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Existence of measurable selectors and parametrizations for G_δ-valued multifunctions

by

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Abstract. In this paper we establish the existence of measurable selectors and parametrizations or G_δ-valued multifunctions. Examples are given to show that certain reasonable conjectures fre false.

1. Introduction. In recent articles, Srivastava [6] and Sarbadhikari and Srivastava [5] have established the following facts about measurable G_δ-valued multifunctions:

THEOREM 1.1. *Let T and X be Polish spaces and A a countably generated sub σ-field of the Borel σ-field B_T on T. Suppose $F: T \to X$ is a multifunction such that F is A-measurable,* $\mathrm{Gr}(F) \in A \otimes B_X$ *and $F(t)$ is a G_δ in X for each $t \in T$. Then there is an A-measurable selector for F, that is, there is an A-measurable function $f: T \to X$ such that $f(t) \in F(t)$ for each $t \in T$.*

THEOREM 1.2. *Let T, X, A, F satisfy the hypotheses of Theorem* 1.1. *Then there is a map $f: T \times \Sigma \to X$ such that for each $t \in T$, the map $f(t, .)$ is continuous, open and onto $F(t)$ and for each $\sigma \in \Sigma$, $f(., \sigma)$ is A-measurable, where Σ is the space of irrationals.*

In [5], Sarbadhikari and Srivastava raised a question regarding the converse of Theorem 1.2, viz., whether all multifunctions induced by maps $f: T \times \Sigma \to X$ of the above kind necessarily satisfy the hypotheses of Theorem 1.1.

Theorems 1.1 and 1.2 hold, as has been shown by the above-mentioned authors, even when T is an analytic set and A any sub σ-field of B_T. This is easily deduced from the above theorems. The present article is motivated by the question whether the above results can be extended to the case where (T, A) is an arbitrary measurable space or the even more general framework of the Kuratowski–Ryll-Nardzewski selection theorem. Debs [1] has already considered this problem. By assuming that the graph of the multifunction is of a certain form, Debs was able to establish Theorem 1.1 in the set-up of Kuratowski and Ryll-Nardzewski. In this article we prove the parametrization theorem (Theorem 1.2) in this situation and settle the question raised in [5], mentioned above, in the negative by means of an example. We then go on to settle another natural question, arising therefrom, by means of another example.

The paper is organized as follows: Section 2 is devoted to definitions and notation. In Section 3 we prove Theorem 1.1 in the set-up of Maitra and Rao [4] It should be mentioned that though this is only a slight extension of Debs' result, our proof is simpler and more transparent than that of Debs. In Section 4 we prove the main result, the parametrization theorem. In Section 5 we give our counter-examples.

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2. Definitions and notation. We denote by N the set of all natural numbers and S will denote the set of all finite sequences of natural numbers, including the empty sequence e. For $k \in N$, S_k will be the set of all elements of S of length k. For $s \in S$, $|s|$ will denote the length of s and if $i<|s|$ is a natural number, s_i will denote the ith coordinate of s and, for $n \in N$, sn will denote the catenation of s and n. If X is a non-empty set then a function $A: S \to P(X)$ from S into the power set of X is called a *system of sets in* X and will usually be denoted by $\{A(s), s \in S\}$ or simply by $\{A(s)\}$. A system of sets $\{A(s)\}$ in X is called *regular* if $A(sn) \subseteq A(s)$ for each $s \in S$ and $n \in N$. We put $\Sigma = N^N$. Endowed with the product of discrete topologies on N, Σ becomes a homeomorph of the irrationals. For $\sigma \in \Sigma$ and $i \in N$, σ_i will denote the ith coordinate of σ and $\sigma|i$ will denote the finite sequence $(\sigma_0, \sigma_1, \ldots, \sigma_{i-1})$; here, if $i = 0$, $\sigma|i$ will just be the empty sequence. If $s \in S$ then the set $\{\sigma \in \Sigma: \sigma_i = s_i$ for $i<|s|\}$ will be denoted by Σ_s. In particular, $\Sigma_e = \Sigma$.

