On the Variance of the Ratio Estimator for Midzuno-Sen Sampling Scheme

By T. J. Rao, Calcutta[1])

1. Summary

For the sampling scheme of Midzuno [3] and Sen [4], which provides unbiased ratio estimators an expression for the variance of the estimator does not seem to be available in literature. An expression for the same is derived in this note.

2. Introduction

The ratio method of estimation of population ratio or total consists in taking the ratio of unbiased estimators of the numerator and the denominator and in the latter case multiply it by the population total of the auxiliary variate taken in the denominator. For many common selection procedures employed in surveys, this estimator is biased and so attempts have been made to construct selection and estimation procedures which provide unbiased ratio estimators. Lahiri [2] showed that the ordinary ratio estimator $\sum_{i \in s} y_i / \sum_{i \in s} x_i$ is unbiased if the sample is drawn with probability proportional to its total size $\sum_{i \in s} x_i$. Midzuno [3] and Sen [4] have independently given a simple procedure which makes this ratio unbiased. The method consists in drawing the first unit of the sample with probability proportional to size and the rest of the $(n - 1)$ units with equal probability without replacement from the remaining $(N - 1)$ units of the population.

It has been pointed out by Cochran [1] that an exact expression for the variance of the ratio estimator under the above scheme has not been found. In the next section, an expression for the same is derived.

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[1]) Indian Statistical Institute, Calcutta.

3. Variance of the Estimator

It is easy to see that the probability that a sample with a specified value of $\sum_{i\in s} x_i$, will be drawn is given by

$$P = \sum_{i\in s} x_i \Big/ \binom{N-1}{n-1} X$$

and for this method of selection, it can be easily shown that

$$\hat{R} = \frac{\sum_{i\in s} y_i}{\sum_{i\in s} x_i}$$

is unbiased for R.

For,

$$E(\hat{R}) = \sum_{s\in S} \left\{ \frac{\sum_{i\in s} y_i}{\sum_{i\in s} x_i} P \right\} = \binom{N-1}{n-1} Y \Big/ \binom{N-1}{n-1} X = R.$$

We next have

$$V(\hat{R}) = E(\hat{R}^2) - R^2 =$$

$$= E\left\{ \frac{\sum_{i\in s} y_i}{\sum_{i\in s} x_i} \right\}^2 - R^2 =$$

$$= \sum_{s\in S} \left\{ \frac{\left(\sum_{i\in s} y_i\right)^2}{\left(\sum_{i\in s} x_i\right)^2} \right\} P - R^2 =$$

$$= \frac{1}{\binom{N-1}{n-1} X} \sum_{s\in S} \frac{\left(\sum_{i\in s} y_i\right)^2}{\left(\sum_{i\in s} x_i\right)} - R^2.$$

Consider the set of all $(n-1)$-tuples of distinct X's where X_i does not belong to any of them and for any particular $(n-1)$-tuple the other $((n-1)!-1)$ permutations do not occur in the set. Index this by a set G. For each $g \in G$, let X_g^i denote the sum of the elements of the $(n-1)$-tuple corresponding to g.

Set

$$J_i = \sum_{g\in G} \frac{1}{X_i + X_g^i}.$$

Next, consider the set of all $(n-2)$-tuples of distinct X's where X_i and X_j do not belong to any of them and for any particular $(n-2)$-tuple the other $((n-2)!-1)$ permutations do not occur in the set. Index this by a set G'. For each $g \in G'$, let X_g^{ij} denote the sum of the elements of the $(n-2)$-tuple corresponding to g.

Set

$$J_{ij} = \sum_{g \in G'} \frac{1}{X_i + X_j + X_g^{ij}}.$$

Let

$$T_i = \frac{J_i}{\binom{N-1}{n-1} X} - \frac{1}{X^2}$$

and

$$T_{ij} = \frac{J_{ij}}{\binom{N-1}{n-1} X} - \frac{1}{X^2}.$$

Then, collecting the coefficients of Y_i^2 and $Y_i Y_j$ from the terms on the r.h.s. of $V(\hat{R})$, we have

$$V(\hat{R}) = \frac{1}{\binom{N-1}{n-1} X} \left[\sum_1^N Y_i^2 J_i + \sum_{i \neq j}^N \sum Y_i Y_j J_{ij} \right] - R^2 =$$

$$= \sum_{i=1}^N T_i Y_i^2 + \sum_{i \neq j}^N \sum Y_i Y_j T_{ij}.$$

An expression for the estimate of this variance is known to be

$$\hat{V}(\hat{R}) = \hat{R}^2 - (\hat{R}^2) = \hat{R}^2 - \frac{\sum_i^n y_i^2 + 2 \frac{N-1}{n-1} \sum_{i>j}^n y_i y_j}{N n \bar{x} X}.$$

References

[1] Cochran, W. G.: *Sampling Techniques*, 2nd Ed., (1963)., Wiley, New York.

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