THE MOMENTS OF A DOUBLY NONCENTRAL t-DISTRIBUTION

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This paper gives analytic expressions for the moments and recurrence relations for the first four raw moments of a doubly noncentral t-distribution with ν degrees of freedom and noncentrality parameters δ and λ. Table I provides numerical values of these moments for $\nu=2$ (1) 20, $\lambda=2$ (2) 8 (4) 20 and any suitable δ. Two approximations to the t''-distribution, involving the moments, are considered.

1. INTRODUCTION

LET X be a normal variate with mean δ and unit variance and Y^2 be an independent chi-square variate with ν degrees of freedom and noncentrality parameter λ. Then the ratio

$$t'' = X\sqrt{\nu}/Y \tag{1}$$

follows a doubly noncentral t-distribution with ν degrees of freedom and noncentrality parameters δ and λ.

Robbins [12] shows that when the population means are unequal, Student's t transforms to this t''-distribution. Patnaik [10] also considers this distribution in tests for standardised means from non-homogeneous normal populations. However, tables of the probability integral of t'' are not available. These would have to be computed from double infinite series of integrals, involving three parameters ν, δ and λ. Values of the moments of t'' would be useful, since most approximations to this distribution would require them. This paper gives new expressions and recursion formulae for the first four raw moments of the t''-distribution. Numerical values of these moments for $\nu=2$ (1) 20, $\lambda=2$ (2) 8 (4) 20 and any suitable δ, could be evaluated using Table I. Two approximations to the t''-distribution, making use of these moments, are considered.

When $\lambda=0$, we have the singly noncentral t-distribution of Johnson and Welch [6]. Tables for this distribution have been published, for instance, by Resnikoff and Lieberman [11] and Constance van Eeden [1].

When $\delta=0$, we have a different noncentral t-distribution defined by Marakathavalli [8]. This is also a particular case ($\alpha=0$) of the distribution considered by Robbins [12].

When $\delta=0$, $\lambda=0$, we have Student's t.

2. THE t''-DISTRIBUTION AND MOMENTS

The frequency function of X is

$$g(X) = \exp\left[-(X-\delta)^2/2\right]/\sqrt{2\pi},$$

that of Y is

$$h(Y) = \exp\left[-(Y^2+\lambda)/2\right]\sum_{j=0}^{\infty}\frac{(\lambda/2)^j Y^{(\nu+2j-1)}}{j!2^{(\nu/2+j-1)}\Gamma(\nu/2+j)},$$

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and hence that of $U = X/Y$ is $\int_0^\infty h(Y)\, g(UY)\, Y\, dY$, which after integration, reduces to

$$\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} e^{-(\lambda+\delta^2)/2} \frac{(\lambda/2)^j(\delta U\sqrt{2})^i\Gamma[(\nu+i+1+2j)/2](1+U^2)^{-(\nu+i+1+2j)/2}}{j!i!\sqrt{\pi}\Gamma(\nu/2+j)}.$$

Hence the probability density of t'' is

$$p(t'') = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} e^{-(\lambda+\delta^2)/2} \frac{(\lambda/2)^j(\delta t''\sqrt{2}/\sqrt{\nu})^i\Gamma[(\nu+i+1+2j)/2]}{j!i!\sqrt{\nu\pi}\Gamma(\nu/2+j)(1+t''^2/\nu)^{(\nu+i+1+2j)/2}}. \qquad (2)$$

(Robbins [12] uses n, 2λ, $\sqrt{2}\alpha$ for our ν, λ, δ respectively.)

Since X and Y are statistically independent,

$$E[(t'')^k] = \nu^{k/2}E(X^k)\cdot E(Y^{-k}).$$

Now

$$E(X^k) = \int_{-\infty}^{\infty} e^{-(X-\delta)^2/2}\frac{X^k}{\sqrt{2\pi}}dX = \int_{-\infty}^{\infty} e^{-t^2/2}\frac{(t+\delta)^k dt}{\sqrt{2\pi}}$$

$$= \frac{1}{\sqrt{\pi}}\sum_{r\in E}\binom{k}{r}2^{r/2}\delta^{k-r}\Gamma\left(\frac{r+1}{2}\right)$$

