Sankhyā : The Indian Journal of Statistics
1990, Volume **52**, Series B, Pt. 1, pp. 82–92.

COMPUTER CONSTRUCTION OF SOME GROUP DIVISIBLE DESIGNS

By ANUP K. DE and BIMAL K. ROY
Indian Statistical Institute

SUMMARY. An algorithm to construct group divisible designs starting from balanced incomplete block designs has been developed. Using this algorithm five group divisible designs with parameters (12, 6, 2, 30, 10, 4, 0, 3), (16, 8, 2, 16, 7, 7, 0, 3), (20, 10, 2, 20, 9, 9, 0, 4), (45, 15, 3, 45, 7, 7, 0, 1) and (38, 19, 2, 38, 9, 9, 0, 2) have been constructed. These designs were so far unknown.

1. Introduction

Group divisible design is an important concept in statistical design of experiments. For the existence of such a design there are some restrictions on its parameters. But unfortunately all the existing conditions on parameters are necessary and none of them are sufficient. So, given a set of parameters satisfying all the necessary conditions, it is yet a problem to say whether such a design exists or not. In this paper we have developed an algorithm which looks for the existence of a group divisible design for a given set of parameters satisfying the necessary conditions and $\lambda_1 = 0$.

Bhaskar Rao designs have been studied by a number of authors including Bhaskar Rao (1966, 1970), Seberry (1984), Singh (1982), Street (1982), Street and Rodger (1980) and Vyas (1982). Generalised Bhaskar Rao designs (GBRD) were introduced by Seberry (1981) and have subsequently been studied by Lam and Seberry (1984) and de Launey and Seberry (1982), 1984). de Launey and Sarvate (1983) studied the non-existence and uniqueness of certain generalised Bhaskar Rao designs. Such questions fall under the general problem of signing (O,H)-matrices (matrices whose non-zero entries are taken from a group H) over another group G. A computer program to deal with this problem when $H = \{1\}$ has been developed by Gibbons and Mathon (1987). They have described some mathematical and computational techniques for constructing and enumerating generalised Bhaskar Rao designs. In this paper we have tried to design an algorithm based on a similar idea.

Construction of a list of GDD's whose existences are so far unknown as given by Clatworthy (1973) were tried with our algorithm. In most of the cases signing was not possible. We could find five GDD's which are given in section 4.

The signing algorithm, given in Section 3, if successful gives the design, but its failure does not tell us anything about the existence or non-existence of the design.

AMS (1980) *subject classification :* 62K05.
Key words and phrases : BIBD, GDD, GBRD, Incidence Matrix, Signing a matrix.

2. Definitions and notations

(a) A BIBD (v, b, r, k, λ) is an arrangement of v elements in b blocks such that each block contains exactly k distinct elements, each element occurs in exactly r different blocks and every pair of distinct elements occurs together in exactly λ blocks.

If $v = b$ and $r = k$, the design is called symmetric and is represented by SBIBD (v, k, λ).

For a matrix $X_{v \times b}$ whose non-zero elements are taken from $G(p)$, a group of p elements, we will use an equation like $\sum_{l=1}^{b} x_{il}\, x_{jl}^{-1} = \frac{\lambda}{p}\,(g_1 + \ldots + g_p)$, $1 \leqslant i \neq j \leqslant v$ where λ is a constant, g_i's are from $G(p)$. By this we mean

(i) $x_{il}\, x_{jl} = 0$ if at least one of them is zero and

(ii) the "Σ" or "$+$" sign merely represents a collection of nonzero product terms so that the R.H.S. says all the group elements appear λ/p times in that collection.

(b) GBRD : Let $G = \{e = g_1, g_2, \ldots, g_p\}$ be a finite group of order p. Let X be a matrix whose non-zero entires are taken from G. Let N be the (0, 1) matrix obtained by replacing every non-zero entry of X by a 1. Then X is a GBRD $(v, b, r, k, \lambda : G(p))$ if

(i) $XX^* = r.I + (\lambda/p)\,(g_1 + \ldots + g_p).\ \ (J - I)$

and (ii) $NN^T = (r - \lambda)I + \lambda.J$

where X^* is obtained from X^T by replacing each non-zeio entry by its inverse.