Let T and X be non-empty sets and $A \subseteq P(T)$. A *multifunction* $F: T \to X$ is a function whose domain is T and whose values are non-empty subsets of X. For $E \subseteq X$, we denote by $F^{-1}(E)$ the set $\{t \in T: F(t) \cap E \neq \emptyset\}$. We denote by $\mathrm{Gr}(F)$ the set $\{(t, x) \in T \times X: x \in F(t)\}$, and call it the *graph* of F. A function $f: T \to X$ is called a *selector* for F if $f(t) \in F(t)$, $t \in T$. For $A \subseteq P(T)$, A^c will denote the set $\{A \subseteq T: A^c \in A\}$ and the smallest countably additive (resp. countably multiplicative) family of subsets of T containing A will be denoted by A_σ (resp., by A_δ). If A and B are σ-fields on T and X respectively, then $A \otimes B$ will denote the product σ-field on $T \times X$. Further if L and M are families of subsets of T and X respectively, then $L \times M$ denotes the family $\{A \times B: A \in L$ and $B \in M\}$. If $W \subseteq T \times X$ then, for $t \in T$, W^t is the set $\{x: (t, x) \in W\}$ and will be called the *section* of W at t.

Now suppose T is a non-empty set and $A \subseteq P(T)$. Suppose X and Y are metric spaces. A multifunction $F: T \to X$ is called *A-measurable* if $F^{-1}(V) \in A$ for every open $V \subseteq X$. Similarly, a point map $f: T \to X$ is *A-measurable* if $F^{-1}(V) \in A$ for every open set $V \subseteq X$. A map $f: T \times Y \to X$ is called a *Carathéodory map* if, for each $t \in T$, the map $f(t, .): Y \to X$ is continuous and the map $f(., y): T \to X$ is A-measurable for each $y \in Y$. A Carathéodory map is said to be *open* (resp. *closed*) if, for each $t \in T$, $f(t, U)$ is relatively open (resp. closed) in the range of $f(t, .)$ for each open (resp. closed) set $U \subseteq Y$. If $F: T \to X$ is a multifunction, a Carathéodory map $f: T \times Y \to X$ is said to *induce* F if $F(t) = f(t, Y)$ for each $t \in T$.

We say $L \subseteq P(T)$ satisfies the *weak reduction principle* (and we write $\mathrm{WRP}(L)$)

if, for any sequence of sets $L_1, L_2, \ldots$ from L such that $\bigcup_{n=1}^{\infty} L_n = T$, we can find disjoint $L'_1, L'_2, \ldots$ from L such that $L'_i \subseteq L_i$ for each i and $\bigcup_{i=1}^{\infty} L'_i = T$.

A *field* on T is a family of subsets of T containing $\emptyset$ and closed under finite intersections and complementation. We remark here that it is well known that, if L is a field, then $\mathrm{WRP}(L_\sigma)$. For $E \subseteq X$, $\mathrm{cl}\,E$ or $\bar{E}$ will denote the closure of E, and $\partial(E)$ will denote the diameter of E.

For terminology not defined we refer the reader to Kuratowski [2].

3. **A Selection Theorem.** Before proceeding to prove our results, we first note that, as every Polish space can be embedded in a compact metric space in which it is automatically a G_δ, we will find it sufficient to prove our theorems when X is a compact metric space. This assumption will be made, when required, without loss of generality.

We now fix some notation. In what follows, X will denote a Polish space with a metric d such that $\partial(X) < 1$. The topology on X will be denoted by U. We fix a base $\{V_n : n \in N\}$ for X such that $V_0 = X$ and $V_n \neq \emptyset$ for each n. Also, in what follows, T will be a non-empty set and L a family of subsets of T containing $\emptyset$ and T, closed under finite intersections and countable unions and such that, moreover, $\mathrm{WRP}(L)$. In the sequel, $F\colon T \to X$ will be an L-measurable multifunction such that $\mathrm{Gr}(F) \in (L \times U)_{\sigma\delta}$. Set $G = \mathrm{Gr}(F)$ and write $G = \bigcap_{n=1}^{\infty} G_n$, where $G_n \supseteq G_{n+1}$ and $G_n = \bigcup_{m=1}^{\infty} (L_{nm} \times U_{nm})$ with $L_{nm} \in L$ and $U_{nm} \in U$, $n, m \geqslant 1$.