where

$$E = \{r : r \text{ even} \le k\},$$

and

$$E(Y^{-k}) = \int_0^\infty e^{-(Y^2+\lambda)/2}\sum_{j=0}^{\infty}\frac{(\lambda/2)^jY^{(\nu-k+2j-1)}dY}{j!2^{\nu/2+j-1}\Gamma(\nu/2+j)}$$

$$= \sum_{j=0}^{\infty} e^{-\lambda/2}\frac{(\lambda/2)^j\Gamma\left(\frac{\nu-k}{2}+j\right)}{j!2^{k/2}\Gamma(\nu/2+j)}$$

$$= \Gamma\left(\frac{\nu-k}{2}\right)H\left(\frac{\nu-k}{2}, \frac{\nu}{2}; \frac{\lambda}{2}\right)\cdot e^{-\lambda/2}/2^{k/2}\Gamma(\nu/2),$$

where the hypergeometric function

$$H(a, b; x) = \frac{\Gamma(b)}{\Gamma(a)}\sum_{j=0}^{\infty}\frac{\Gamma(a+j)x^j}{\Gamma(b+j)j!}.$$

Using Kummer's formula ([13], page 6)

$$e^{-x}H(a, b; x) = H(b-a, b; -x),$$

$$E(Y^{-k}) = \Gamma\left(\frac{\nu-k}{2}\right)H(k/2, \nu/2; -\lambda/2)/2^{k/2}\Gamma(\nu/2), \qquad (\nu > k).$$

Hence the kth raw moment of t''

$$m_k(t''/v, \delta, \lambda) = (\nu/2)^{k/2}\Gamma\left(\frac{\nu-k}{2}\right)\frac{H(k/2, \nu/2; -\lambda/2)}{\Gamma(\nu/2)\sqrt{\pi}} \sum_{r\in E}\binom{k}{r}2^{r/2}\delta^{k-r}\Gamma\left(\frac{r+1}{2}\right), \qquad (3)$$

where

$$E = \{r: r \text{ even} \leq k\} \qquad (\nu > k).$$

Making appropriate substitutions for k in (3), we have for the first four raw moments of t'':

$$m_1 = \delta\sqrt{(\nu/2)}\Gamma\left(\frac{\nu-1}{2}\right)H(1/2, \nu/2; -\lambda/2)/\Gamma(\nu/2), \qquad (\nu > 1), \qquad (4)$$

$$m_2 = \{\delta^2\Gamma(1/2) + 2\Gamma(3/2)\}(\nu/2)\Gamma\left(\frac{\nu-2}{2}\right)H(1, \nu/2; -\lambda/2)/\sqrt{\pi}\,\Gamma(\nu/2)$$
$$= (\delta^2+1)\nu H(1, \nu/2; -\lambda/2)/(\nu-2), \qquad (\nu > 2), \qquad (5)$$

$$m_3 = \{\delta^3\Gamma(1/2) + 6\delta\Gamma(3/2)\}\Gamma\left(\frac{\nu-3}{2}\right)H(3/2, \nu/2; -\lambda/2)(\nu/2)^{3/2}/\sqrt{\pi}\,\Gamma(\nu/2)$$
$$= \delta(\delta^2+3)(\nu/2)^{3/2}H(3/2, \nu/2; -\lambda/2)\Gamma\left(\frac{\nu-3}{2}\right)\Big/\Gamma(\nu/2), \quad (\nu > 3), \qquad (6)$$

and

$$m_4 = \{\delta^4\Gamma(1/2 + 12\delta^2\Gamma(3/2) + 4\Gamma(5/2)\}$$
$$\cdot(\nu/2)^2\Gamma\left(\frac{\nu-4}{2}\right)H(2, \nu/2; -\lambda/2)/\sqrt{\pi}\,\Gamma(\nu/2) \qquad (7)$$
$$= (\delta^4 + 6\delta^2 + 3)\nu^2 H(2, \nu/2; -\lambda/2)/(\nu-2)(\nu-4), \qquad (\nu > 4).$$

