^2 and σ_b^2 within and between sub-blocks are obtained from suitable expressions in the following analysis of variance tables.

TABLE 5.

ANALYSIS OF VARIANCE FOR α-VARIETIES ONLY
(as a BIBD in blocks of size k)

	d.f.	S.S.	Expectation
Blocks	$b-1$	$B(\alpha)$	—
Varieties	v_1-1	$V(\alpha)$	—
Error	$bk-b-v_1+1$ $=c_1$	$E(\alpha)$	$c_1\sigma_w^2$

TABLE 6.

ANALYSIS OF VARIANCE FOR β-VARIETIES ONLY
(as a BIBD in blocks of size k)

	d.f.	S.S.	Expectation
Blocks	$b-1$	$B(\beta)$	—
Varieties	v_2-1	$V(\beta)$	—
Error	$bk-b-v_2+1$ $=c_2$	$E(\beta)$	$c_2\sigma_w^2$

TABLE 7.

ANALYSIS OF VARIANCE FOR α- AND β-VARIETIES
TOGETHER IGNORING SUB-BLOCKS
(as a IIGB in blocks of size 2k)

	d.f.	S.S.	Expectation
Blocks	$b-1$	$B(\alpha,\beta)$	—
Varieties	v_1+v_2-1	$V(\alpha,\beta)$	—
Error	$2bk-v_1-v_2-b+1$	$E(\alpha,\beta)$	$g_1\sigma_b^2+g_2\sigma_w^2$

$$g_2 = 2b(k-1)\left\{1-\frac{k}{\lambda_{11}v_1+\lambda_{12}v_2}-\frac{k}{\lambda_{12}v_1+\lambda_{22}v_2}\right\}$$

$$g_1 = bk\left\{1-\frac{k}{\lambda_{11}v_1+\lambda_{12}v_2}\left[1+\frac{(\lambda_{11}-\lambda_{12})k}{(v_1+v_2)\lambda_{12}}\right]-\frac{k}{\lambda_{12}v_1+\lambda_{22}v_2}\left[1+\frac{(\lambda_{22}-\lambda_{12})k}{(v_1+v_2)\lambda_{12}}\right]\right\}$$

The estimates of σ_w^2 and σ_b^2 are

$$\hat{\sigma}_w^2 = \{E(\alpha)+E(\beta)\}\div(c_1+c_2)$$

$$\hat{\sigma}_b^2 = \{E(\alpha,\beta)-g_2\hat{\sigma}_w^2\}\div g_1$$

To obtain the estimates of varietal differences, we have to construct the normal equations. For this purpose we first obtain the equations separately for the α and β-varieties and those arising out of the observations on the differences between sub-block totals.

Let $Q_i(x)$ = sum of yields of variety x_i minus the sum of sub-block means in which it occurs.

$P_i(x)$ = sum of sub-block totals in which x_i occurs minus the sum of means of whole blocks in which it occurs.

The equations for the α-varieties are

$$Q_i(\alpha) = \frac{\lambda_{11}v_1}{k}\alpha_i - \frac{\lambda_{11}}{k}\Sigma\alpha_j \tag{2.1}$$

and for the β-varieties

$$Q_i(\beta) = \frac{\lambda_{22}v_2}{k}\beta_i - \frac{\lambda_{22}}{k}\Sigma\beta_j \tag{2.2}$$

and those depending on sub-block differences

(2.3)
$$P_i(\alpha) = \frac{(r_1 - \lambda_{11})}{2}\alpha_i + \frac{\lambda_{11}}{2}\Sigma\alpha_j - \frac{\lambda_{12}}{2}\Sigma\beta_j$$

$$P_i(\beta) = \frac{(r_2 - \lambda_{22})}{2}\beta_i - \frac{\lambda_{12}}{2}\Sigma\alpha_j + \frac{\lambda_{22}}{2}\Sigma\beta_j$$