(c) GDD : A GDD$(v\,; m, n, b, r, k, \lambda_1, \lambda_2)$ is an arrangement of v elements grouped in m groups each containing n elements, in b blocks of size $k < v$ such that

(i) each element appears in exactly r blocks

(ii) each pair of distinct elements of the same group appear together in exactly λ_1 blocks,

(iii) each pair of distinct elements of different groups appear exactly λ_1 blocks.

It is easy to see that the parameters v, b and r can be uniquely determined from m, n, k, λ_1 and λ_2.

If for a GDD, $v = b$ i.e. $r = k$, it is called a symmetric group divisible design or SGDD. If the dual of an SGDD is also another SGDD with the same parameters, then the SGDD is said to posses dual property. This concept of SGDD possessing dual property was introduced by Bose and Shrikhande (1975). Four of the five designs constructed by us are SGDD's possessing dual property.

It is interesting to note that R. M. Wilson's definition of SGDD as given in Marshal Hall, Jr.'s book (1986) is actually Bose and Shrikhande's concept of SGDD possessing dual property with another added condition, viz., $\lambda_1 = 0$. The 4 designs with parametrrs (16, 8, 2, 16, 7, 7, 0, 3), (20, 10, 2, 20, 9, 9, 0, 4), (45, 15, 3, 45, 7, 7, 0, 1) and (38, 19 2, 38, 9, 9, 0, 2) are also SGDD's according to Wilson's definition.

Mukhopadhyay (1988) has developed a few techniques exploiting the properties of SGDD possessing dual property and with $\lambda_1 = 0$ systematically for constructing series of SGDD's and BIBD's.

(d) SIGN : Let G be a group. Let N be a (0, 1) matrix. Suppose the non-zero entries of N are replaced by g, for some $g \in G$, so as to produce $(0, G)$ matrix X, such that

$$XX^* = r.I + \frac{\lambda}{p}(g_1 + \ldots + g_p).\ (J-I)$$

where $G = \{g_1, g_2, \ldots, g_p\}$.
Then N is said to be signable over G.

Any parameter subscripted by G (respectively B) will indicate that it is a parameter of a GDD (respectively BIBD).

3. The algorithm

We may now proceed to describe our algorithm which is concerned with a particular type of GDD for which $\lambda_1 = 0$. Suppose we are given a set of parameters $(V_G, m, n, b_G, r_G, k_G, \lambda_1 = 0, \lambda_2)$. To construct a GDD with these parameters we first need a BIBD $(v_B = v_G/n, b_B = b_G/n, r_B = r_G, k_B = k_G, \lambda_2 n)$. Let N_B be the incidence matrix of this BIBD. Now the algorithm has two phases, (i) to sign this N_B over $G(n)$, a cyclic group of order n, thus getting a GBRD$(v_{GBRD} = v_B,\ b_{GBRD} = b_B,\ r_{GBRD} = r_B,\ k_{GBRD} = k_B, \lambda_{GBRD} = \lambda/n : G(n))$; (ii) to construct the required GDD from this GBRD. We first describe the second phase which is easier and straightforward.

From a GBRD $(v, b, r, k, \lambda : G(n))$ we can form a GDD $(vn, v, n, bn, r, k, \lambda_1 = 0, \lambda_2 = \lambda/n)$ as follows :

Let $G = \{x, x^2, x^3, \ldots, x^n = e\}$ be a cyclic group. Let P_1 be the permutation matrix obtained by rotating the rows of a $n \times n$ identity matrix cyclically one step upward and let $P_i = P_1^i$ for $i = 2, \ldots, n$. Let x^i correspond to P_i for $i = 1, \ldots, n$. If X is the $v \times b$ GBRD, let N be the $vn \times bn$ (0, 1)-matrix obtained from X by replacing any group element by the corresponding permutation matrix and any zero element by a $n \times n$ all zero matrix. Then N is the incidence matrix of a GDD $(vn, v, n, bn, r, k, \lambda_1 = 0, \lambda/n.)$.