The following is well known:

LEMMA 3.1. *Let $f_n\colon T \to X$, $n \in N$, be a sequence of L-measurable functions. If f_n converges uniformly to a function $f\colon T \to X$, then f is L-measurable.*

The next lemma is implicit in [1].

LEMMA 3.2. *Let X be compact. If $H \subseteq T \times X$ is such that $H \in (L \times U)_\sigma$, then or any closed set $C \subseteq X$, the set $\{t \in T : C \subseteq H^t\}$ belongs to L.*

Proof. Let $H = \bigcup_{n=1}^{\infty} (L_n \times U_n)$, where $L_n \in L$ and $U_n \in U$. Then, as C is compact, $\{t \in T : C \subseteq H^t\} = \bigcup (L_{n_1} \cap \ldots \cap L_{n_k})$, where the union runs over all finite sequences $(n_1, \ldots, n_k)$ such that $C \subseteq U_{n_1} \cup U_{n_2} \cup \ldots \cup U_{n_k}$. As L is closed under finite intersections and countable unions, the proof is complete.

LEMMA 3.3. *Let $\{T(s)\}$ and $\{U(s)\}$ be regular systems of sets belonging to L and U respectively such that:*

(i) $T(e) = T$,

(ii) $T(s) = \bigcup_{n=1}^{\infty} T(sn)$ *for each* $s \in S$,

(iii) $s, t \in S$, $s \neq t$, $|s| = |t| \to T(s) \cap T(t) = \emptyset$,

(iv) $\partial(U(s)) < 2^{-|s|}$ *for each* s,

(v) $T(s) \neq \emptyset \Rightarrow U(s) \neq \emptyset$.

Put $M_k = \bigcup_{s \in S_k} \big(T(s) \times \mathrm{cl}(U(s))\big)$ *and* $M = \bigcap_{k=1}^{\infty} M_k$. *Then* M *is the graph of an* L*-measurable function* $f\colon T \to X$. *Further, if each* $T(s) \in (L \cap L^c)$, *then* f *is* $(L \cap L^c)_\sigma$*-measurable.*

Proof. Let $t \in T$. Then there is a unique $\sigma \in \Sigma$ such that $t \in T(\sigma|n)$ for each n. Then $\mathrm{cl}(U(\sigma|n))$ is a decreasing sequence of non-empty closed sets of diameters tending to zero. We put $f(t)$ to be the unique point belonging to $\bigcap_{n=1}^{\infty} \mathrm{cl}(U(\sigma|n))$. Then M is the graph of f.

We shall now define a sequence of L-measurable functions $f_k\colon T \to X$ which converge uniformly to f. First, for each $s \in S$ such that $U(s) \neq \emptyset$, choose and fix a point $x_s \in \mathrm{cl}(U(s))$. Define $f_k\colon T \to X$ by $f_k(t) = x_s$, where s is the unique element of S of length k such that $t \in T(s)$. (Observe that in this situation $U(s) \neq \emptyset$.) It is easily checked that f_k is L-measurable and that $f_k \to f$ uniformly. Lemma 3.1 now shows that f is L-measurable. This completes the proof.

Lemmas 3.1, 3.2 and 3.3 hold even without the assumption that the weak reduction principle holds for L.

Lemma 3.4. *There exist systems* $\{T(s)\}$ *and* $\{U(s)\}$ *of sets in* T *and* X, *respectively, satisfying conditions* (i)–(v) *of Lemma* 3.3 *and further satisfying:*

(vi) $\mathrm{cl}(U(s)) \subseteq G_k^t$ *for each* $s \in S_k$ *and* $t \in T(s)$,

(vii) $U(s) \cap G^t \neq \emptyset$ *for each* $t \in T(s)$,

(viii) $T(s) \in L \cap L^c$ *for each* $s \in S$.