3. RECURRENCE RELATIONS FOR MOMENTS

Recursion relations in the moments as functions of ν, help in their numerical evaluation. Following are forward recursion formulae for the first four raw moments of t'':

$$m_1(\nu) = \sqrt{\nu}\left[\left(1 - \frac{\nu-4}{\lambda}\right)\frac{m_1(\nu-2)}{\sqrt{(\nu-2)}} + \frac{(\nu-5)m_1(\nu-4)}{\lambda\sqrt{(\nu-4)}}\right], \qquad (\nu > 5) \qquad (8)$$

$$m_2(\nu) = \frac{\nu}{\lambda}\left[(\delta^2+1) - (\nu-4)m_2(\nu-2)/(\nu-2)\right], \qquad (\nu > 4) \qquad (9)$$

$$m_3(\nu) = (\delta^2+3)\nu^{3/2}\left[m_1(\nu-2)/\sqrt{(\nu-2)} - m_1(\nu)/\sqrt{\nu}\right], \qquad (\nu > 3) \qquad (10)$$

$$m_4(\nu) = (\delta^4 + 6\delta^2 + 3)\nu^2\left[m_2(\nu-2)/(\nu-2) - m_2(\nu)/\nu\right]/2(\delta^2+1), \ (\nu > 4). \qquad (11)$$

where $m_k(\nu)$ stands for the k^{th} moment of t'' with ν degrees of freedom, and parameters δ and λ.

These relations can be derived using the following identities in the hypergeometric functions: (see page 19, [13])

$$b(b-1)H(a, b-1; x) - b(b-1+x)H(a, b; x) + (b-a)xH(a, b+1; x) = 0,$$
$$\text{with } a = 1/2,\ b = \nu/2 - 1,\ x = -\lambda/2$$

for formula (8),

$$bH(a, b; x) - bH(a-1, b; x) - xH(a, b+1; x) = 0,$$
$$\text{with } a = 1,\ b = \nu/2 - 1,\ x = -\lambda/2, \quad \text{for (9)}$$

and

$$(1 + a - b)H(a, b; x) - aH(a+1, b; x) + (b-1)H(a, b-1; x) = 0$$

with

$a = 1/2,\ b = \nu/2,\ x = -\lambda/2$ for (10) and $a = 1,\ b = \nu/2,\ x = -\lambda/2$ for (11).

4. LIMITING CASE

Using the formula (Bateman, [2], page 47)

$$\Gamma(z) = e^{-z}e^{(z-1/2)\log z}\sqrt{2\pi}\,[1 + 1/12z + 1/288z^2 - 0(1/z^3)]$$

and expressing the Gamma and Hypergeometric functions in powers of $(1/\nu)$, we get

$$\begin{aligned}
m_1 &= \delta\left[1 + \frac{1}{\nu}(3/4 - \lambda/2) + 0(1/\nu^2)\right],\\
m_2 &= (\delta^2 + 1)[1 + (2 - \lambda)/\nu + 0(1/\nu^2)],\\
m_3 &= \delta(\delta^2 + 3)\left[1 + \frac{3}{\nu}(5/4 - \lambda/2) + 0(1/\nu^2)\right],\\
m_4 &= (\delta^4 + 6\delta^2 + 3)[1 + (6 - 2\lambda)/\nu + 0(1/\nu^2)].
\end{aligned} \tag{12}$$

When $\nu \to \infty$, the first four raw moments of t'' tend to

δ, $(\delta^2 + 1)$, $\delta(\delta^2 + 3)$, $(\delta^4 + 6\delta^2 + 3)$ respectively, $(\lambda, \delta$, finite).

The first four central moments of t'' therefore tend to those of a normal variate with mean δ and unit variance.

When $\lambda \to \infty$ (ν, δ, finite), the moments tend to zero, since

$$\operatorname*{Lt}_{x\to\infty} e^{-x}H(a, b; x) = 0,\ (b > a),\ \text{(page 60), [13].}$$

5. TABLE OF MOMENTS-COMPUTATIONAL PROCEDURES

The moments of the t''-distribution could be evaluated using tables of the hypergeometric and gamma functions. However, existing tables for $H(a, b; x)$ are for restricted small values of a, b and positive x: for example, the ranges of a, b, and x are $(-1, 1)$, $(0, 1)$, $(0, 10)$ or $(-11, 2)$, $(-4, 1)$, 1 in [13], and