Now the first phase is described : In this case the algorithm accepts the matrix N_B, scalars v, b, r, k, λ and group size n, and tries to sign N_B over $G(n)$. N_B should be arranged in the following way. Order the columns of N_B in the lexicographic order, such that the smallest one becomes the leftmost column and the largest one becomes the right most column. Since multiplication by any group element to any row or any column will not change the property (as a GBRD) of a signed matrix, so we may assume without loss of generally that, the first non-zero element in each row and column is 1. So the first row of the signed matrix looks like

$$0\ 0\ \ldots\ 0\ 1\ 1\ \ldots\ 1.$$

By the property of BIBD and by the first condition of GBRD we may assume the structure of the second row as (0, 1) combination 0...0 1... 1 2 ... 2 ... n ... n since permutation of columns will not change the GBRD properties. So the algorithm starts from the third row. It is a back track procedure. Note that the elements of $G(n)$ are denoted by 1, 2, ..., n. We define an ordering on these elements as $1 < 2 < \ldots < n$. We shall proceed to construct X from N_B, row by row replacing 1's in N_B by elements from $G(n)$ subject to the constraint

$$\sum_{l=1}^{v} x_{il}\, x_{jl}^{-1} = \frac{\lambda}{n}(1+2+\ldots+n) \text{ for } i \neq j.$$

The search time in a computer can be shortened by inputting the rows, starting from the third row onward, not as (0,1) row but as a row similar to the second row.

Step by step description of the phase 1 *of the algorithm* : Given a $V \times B$ incidence matrix N of a BIBD whose elements are 0 or 1 and a group of size P, it is required to sign the matrix N by the group elements 1, 2, ... P. This algorithm either outputs the signed matrix or declares that the matrix N can not be signed with the given group of size P. R is the current row and I, J are index variables.

Step 1 : Input the values of P, V, B and the matrix N.

Step 2 : [Initialise] $R \leftarrow 3$

Step 3 : If the R-th row of N is lexicographically largest [i.e.. STEP 4 can not be performed] then go to Step 5.

Step 4 : Replace the R-th row by the next lexicographically higher row Go to Step 6.

Step 5 : Replace the R-th row by the original row. $R \leftarrow R-1$ Go Step 12.

Step 6 : $I \leftarrow R, J \leftarrow 1$

Step 7 : If Row I and Row J are compatible then continue else go to Step 3.

Step 8 : $J \leftarrow J+1$

Step 9 : If $J < R$ then go to Step 7.

Step 10 : $R \leftarrow R+1$ [At this stage the first R rows of the matrix N are signed]

Step 11 : [JOB complete ?] If $R \leqslant V$ go to Step 3 ; otherwise go to Step 14.

Step 12 : If $R > 2$ then go to Step 3.

Step 13 : Output the message that the solution does not exist. Go to Step 15.

Step 14 : Output the matrix N, which contains the required signed matrix.

Step 15 : Terminate the algorithm.

Note that compatible means $\sum_{l=1}^{b} x_{il} \, x_{jl}^{-1} = \frac{\lambda}{p} \, (g_1+g_2+\ldots+g_p)$ for $i \neq j$.

Let us illustrate the algorithm by a small example. Suppose we want to find a GDD ($8 = 4\times 2$, 8, 3, 3, 0, 1). Then we start with a SBIBD (4, 3, 2). The IM of this SBIBD is

$$N_B = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

, We try to sign it over $G(2) = \{1, 2\}$.

The product rule for this group is $1.1 = 2.2 = 1$ and $1.2 = 2.1 = 2$.

So the input matrix of our algorithm will look like

$$N_B = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Replace the 3rd row by 1 1 0 2, and this is compatible with 1st row but not with the 2nd row. So replace the 3rd row by 1 2 0 1, and this is compatible with both 1st row and 2nd row. At this stage the matrix looks like

$$\begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{matrix}$$

So we proceed to 4th row and replace it by 1 1 2 0 which is compatible with all the previous three rows. So this is the required output of the phase 1 of the algorithm. So the GBRD (4, 3, 2 : G(2)) is given by

$$X = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 2 & 0 \end{bmatrix}$$

Now starts the phase 2 of the algorithm :

Take P_1, P_2 and $P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $P_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then the required GDD is given by

$$N_G = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

So there are 8 treatments divided in 4 groups, as {1,2}, {3,4}, {5,6}, {7,8}. The 8 blocks are as follows :

(3,5,7), (4,6,8), (1,6,7), (2,5,8).