We assume here that X *is compact.*

Proof. The construction is by induction on $k = |s|$. For $k = 0$, put $T(e) = T$ and $U(e) = X$. Suppose $T(s)$, $U(s)$ have been defined for $s \in S_k$. We will now define $T(sn)$, $U(sn)$, for all $n \geq 0$. Define

$$A^m(s) = \begin{cases} T(s) \cap \{t\colon V_m \cap F(t) \neq \emptyset\} \cap \{t\colon \bar{V}_m \subseteq G_{k+1}^t\}, & \\ \qquad \text{if } V_m \subset U(s) \text{ and } \partial(V_m) < \dfrac{1}{2^{k-1}}, & \\ \emptyset \quad \text{otherwise.} & \end{cases}$$

As F is L-measurable, $\{t\colon V_m \cap F(t) \neq \emptyset\} \in L$. By Lemma 3.2, X being compact, we have, $\{t\colon \bar{V}_m \subseteq G_{k+1}^t\} \in L$. Finally, as L is closed under finite intersections, it follows that $A^m(s) \in L$, for each $m \geq 0$. Further, by the induction hypothesis and (vii), we have $\bigcup_{m \geq 0} A^m(s) = T(s)$. As WRP($L$), we obtain a disjoint family

$$\{B^m(s)\colon m \geq 0\} \subset L$$

such that for each $m \geq 0$, $B^m(s) \subseteq A^m(s)$ and $\bigcup_{m \geq 0} B^m(s) = T(s)$. Since L is closed under countable unions, it is easy to see that $B^m(s) \in L \cap L^c$ for each $m \geq 0$. Put

$$T(sn) = B^n(s), \quad n \geq 0$$

and

$$U(sn) = \begin{cases} V_n, & \text{if } T(sn) \neq \emptyset, \\ \emptyset & \text{otherwise}. \end{cases}$$

It is easily seen that $\{T(s)\}$ and $\{U(s)\}$ as defined above satisfy conditions (i)–(viii). This completes the proof of the lemma.

We now prove the main theorem of this section. This is essentially the theorem in [1].

THEOREM 3.1. *Let X be a Polish space, T a non-empty set and L a finitely multiplicative, countably additive family of subsets of T, containing $\emptyset$ and T and satisfying* WRP(L). *Let $F: T \to X$ be an L-measurable multifunction with* $\mathrm{Gr}(F) \in (L \times U)_{\sigma\delta}$, *$U$ being the topology of X. Then F has an $(L \cap L^c)_\sigma$-measurable selector.*

Proof. As remarked earlier, we can, without loss of generality, take X to be compact. Use Lemma 3.4 to obtain systems $\{T(s)\}$ and $\{U(s)\}$ satisfying conditions (i)–(viii). By Lemma 3.3 we obtain an $(L \cap L^c)_\sigma$-measurable function $f: T \to X$, whose graph is M. From condition (vi) it follows that $M \subseteq G$. Thus, f is an $(L \cap L^c)_\sigma$-measurable selector for F. The theorem is proved.

Remark 3.1. If M is a field on T and we take $L = M_\sigma$ in the above, then as remarked in Section 2, we have WRP(L). Thus we obtain the theorem in [1]. Notice that while in [1] M has been taken to be a *clan*, that is, a family closed under finite intersections and pairwise differences, no greater level of generality has really been attained. For the multifunction F being L-measurable and non-empty set valued, $F^{-1}(X) = T \in L = M_\sigma$. Consequently, M now being closed under differences, we can write $T = \bigcup_{n\geq 1} M_n$, $\{M_n\}$ being a pairwise disjoint family of subsets of T. Further, M restricted to each M_n is a field. Thus our theorem applies to F restricted to M_n, for every $n \geq 1$. The theorem in [1] is an immediate consequence. To see that Theorem 1.1 follows we have to show that $\mathrm{Gr}(F)$ may be written as $\bigcap_{n\geq 1}\bigcup_{m\geq 1}(T_{nm} \times U_{nm})$ with $T_{nm} \in A$ and U_{nm} open in X. But this is implicit in the proof of Lemma 3.8 of [6]. The only additional observation one need make is the following:

Let T be a Polish space, and X a compact metric space. Let B be a Borel set with open sections contained in $T \times X$. Then $B = \bigcup_{n\geq 1}(B_n \times U_n)$, with B_n, $n \geq 1$, Borel in T and $\{U_n, n \geq 1\}$, form a base for X. This, X being compact, is an easy consequence of the well-known theorem of Kunugui–Novikov.