TABLE I

	λ=2				λ=4			
ν	c_1	c_2	c_3	c_4	c_1	c_2	c_3	c_4
2	1.143295				.826537			
3	1.032093	1.614238			.826621	.939982		
4	1.004476	1.264241	2.440549		.844320	.864064	1.292664	
5	.994384	1.154801	1.090210	3.838990	.860738	.850007	1.032135	1.874006
6	.990122	1.103638	1.440628	2.378170	.874606	.851501	.967421	.330480
7	.988262	1.074917	1.318163	1.890316	.885951	.857492	.927306	1.163813
8	.987620	1.050904	1.246197	1.658213	.895530	.864604	.914051	1.082082
9	.987337	1.044008	1.190228	1.617077	.903655	.871887	.908274	1.037710
10	.987444	1.036383	1.166374	1.424111	.910608	.878753	.906318	1.010386
11	.987703	1.030114	1.142228	1.358476	.916627	.885129	.906412	.992810
12	.988040	1.025360	1.123808	1.309796	.921885	.890901	.907637	.981078
13	.988412	1.021664	1.100342	1.272333	.926517	.896360	.909489	.973058
14	.988797	1.018730	1.097712	1.242663	.930630	.901275	.911677	.967499
15	.989180	1.016361	1.088179	1.218616	.934304	.905778	.914028	.963626
16	.989553	1.014418	1.080239	1.108757	.937608	.909912	.916430	.960935
17	.989913	1.012805	1.073533	1.182095	.940593	.913714	.918839	.959096
18	.990257	1.011451	1.067802	1.167927	.943305	.917220	.921199	.957877
19	.990585	1.010302	1.062854	1.155740	.945780	.920461	.923401	.957115
20	.990896	1.009319	1.058543	1.145153	.948046	.923464	.925703	.956694
	λ=6				λ=8			
ν	c_1	c_2	c_3	c_4	c_1	c_2	c_3	c_4
2	.651259				.546816			
3	.696990	.630634			.609507	.462010		
4	.735034	.633475	.743938		.656457	.490842	.467431	
5	.764847	.658157	.676820	.982251	.692597	.530831	.471371	.556209
6	.788155	.683202	.672446	.800851	.721384	.565934	.495854	.510087
7	.807261	.705956	.682367	.754120	.744942	.596313	.521848	.513974
8	.823109	.725989	.695705	.740109	.764630	.622710	.546816	.527473
9	.836470	.743617	.709825	.738184	.781306	.645819	.569878	.543013
10	.847020	.759180	.723437	.741529	.795786	.666208	.590979	.560898
11	.857820	.772980	.736270	.747321	.808354	.684332	.610245	.577515
12	.866408	.785311	.748220	.754230	.819417	.700549	.627864	.593407
13	.874151	.796367	.759290	.761578	.829237	.716149	.643981	.608443
14	.880959	.806339	.769527	.769008	.838015	.728365	.658792	.622594
15	.887056	.815377	.778994	.776310	.845914	.740387	.672432	.635883
16	.892550	.823608	.787759	.783393	.853062	.751373	.685030	.648354
17	.897526	.831129	.795880	.790200	.859562	.761452	.696698	.660059
18	.902057	.838034	.803436	.796705	.865501	.770733	.707533	.671053
19	.906199	.844393	.810463	.802902	.870950	.779309	.717621	.681389
20	.910000	.850269	.817016	.808704	.875967	.787258	.727035	.691216

(0; 100), 1, (0, .1) in [9]. Although recursive relations aid in evaluating $H(a, b; x)$ for untabulated values of the arguments, they could hardly be used for the moments of t'', which might involve large values of b and negative x. We have therefore tabulated the first four moments of t''. To simplify tabulation with results not involving δ, the following transformations were made:

$$\begin{aligned} c_1 &= m_1/\delta, \\ c_2 &= m_2/(\delta^2 + 1), \\ c_3 &= m_3/\delta(\delta^2 + 3), \\ c_4 &= m_4/(\delta^4 + 6\delta^2 + 3). \end{aligned} \tag{13}$$

TABLE I (*continued*)