Using the weight $w = 1/\sigma_w^2$ for equations (2.1, 2.2) and $w' = 2/k^2\sigma_b^2$ for (2.3), the combined equations for estimating the varietal differences are

$$R_i(\alpha) = \xi\alpha_i - \frac{\Delta_{12}}{k}\Sigma\alpha_j - \frac{\Delta_{12}}{k}\Sigma\beta_j$$

$$R_i(\beta) = \eta\beta_i - \frac{\Delta_{12}}{k}\Sigma\alpha_j - \frac{\Delta_{22}}{k}\Sigma\beta_j$$

where

$$\xi = \frac{w\lambda_{11}v_1}{k} + \frac{w'}{2}(r_1 - \lambda_{11})$$

$$\eta = \frac{w\lambda_{22}v_2}{k} + \frac{w'}{2}(r_2 - \lambda_{22})$$

$$\Delta_{11} = (w - \frac{kw'}{2})\lambda_{11}$$

$$\Delta_{12} = \frac{kw'}{2}\lambda_{12}$$

$$\Delta_{22} = (w - \frac{kw'}{2})\lambda_{22}$$

$$R_i(x) = wQ_i(x) + w'P_i(x)$$

The estimates of varietal effects for obtaining the comparisons are:

$$\hat{\alpha}_i = \frac{1}{\xi}\left[R_i(\alpha) + \frac{\Lambda_{11}}{k}\frac{2R.(\alpha)}{vw'\lambda_{12}} + \frac{\Lambda_{12}}{k}\frac{2R.(\beta)}{vw'\lambda_{12}}\right]$$

$$\hat{\beta}_i = \frac{1}{\eta}\left[R_i(\beta) + \frac{\Lambda_{12}}{k}\frac{2R.(\alpha)}{vw'\lambda_{12}} + \frac{\Lambda_{22}}{k}\frac{2R.(\beta)}{vw'\lambda_{12}}\right]$$

where $v = v_1 + v_2$, $R.(\alpha) = \Sigma R_i(\alpha)$, $R.(\beta) = \Sigma R_i(\beta)$

There are three types of comparisons with their associated variances as given below :

$$V[d(\alpha_i - \alpha_j)] = \frac{2}{\xi},\ V[d(\beta_i - B_j)] = \frac{2}{\eta}$$

$$V[d(\alpha_i - \beta_j)] = \frac{1}{\xi}\left[1 + \frac{2(\Lambda_{11} - \Lambda_{12})}{w'kv\,\lambda_{12}}\right] + \frac{1}{\eta}\left[1 + \frac{2(\Lambda_{22} - \Lambda_{12})}{w'kv\,\lambda_{12}}\right]$$

REFERENCES

[1] NAIR, K. R. and RAO, C. R. (1942) : Incomplete block designs for experiments involving several groups of varieties. Science and Culture, 7, 625.

[2] RAO, C.R. (1947) : General methods of analysis for incomplete block designs J.A.S.A., 42, 541.

RESUME

Plans d'experience a arrangement au hasard restreint.

Ce communiqué traite de quelques plans d'expérience comprenant plusieurs groupes de traitements et fournissant des précisions de différents ordres sur les comparaisons à l'intérieur d'un même groupe et entre différents groupes. Un plan ayant cette propriété, appelé plan d'expérience compensé entre groupes, fut présenté par Nair et Rao (1947) et développé ultérieurement par Rao (1948).

Dans le plan que nous examinons, les paramètres l association pour des traitements appartenant au même groupe sont différents de ceux utilisés pour des traitements appartenant à différents groupes. Dans ce travail nous avons adopté, sans restrictions, l'arrangement au hasard restreint de traitements à l'intérieur d'un bloc. Nous signalons, par exemple, des traitements dont les différences doivent être mesurees avec une très grande précision. Nous examinons également un certain nombre de types d'arrangements expérimentaux à aléatoires restreints.