Remark 3.2. If T is a Polish space, for L we can take the family of sets of the additive class α, for any ordinal $\alpha > 0$, or the family of coanalytic sets to obtain selectors of the respective classes, and in the last case, a Borel measurable selector.

4. A representation theorem. In this section T and X will be as before. However, in addition to the assumptions made in Section 3, we will further assume that each V_n appears in $\{V_n: n \in N\}$ infinitely often. We will also require that $L = M_\sigma$, where M is a field on T. Such an L will satisfy the earlier conditions, as observed in Remark 3.1. We will take F, G, L_{nm}, U_{nm}, $n, m \geqslant 1$ as before. We first prove a lemma:

LEMMA 4.1. *Let X be compact. For each $s \in S$, there is a map $p(.,s)\colon T \to N$ such that:*

(a) $p(.,s)$ *is L-measurable,*

(b) $\partial(V_{p(t,s)}) < \frac{1}{2^k}$ *for $s \in S_k$ and $t \in T$,*

(c) $\bar{V}_{p(t,sn)} \subseteq G^t_{k+1} \cap V_{p(t,s)}$, $n \geqslant 0$,

(d) $F(t) \cap V_{p(t,s)} \neq \emptyset$,

(e) $F(t) \cap V_{p(t,s)} \subseteq \bigcup_{n=0}^{\infty} V_{p(t,sn)}$,

(f) $F(t) \subseteq V_{p(t,e)}$.

Proof. The proof will be by induction on $|s|$. Define $p(t,e) \equiv 0$. Suppose $p(t,s)$ has been defined for $s \in S_k$. We shall now define $p(t,sn)$, for $n \geqslant 0$. Put

$$R_m = \begin{cases} \emptyset, & \text{if } \partial(V_m) \geqslant \frac{1}{2^{k+1}}, \\ \{t \in T\colon F(t) \cap V_m \neq \emptyset,\ \bar{V}_m \subseteq V_{p(t,s)} \text{ and } \bar{V}_m \subseteq G^t_{k+1}\}, & \\ \qquad \text{if } \partial(V_m) < \frac{1}{2^{k+1}}. & \end{cases}$$

By Lemma 3.2, $\{t\colon \bar{V}_m \subseteq G^t_{k+1}\} \in L$.

Now, $\{t\colon \bar{V}_m \subseteq V_{p(t,s)}\} = \bigcup \{t\colon p(t,s) = l\}$, the union running over all l such that $\bar{V}_m \subseteq V_l$. As $p(.,s)$ is L-measurable, we have $\{t\colon \bar{V}_m \subseteq V_{p(t,s)}\} \in L$. As F is L-measurable, it follows that $R_m \in L$. Let $R_m = \bigcup_{l \geqslant 0} Q_{ml}$, $Q_{ml} \in M$. Let $i \to (m_i, l_i)$ be a 1-1 mapping from N onto $N \times N$. Put $P_i = Q_{m_i l_i} \in M$. Observe that $\bigcup_{m=0}^{\infty} R_m = T$. Further, as the base $\{V_n\}$ has been chosen so that each V_n appears infinitely often, it follows that, for each fixed $t \in T$, $\{m\colon t \in R_m\}$ is infinite, and consequently that $\{i\colon t \in P_i\}$ is infinite. Define $p(t,sn) = m_i$, where i is the $(n+1)$st integer j such that $t \in P_j$. As $\{i\colon t \in P_i\}$ is infinite, $p(t,sn)$ is defined on the whole of T for each $n \geqslant 0$.