(1,3,8), (2,4,8), (1,4,5), (2,3,6).

The proof that the algorithm, if successful, will give the required GDD is obvious and is omitted.

4. The designs

(a) $v = 12$, $m = 6$, $n = 2$, $b = 30$, $r = 10$, $k = 4$, $\lambda_1 = 0$, $\lambda_2 = 3$

Treatment Groups : $\{i, i+1\}$, $i = 1, 3, 5, 7$.

block	treatments			
1	5	7	9	11
2	6	8	10	12
3	3	8	10	12
4	4	7	9	11
5	3	5	9	12
6	4	6	10	11
7	3	5	7	12
8	4	6	8	11
9	3	5	7	10
10	4	6	8	9
11	1	7	9	11
12	2	8	10	12
13	1	6	9	12
14	2	5	10	11
15	1	6	7	12
16	2	5	8	11
17	1	6	7	10
18	2	5	8	9
19	1	3	10	11
20	2	4	9	12
21	1	3	8	11
22	2	4	7	12
23	1	3	8	9
24	2	4	7	10
25	1	4	5	12
26	2	3	6	11
27	1	4	5	10
28	2	3	6	9
29	1	4	5	8
30	2	3	6	7

(b) $v = b = 16, r = k = 7, m = 8, n = 2, \lambda_1 = 0, \lambda_2 = 3$

Treatment Groups : $\{i, i+1\}$, $i = 1, 3, 5, 7, 9, 11, 13, 15$.

block	treatments						
1	3	5	7	9	11	13	15
2	4	6	8	10	12	14	16
3	1	6	8	10	11	13	15
4	2	5	7	9	12	14	16
5	1	3	8	9	12	14	15
6	2	4	7	10	11	13	16
7	1	3	5	10	12	13	16
8	2	4	6	9	11	14	15
9	1	3	6	7	11	14	16
10	2	4	5	8	12	13	15
11	1	4	5	7	10	14	15
12	2	3	6	8	9	13	16
13	1	4	5	8	9	11	16
14	2	3	6	7	10	12	15
15	1	4	6	7	9	12	13
16	2	3	5	8	10	11	14

(c) $v = 20 = b, r = 9 = k, m = 10, n = 2, \lambda_1 = 0, \lambda_2 = 4$

Treatment Groups : $\{i, i+1\}$, $i = 1, 3, 5, \ldots 19$

block	treatments								
1	3	5	7	9	11	13	15	17	19
2	4	6	8	10	12	14	16	18	20
3	1	5	7	9	11	14	16	18	20
4	2	6	8	10	12	13	15	17	19
5	1	3	7	10	12	13	15	18	20
6	2	4	8	9	11	14	16	17	19
7	1	3	5	10	12	14	16	17	19
8	2	4	6	9	11	13	15	18	20
9	1	3	6	8	11	13	16	17	20
10	2	4	5	7	12	14	15	18	19
11	1	3	6	8	9	14	15	18	19
12	2	4	5	7	10	13	16	17	20
13	1	4	5	8	9	12	15	17	20
14	2	3	6	7	10	11	16	18	19
15	1	4	5	8	10	11	13	18	19
16	2	3	6	7	9	12	14	17	20
17	1	4	6	7	9	12	13	16	19
18	2	3	5	8	10	11	14	15	20
19	1	4	6	7	10	11	14	15	17
20	2	3	5	8	9	12	13	16	18