	λ = 12				λ = 16			
ν	c_1	c_2	c_3	c_4	c_1	c_2	c_3	c_4
2	.430706				.366901			
3	.499733	.280570			.432085	.202797		
4	.551241	.332507	.231480		.483469	.249916	.141629	
5	.591759	.377698	.260971	.224796	.524082	.291375	.174494	.110550
6	.624775	.416873	.302127	.245662	.557829	.328140	.205785	.140200
7	.652360	.451138	.334731	.271737	.586508	.361013	.235163	.164189
8	.675845	.481389	.364663	.297765	.611293	.390619	.262657	.187604
9	.696138	.508318	.392110	.322727	.632996	.417449	.288374	.210190
10	.713885	.532464	.417335	.346361	.652202	.441896	.312443	.231888
11	.729559	.554254	.440550	.368630	.669349	.464280	.334993	.252647
12	.743521	.574028	.462004	.389577	.684772	.484861	.356150	.272487
13	.756048	.592062	.481862	.409276	.698734	.503850	.376028	.291433
14	.767359	.608583	.500299	.427810	.711443	.521454	.394733	.309520
15	.777628	.623780	.517402	.445265	.723070	.537804	.412363	.326788
16	.786999	.637809	.533476	.461721	.733752	.553038	.429003	.343278
17	.795587	.650803	.548454	.477255	.743606	.567272	.444732	.359033
18	.803489	.662875	.562493	.491936	.752727	.580602	.459622	.374092
19	.810786	.674122	.575678	.505830	.761198	.593115	.473736	.388495
20	.817547	.684628	.588086	.518996	.769087	.604888	.487134	.402279

	λ = 20			
ν	c_1	c_2	c_3	c_4
2	.325317			
3	.387295	.159112		
4	.435543	.199990	.098102	
5	.475000	.236740	.124978	.071118
6	.508259	.270001	.151017	.089955
7	.536884	.300284	.176007	.109033
8	.561902	.327999	.199879	.128008
9	.584033	.353479	.222627	.146700
10	.603798	.377000	.244378	.164997
11	.621593	.398789	.264874	.182830
12	.637722	.419039	.284467	.200161
13	.652428	.437916	.303111	.216966
14	.665900	.455560	.320861	.233230
15	.678300	.472091	.337771	.248978
16	.689759	.487615	.353892	.264192
17	.700384	.502226	.369272	.278891
18	.710268	.516002	.383958	.293089
19	.719491	.529016	.397993	.306803
20	.728118	.541331	.411416	.320048

Then the recurrence relations become

$$\begin{aligned} c_1(\nu) &= \nu^{1/2}[(1-(\nu-4)/\lambda)c_1(\nu-2)/\sqrt{(\nu-2)} \\ &\qquad + (\nu-5)/\lambda)c_1(\nu-4)/\sqrt{(\nu-4)}], \qquad (\nu > 5) \\ c_2(\nu) &= (\nu/\lambda)[1-((\nu-4)/(\nu-2))c_2(\nu-2)], \qquad (\nu > 4) \qquad (14) \\ c_3(\nu) &= \nu^{3/2}[c_1(\nu-2)/\sqrt{(\nu-2)} - c_1(\nu)/\sqrt{\nu}], \qquad (\nu > 3), \\ c_4(\nu) &= \nu^2[c_2(\nu-2)/(\nu-2) - c_2(\nu)/\nu]/2, \qquad (\nu > 4), \end{aligned}$$

and depend only on ν and λ. Table I gives the results for c_i ($i=1, 2, 3, 4$) for $\nu=2$ (1) 20, $\lambda=2$ (2) 8 (4) 20. The first four raw moments of t'' could easily be obtained from corresponding values of the c's, for any value of δ and the tabulated values of ν and λ, from (13).

When $\delta=0$, m_1 and m_3 become zero.

Values for the case $\lambda=0$ have been given by Hogben, Pinkham and Wilk [5], and have not been included here.

When $\nu\rightarrow\infty$, $c_i\rightarrow 1$; when $\lambda\rightarrow\infty$, $c_i\rightarrow 0$; $i=1, 2, 3, 4$.

c_2, c_3, c_4 are defined only for $\nu\geq 3, 4, 5$ respectively: undefined values have been left blank in Table I.

All computations were performed on the IBM 1620 using a modulus of 5 and a mantissa of 28. Two runs were necessary to obtain the results. The first run was used to calculate c_1 for $\nu=2, 3, 4, 5$ and c_2 for $\nu=3, 4$ and $\lambda=2$ (2) 8 (4) 20 using formulae (13). The second run used these values in the recurrence formulae (14) to get c_i, $i=1$ to 4, for higher values of ν. The final answers were truncated to 6 decimal places. 11 terms ($\lambda=2$) to 40 terms ($\lambda=20$) of the hypergeometric function were used for the initial values of the moments ($\nu=2, 3, 4, 5$) to give numerical accuracy to about 28 decimal places. This was necessary, since errors in the last decimal place tended to shift to the left as ν increased in formulae (14), giving accuracy to only about 7 decimal places for ν near 20. Four-point Lagrangian interpolation in λ could be used in Table I to get moments for intermediate values of λ. The moments for larger degrees of freedom, say $\nu=22$ (2) 50, could be obtained using the recursive formulae.