Now,

$$p(t,so) = m \leftrightarrow (\exists i)[m = m_i \text{ and } t \in P_i \text{ and } (\forall j < i)(t \notin P_j)].$$

As each $P_i \in M$ and M is a field, it follows that $\{t\colon p(t,so) = m\} \in M_\sigma = L$. Further, for $n \geqslant 1$,

$$\begin{aligned} p(t,sn) = m \leftrightarrow (\exists i)[m_i = m \text{ and } t \in P_i \text{ and } \\ (\exists j_1 < j_2 < \ldots < j_n < i)(t \in P_{j_1} \cap \ldots \cap P_{j_n} \\ \text{and } (\forall j)\ (j < i \text{ and } j \notin \{j_1, \ldots, j_n\} \to t \notin P_j))]. \end{aligned}$$

Here again the expression within square brackets is a Boolean combination of the P_i's. Thus, as M is a field, we have

$$\{t\colon p(t, sn) = m\} \in M_\sigma = L.$$

Consequently, $p(., sn)$ is L-measurable for each $n \geqslant 0$. One easily checks that the system of functions $p(., s)$ as defined above satisfies conditions (a)–(f). This proves the lemma.

We now prove the representation theorem:

THEOREM 4.1. *Let X be a Polish space, T a non-empty set and M a field on T. Put $L = M_\sigma$. Let $F\colon T \to X$ be an L-measurable multifunction with* $\mathrm{Gr}(F) \in (L \times U)_{\sigma\delta}$, *$U$ being the topology of X. Then there exists a map $f\colon T \times \Sigma \to X$ satisfying:*

(i) *For each $t \in T$, the map $f(t,.)\colon \Sigma \to X$ is continuous, open and onto $F(t)$ (here $f(t, U)$ is relatively open in $F(t)$ for each U open in Σ).*

(ii) *For each $\sigma \in \Sigma$, the map $f(., \sigma)\colon T \to X$ is L-measurable.*

Proof. As before, without loss of generality we take X to be compact. By Lemma 4.1, we have a system $p(., s)$ of functions satisfying conditions (a)–(f).

Define $f\colon T \times \Sigma \to X$ by:

$$f(t, \sigma) = \bigcap_{k=1}^{\infty} \bar{V}_{p(t, \sigma|k)} \subseteq \bigcap_{k=1}^{\infty} G_k^t.$$

It is easy to see that, for each t and σ, the intersection $\bigcap_{k=1}^{\infty} \bar{V}_{p(t,\sigma|k)}$ reduces to a singleton. The map f is therefore well-defined. Fix $\sigma \in \Sigma$. For each $s \in S_k$, define

$$T(s) = \{t \in T\colon p(t, \sigma|1) = s_0,\ p(t, \sigma|2) = s_1, \ldots, p(t, \sigma|k) = s_{k-1}\}$$

and $U(s) = V_{s_{k-1}}$, if $T(s) \neq \emptyset$ and $U(s) = \emptyset$, otherwise. Observe that, so defined, $U(s) = V_{p(t,\sigma|k)}$, for $t \in T(s)$. Then, as is easily checked, $\{T(s)\}$ and $\{U(s)\}$ are systems satisfying conditions (i)–(vii) of Lemmas 3.3 and 3.4. It follows that $\bigcap_{k=1}^{\infty} \bigcup (T(s) \times U(s))$, where the inner union runs through all $s \in S_k$, is the graph of an L-measurable selector for F, say $f_\sigma\colon T \to X$ (by Lemma 3.3 and condition (vi) of Lemma 3.4). Also, note that $f_\sigma(t) = f(t, \sigma)$. Thus, for each fixed $\sigma \in \Sigma$, $f(., \sigma)$ is an L-measurable function on T into X and $f(t, \sigma) \in F(t)$. Now fix $t \in T$. It follows from (b), (c) and (e) that the map $f(t,.)\colon \Sigma \to X$ is continuous and open onto $F(t)$. The proof of the theorem is complete.

Remark 4.1. The above proof does not go through under the weaker assumptions on the family L made in Theorem 3.1. Indeed, Lemma 4.1 makes essential use of the fact that L is of the type M_σ, where M is a field on T.

Remark 4.2. If we take T to be metric and L to be the family of sets of additive class α, where $\alpha > 0$, we obtain a representation of the above type, where the maps $f(., \sigma)$, for fixed $\sigma \in \Sigma$, are of class α.

Remark 4.3. By the observations made in Remark 3.1, Theorem 1.2 follows from Theorem 4.1.