(d) $v = b = 45$, $r = k = 7$, $m = 15$, $n = 3$, $\lambda_1 = 0$, $\lambda_2 = 1$
Treatment Groups : $\{i, \ i+1, i+2\}$, $i = 1, 4, 7, \ldots 43$

block	treatments						
1	13	16	19	25	28	37	43
2	14	17	20	26	29	38	44
3	15	18	21	27	30	39	45
4	10	13	17	22	27	34	40
5	11	14	18	23	25	35	41
6	12	15	16	24	26	36	42
7	7	13	30	31	36	41	44
8	8	14	28	32	34	42	45
9	9	15	29	33	35	40	43
10	7	10	15	20	23	32	37
11	8	11	13	21	24	33	38
12	9	12	14	19	22	31	39
13	4	19	24	27	32	35	44
14	5	20	22	25	33	36	45
15	6	21	23	26	31	34	43
16	4	10	26	28	33	39	41
17	5	11	27	29	31	37	42
18	6	12	25	30	32	38	40
19	4	7	18	22	38	42	43
20	5	8	16	23	39	40	44
21	6	9	17	24	37	41	45
22	4	9	11	16	20	30	34
23	5	7	12	17	21	28	35
24	6	8	10	18	19	29	36
25	1	17	19	23	30	33	42
26	2	18	20	24	28	31	40
27	3	16	21	22	29	32	41
28	1	10	16	31	35	38	45
29	2	11	17	32	36	39	43
30	3	12	18	33	34	37	44
31	1	7	24	25	29	34	39
32	2	8	22	26	30	35	37
33	3	9	23	27	28	36	38
34	1	8	12	20	27	41	43
35	2	9	10	21	25	42	44
36	3	7	11	19	26	40	45
37	1	4	14	21	36	37	40
38	2	5	15	19	34	38	41
39	3	6	13	20	35	39	42
40	1	6	11	15	22	28	44
41	2	4	12	13	23	29	45
42	3	5	10	14	24	30	43
43	1	5	9	13	18	26	32
44	2	6	7	14	16	27	33
45	3	4	8	15	17	25	31

(e) $v = b = 38, r = k = 9, M = 19, n = 2, \lambda_1 = 0, \lambda_2 = 2$

Treatment Groups : $\{i, i+1\}, i = 1, 3, 5, \ldots 37$

block	treatments							
1	11	13	17	19	21	29	31	35	37
2	12	14	18	20	22	30	32	36	38
3	9	14	15	20	23	27	31	33	37
4	10	13	16	19	24	28	32	34	38
5	9	11	15	17	25	28	30	34	36
6	10	12	16	18	26	27	29	33	35
7	5	7	18	20	24	25	28	35	37
8	6	8	17	19	23	26	27	36	38
9	5	7	11	13	23	26	30	32	33
10	6	8	12	14	24	25	29	31	34
11	3	8	16	19	22	25	30	33	37
12	4	7	15	20	21	26	29	34	38
13	3	8	9	14	21	26	28	32	35
14	4	7	10	13	22	25	27	31	36
15	3	5	16	18	21	23	31	34	36
16	4	6	15	17	22	24	32	33	35
17	3	5	9	11	22	24	27	29	38
18	4	6	10	12	21	23	28	30	37
19	3	6	7	10	11	14	15	18	19
20	4	5	8	9	12	13	16	17	20
21	1	7	14	16	17	21	24	27	30
22	2	8	13	15	18	22	23	28	29
23	1	7	9	12	19	22	23	34	35
24	2	8	10	11	20	21	24	33	36
25	1	6	11	16	20	22	26	28	31
26	2	5	12	15	19	21	25	27	32
27	1	6	9	13	18	21	25	33	38
28	2	5	10	14	17	22	26	34	37
29	1	5	8	10	15	30	31	35	38
30	2	6	7	9	16	29	32	36	37
31	1	3	12	13	15	24	26	36	37
32	2	4	11	14	16	23	25	35	38
33	1	3	10	17	20	23	25	29	32
34	2	4	9	18	19	24	26	30	31
35	1	4	8	11	18	27	32	34	37
36	2	3	7	12	17	28	31	33	38
37	1	4	5	14	19	28	29	33	36
38	2	3	6	13	20	27	30	34	35

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Paper received : *September*, 1988.

Revised : *March*, 1989.