6. APPROXIMATIONS TO THE t''-DISTRIBUTION BY (1) A MODIFIED r-DISTRIBUTION, (2) A SINGLY NONCENTRAL t.

The first approximation to the t''-distribution is by that of a transformed correlation coefficient, following Harley's method [4] for the singly noncentral t. Suppose then, we approximate the distribution of t'' by that of

$$w = r\sqrt{(n-2)}\,h(\rho)/\sqrt{(1-r^2)} \tag{15}$$

where r is the correlation coefficient in a sample of size n drawn randomly from a bivariate normal population with correlation ρ, and $h(\rho)$ is a function of ρ to be determined suitably, (see [7]).

The first three moments of w when $\rho\neq 0$ are (see [4])

$$\begin{aligned} m_1(w) &= (n-2)^{1/2}\rho h(\rho)/\left[(n-3)\sqrt{(1-\rho^2)}\right], \\ m_2(w) &= (n-2)\left[1+(n-1)\rho^2/(1-\rho^2)\right](h(\rho))^2/(n-4), \\ m_3(w) &= (n-2)^{3/2}\rho\left[3+n\rho^2/(1-\rho^2)\right](h(\rho))^3/\left[(n-3)(n-5)\sqrt{(1-\rho^2)}\right]. \end{aligned} \tag{16}$$

To bring the distribution of $r/\sqrt{(1-r^2)}$ and t'' into correspondence, we have to specify a form for $h(\rho)$ and to relate n and ρ to ν, δ and λ. The link could be made in a number of ways: the method chosen below, following Harley's in [4], was partly because it led to simple relations which did not involve any approximations to the gamma functions, while providing satisfactory results in a few numerical comparisons.

Here, the ratio of the moments $m_3(w)/m_1(w)$ are equated to the ratio $m_3(t'')/m_1(t'')$, and the values of $m_2(w)$, to $m_2(t'')$.

From (16),

$$\frac{m_3(w)}{m_1(w)} = (h(\rho))^2 \frac{(n-2)}{(n-5)}\left(3 + \frac{n\rho^2}{(1-\rho^2)}\right)$$

while from (6) and (4),

$$\frac{m_3(t'')}{m_1(t'')} = \frac{\nu(\delta^2+3)}{(\nu-3)} \frac{H(3/2, \nu/2; -\lambda/2)}{H(1/2, \nu/2; -\lambda/2)}.$$

Equating these moment ratios, as also the second moments in (16) and (5), we get

$$h^2(\rho)\frac{(n-2)}{(n-5)}\left[3 + n\rho^2/(1-\rho^2)\right] = \frac{\nu}{(\nu-3)}(\delta^2+3)\frac{H(3/2, \nu/2; -\lambda/2)}{H(1/2, \nu/2; -\lambda/2)}, \quad (17)$$

and

$$h^2(\rho)\frac{(n-2)}{(n-4)}\left[1 + \frac{(n-1)}{(1-\rho^2)}\rho^2\right] = \frac{\nu}{(\nu-2)}(\delta^2+1)\, H(1, \nu/2; -\lambda/2).$$

If we choose $\nu = n-2$ or

$$n = \nu + 2 \quad (18)$$

a result which gives exact correspondence when $\rho=0$ and $\delta=\lambda=0$, we have on solving the equations in (17),

$$\begin{aligned} \rho &= \left[(3K-L)/\{(n-2)L-(n-3)K\}\right]^{1/2}, \\ h(\rho) &= \left[K/(1+(n-1)\rho^2/(1-\rho^2))\right]^{1/2} \end{aligned} \quad (19)$$

where

$$\begin{aligned} L &= (\delta^2+3)H(3/2, n/2-1; -\lambda/2)/H(1/2, n/2-1; -\lambda/2) \\ &= (n-5)m_3(t''/(n-2), \delta, \lambda)/[(n-2)m_1(t''/(n-2), \delta, \lambda) \end{aligned} \quad (20)$$

and

$$K = (\delta^2+1)H(1, n/2-1; -\lambda/2) = (n-4)m_2(t''/(n-2), \delta, \lambda)/(n-2).$$

n, ρ, $h(\rho)$ can be obtained from the tabulated moments of t'' in Table I.