Remark 4.4. In [5], S. M. Srivastava and H. Sarbadhikari have proved that the multifunctions of the type considered in Theorem 1.2 are of the so-called 'Souslin type', i.e., they prove the following: Let T, X, F, A be as in Theorem 1.2. Then there is an A-measurable closed valued multifunction H: $T \to \Sigma$ and a continuous, open map $f: \Sigma \to X$ such that $F(t) = f(H(t))$. We shall content ourselves with the observation that the same result holds in the more general set-up of Theorem 4.1, with the closed valued multifunction H: $T \to \Sigma$ now being L-measurable. The proof in [5] goes through *mutatis mutandis*.

5. Counterexamples. Let T, X be Polish spaces, and let A, F, B_X be as in Theorem 1.1. As seen above, F is induced by a continuous, open Carathéodory map $f: T \times \Sigma \to X$. Conversely, suppose a multifunction $F: T \to X$ is induced by a continuous, open Carathéodory map f: $T \times \Sigma \to X$. Then, as observed in [5], F is A-measurable. Moreover, by a theorem of Hausdorff, continuous, open images of absolute G_δ sets are absolute G_δ's. It follows that $F(t)$ is a G_δ in X for each $t \in T$. The question has been posed in [5] as to whether in this situation $\mathrm{Gr}(F)$ is necessarily in $A \otimes B_X$. An answer in the affirmative would provide a complete characterization of multifunctions of the type specified in Theorem 1.1 in terms of such Carathéodory maps. We remark here that in [7] it has been shown that such multifunctions are indeed induced by Carathéodory maps where the maps $f(t, .)$, for each $t \in T$, are continuous and closed on Σ onto $F(t)$, and, further, that such closed Carathéodory maps characterize these multifunctions. We show below, by means of an example, that the answer to the above question is in the negative. We then show, by means of another example, that if in Theorem 1.2, $\mathrm{Gr}(F)$ is assumed to be analytic, F need not even admit a measurable selector.

Example 1. Let T, X be uncountable Polish spaces. Let A be an analytic, non-Borel subset of T. Fix $x_0 \in X$ such that x_0 is not an isolated point. Let $G \subseteq T \times X$ be defined by:

$$G = (A \times \{x_0\}) \cup (T \times (X - \{x_0\})).$$

Then G is an analytic, non-Borel subset of $T \times X$. Consider the multifunction F: $T \to X$ defined by:

$$F(t) = G^t$$

Then

(i) As each $F(t)$ is dense in X, F is B_T-measurable, where B_T is the Borel σ-field on T.

(ii) Each $F(t)$ is a G_δ in X.

(iii) $\mathrm{Gr}(F) = G$ is an analytic, non-Borel subset of $T \times X$.

We will now show that F is induced by a continuous, open Carathéodory map $f: T\times\Sigma \to X$ providing us with our counterexample. We will do so by obtaining a subset H of $T\times X\times\Sigma$ satisfying:

(a) For each $t\in T$, $(X-\{x_0\})\times\Sigma\subseteq H^t$.

(b) H is a G_δ in $T\times X\times\Sigma$.

(c) $\pi_{T\times X}(H) = G$, where $\pi_{T\times X}$ is the projection map on $T\times X\times\Sigma$ onto $T\times X$.

Suppose such an H has been obtained. Consider the multifunction $K: T\to X\times\Sigma$ given by:

$$K(t) = H^t.$$

Then by (a), K is B_T-measurable. By (b), K is G_δ-valued and $\mathrm{Gr}(K) = H\in B_T\otimes B_{X\times\Sigma}$. Thus K satisfies the hypotheses of Theorem 1.2. There is therefore a Carathéodory map h: $T\times\Sigma\to X\times\Sigma$ inducing K such that for each fixed $t\in T$, the map $h(t,.)$: $\Sigma\to X\times\Sigma$ is continuous and open onto $K(t)$. Look at the map f: $T\times\Sigma\to X$ defined by:

$$f(t,\sigma) = \pi_X(h(t,\sigma))$$

where π_X is the projection onto X.