In the second approximation to t'', considered by Patnaik [10], the first two moments of t'' are equated to the corresponding moments of ct', where t' is a singly noncentral t with f degrees of freedom and noncentrality parameter δ, c being a scale factor. He obtains

$$\begin{aligned} c(\delta^2/2)^{1/2}\sqrt{f}\,\Gamma(f-1)/2]/\Gamma(f/2) &= m_1(t'') = (\delta^2/2)^{1/2}g, \\ c^2(1+\delta^2)f/(f-2) &= m_2(t'') = (1+\delta^2)h, \end{aligned} \quad (21)$$

and employing Stirling's approximation to the gamma function, gets

$$\begin{aligned} 1/f &= 2[-1+\sqrt{(15-7g^2/h)}]/7, \\ c &= \sqrt{[h(1-2/f)]}. \end{aligned} \quad (22)$$

Here again, c and f can be obtained using the values of the moments in Table I.

7. EVALUATION OF THE PROBABILITY INTEGRAL OF t''

Using (2), the probability integral $\int_{-\infty}^{a} p(t''/\nu, \delta, \lambda)\, dt''$ can be put in the form

$$\frac{1}{2}\sum_{j=0}^{\infty}\sum_{r=0}^{\infty} e^{-\lambda/2}\frac{(\lambda/2)^j}{j!}e^{-\delta^2/2}\frac{(\delta^2/2)^{r/2}}{\Gamma(r/2+1)}\left\{I_{(a^2/(\nu+a^2))}((r+1)/2, (\nu/2+j)) + (1-)^r\right\} \quad (23)$$

and although the incomplete beta functions $I_x(p, q)$ are each less than one, and the Poisson terms get smaller and smaller after a certain stage, calculation of the double infinite series of integrals involving three parameters ν, δ, and λ, is tedious.

When t'' is approximated by $\sqrt{(n-2)}\, rh(\rho)/\sqrt{(1-r^2)}$, the above probability integral is approximately given by

$$\begin{aligned}\Pr(\sqrt{(n-2)}rh(\rho)/\sqrt{(1-r^2)} \le a) &= \Pr(r \le a/\sqrt{(a^2+(n-2)h^2(\rho))})\\ &= \int_{-1}^{r_0} p_n(r/\rho)dr\end{aligned}$$

where

$$r_0 = a/\sqrt{(a^2+(n-2)h^2(\rho))}$$

and n, ρ, $h(\rho)$ are defined as in (18), (19), in terms of ν, δ, λ, or the first three moments of t''. Using David's tables [3] giving the probability integral of r in samples of size n from a bivariate normal population with correlation ρ, and interpolating in the arguments n, ρ, r_0, the approximate value of the probability integral of t'' can be obtained. Four-point Lagrangian interpolation may be used.

Using the second approximation to t'' by ct', the approximate value of $Pr(t'' \le a)$ is $Pr(t' \le a/c)$, where c and the degrees of freedom f of t' are defined in (22). This probability can be obtained from tables of t', [11] or by approximate methods as in [1] or [4].

A few exact values of $\int_{-\infty}^{a} p(t''/\nu, \delta, \lambda)dt''$ have been evaluated and shown in Table II, along with their approximate values.

TABLE II. SHOWING EXACT AND APPROXIMATE VALUES OF

$$\int_{-\infty}^{a} p(t''/\nu, \delta, \lambda)dt''$$

	ν	λ	δ	a	n	ρ	$h^2(\rho)$	c	f	δ	$\int_{-\infty}^{a} p(t'')dt''$.
1)	10	12	2	$\sqrt{10}$							.9702
App. I					12	.60090	.28340				.9749
App. II.								.07217	13.2047	2	.9756
2)	10	10	$\sqrt{3}$	$\sqrt{2.5}$							.6038
App. I.					12	.52083	.35810				.8709
App. II.								.70480	12.3036	$\sqrt{3}$	.6040
3)	12	14	2	2							.7880
App. I.					14	.55723	.31065				.7058
App. II.								.57810	15.9282	2	.7873

ACKNOWLEDGMENT

I would like to thank Dr. Richard N. Schmidt for programming and carrying out the computations in Table I. I am also grateful to the referee for pointing out a shorter method of deriving the moments of t''.

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