Then, clearly, for each fixed $\sigma\in\Sigma$, if h is B_T-measurable, so is $f(.,\sigma)$: $T\to X$. For fixed $t\in T$, $h(t,.)$ is continuous, open and onto H^t. As π_X is continuous and (c) holds, we have

$$f(t,.)\colon \Sigma\to X \text{ is continuous and onto } F(t).$$

Let $U = U_1\times U_2\subseteq X\times\Sigma$, with U_1 open in X and U_2 open in Σ be a basic open set in $X\times\Sigma$. Fix $t\in T$. Then $\pi_X(U\cap H^t)$ is either U_1 or $U_1-\{x_0\}$. In either case, $\pi_X(U\cap H^t)$ is open in X. It follows that π_X is open on the range of $h(t,.)$. As $h(t,.)$: $\Sigma\to X\times\Sigma$ is an open map, it follows that $f = \pi_X\circ h$: $\Sigma\to X$ is an open map. Thus, f as defined above gives the required representation. Finally, it remains to find H. As A is analytic in T, there is a closed subset C of $T\times\Sigma$ such that $\pi_T(C) = A$. Let

$$H = (T\times(X-\{x_0\})\times\Sigma)\cup(\{(t,x,\sigma)\in T\times X\times\Sigma\colon (t,\sigma)\in C \text{ and } x = x_0\}).$$

This H satisfies conditions (a), (b) and (c).

As we have found that the range under such open Carathéodory maps can be an analytic non-Borel set, the question that naturally arises is whether any measurable multifunction, taking G_δ-values, whose graph is analytic is induced by such a Carathéodory map. We show below that such a multifunction need not even admit a measurable selector.

EXAMPLE 2. Let $C\subset\Sigma\times\Sigma\times\Sigma$ be a coanalytic subset of $\Sigma\times\Sigma\times\Sigma$ which is universal for all coanalytic subsets of $\Sigma\times\Sigma$. To fix ideas, we assume that sections of C obtained by fixing the first coordinate run through all the coanalytic subsets of $\Sigma\times\Sigma$. Consider $D = \{(x,z)\in\Sigma\times\Sigma\colon (x,x,z)\in C\}$. Then D is a coanalytic subset of $\Sigma\times\Sigma$. Apply the Kondo Uniformization theorem for coanalytic sets to obtain a coanalytic uniformization for D, i.e., get a coanalytic set $B\subseteq D$ such that B^x is a singleton whenever $D^x\neq\emptyset$. Let $A=(\Sigma\times\Sigma)-B$. Then,

(i) A is an analytic subset of $\Sigma\times\Sigma$.

(ii) For each $x\in\Sigma$, $A^x=\{y\in\Sigma\colon (x,y)\in A\}$ is either Σ or Σ minus a point.

Define a multifunction $F\colon \Sigma\to\Sigma$ by $F(x)=A^x$. Then,

(a) As each A^x is dense in Σ, F is B_Σ-measurable, B_Σ being the Borel σ-field on Σ.

(b) For each $x\in X$, $F(x)$ is open in Σ.

(c) $\mathrm{Gr}(F)=A$ is analytic in $\Sigma\times\Sigma$.

However, F admits no B_Σ-measurable selector. Indeed, A admits no coanalytic uniformization, *a fortiori*, no Borel uniformization. For if not, let $E\subseteq A$ be a coanalytic subset of $\Sigma\times\Sigma$ such that E^x is a singleton for each $x\in X$. As C is universal for the coanalytic subsets of $\Sigma\times\Sigma$ there is $x^*\in\Sigma$ such that $E=C^{x^*}$. Now, there is a unique $y^*\in\Sigma$ such that $(x^*,y^*)\in E$. It follows that $D^{x^*}=y^*$, and consequently, that $(x^*,y^*)\in B$. But $E\subseteq A$. So $(x^*,y^*)\in A=\Sigma\times\Sigma-B$, which leads to a contradiction.

Added in proof: Theorem 1.1 has been extended to an arbitrary measurable space $(T,\mathcal{A})$ by the author in his doctoral dissertation: *Measurable sets in product spaces and their parametrizations*, Indian Statistical Institute, Calcutta 1981.

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