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A MODEL FOR AF ALGEBRAS AND A REPRESENTATION OF THE JONES PROJECTIONS

V. S. SUNDER

A model for approximately finite-dimensional (henceforth abbreviated to AF) algebras is developed here, which may be looked upon as a matrix-theoretic/ergodic theoretic alternative to the model developed in [4]. One advantage of this model is that it leads directly to a certain Borel space and a canonical (tail-) equivalence relation on it, which underlies the GNS representation of the AF-algebra associated with any trace on the algebra that factors through the conditional expectation onto an appropriate Cartan subalgebra.

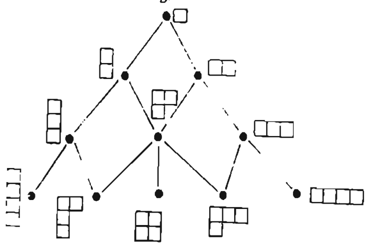
As an application of this model, we construct a representation of a sequence $\{e_n\}$ of projections in the hyperfinite II_1 factor which satisfy: $e_n e_m = e_m e_n$ if $|m-n|>1$, $e_n e_{n\pm 1} e_n = \tau e_n$, and $\mathrm{tr}(we_n) = \tau\,\mathrm{tr}\,w$ for any word w in $1, e_1, \ldots, e_{n-1}$ — where τ^{-1} is the Perron-eigenvalue of a primitive (in the sense of the Perron-Frobenius theory) matrix of the form AA^t where A is a non-negative matrix with non-negative integral entries. Such sequences were encountered in [1] and it is the author's belief that the model developed here could be used in the problem of constructing subfactors of the hyperfinite II_1 factor with trivial relative commutant and index τ^{-1} with τ as above. We obtain explicit formulae for these projections by applying our model to the AF-algebra resulting from an application of what Jones calls his "basic construction" to a pair of finite-dimensional C^*-algebras with inclusion matrix A.

We begin by reviewing some basic facts concerning inclusions of finite-dimensional C^*-algebras, and by setting up the notation to be used in the sequel. Recall that any finite-dimensional C^*-algebra N is of the form $N \cong N_1 \oplus N_2 \oplus \cdots \oplus N_n$, where $N_i \cong M(n_i, \mathbf{C})$; the vector $\mathbf{n} = (n_1, \ldots, n_n)^t$ will be called the dimension-vector of N — it is uniquely determined, up to a permutation, by N. If $N \subset M$ is a unital inclusion of finite-dimensional C^*-algebras, where $M \cong M_1 \oplus \ldots \oplus M_m$, with $M_j \cong M(m_j, \mathbf{C})$, the associated inclusion matrix $A = A_N^M$ is the $n \times m$ $\mathbf{Z}_+$-valued matrix with A_{ij} = the number of simple components of a simple M_j-module when viewed as an N_i-module. (The matrix A is uniquely determined once one has chosen ordered partitions of unity $\{p_1, \ldots, p_n\}$ and $\{q_1, \ldots q_m\}$ into minimal central projections of N and M respectively.) The dimension vectors $\mathbf{n}$ and $\mathbf{m}$ then satisfy $\mathbf{m} = A^t\mathbf{n}$, where A^t denotes the transpose of A.

With M as above, there is a bijective correspondence between faithful traces τ on M and strictly positive vectors $\mathbf{t}$ in $\mathbf{R}^m$, the correspondence being given by $\tau(x_1 \oplus \ldots \oplus x_m) = \sum_i t_i \operatorname{tr} x_i$, where 'tr' denotes the usual trace on matrix algebras. It is known that if a trace τ on M corresponds to $\mathbf{t}$ in $\mathbf{R}^m_+$ and if a trace σ on N corresponds to $\mathbf{s}$ in $\mathbf{R}^n_+$, then $\tau/N = \sigma$ iff $\mathbf{s} = \Lambda\mathbf{t}$.

Remark. For the reader who is more comfortable with Bratteli diagrams, it might be worth mentioning that as far as book-keeping devices go, the Bratteli diagram and the inclusion matrices are equivalent; thus, for instance, if M_n denotes the group algebra of the symmetric group S_n on n letters, the two equivalent ways of describing the tower $M_1 \subsetneq M_2 \subsetneq M_3 \subsetneq M_4$ are:

Diagram



Matrices

$$[1 \quad 1]$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

(Note that multiple edges in the diagram would correspond to entries larger than one in the inclusion matrices.)

Suppose now that $M_1 \subset M_2 \subset \ldots$ (*) is an ascending chain of finite-dimensional C^*-algebras. Once and for all, choose and fix ordered partitions of unity $\{p_1^{(n)}, \ldots, p_{v_n}^{(n)}\}$ into minimal central projections in M_n. With respect to this choice, let us write $\Lambda^{(n)}$ for the inclusion matrix $\Lambda_{M_n}^{M_{n+1}}$. Thus, if $\mathbf{m}^{(n)}$ is the dimension vector of M_n — so that $\dim p_j^{(n)} M_n = (m_j^{(n)})^2$ — we have $\mathbf{m}^{(n+1)} = (\Lambda^{(n)})^t \mathbf{m}^{(n)}$, and in particular, $\mathbf{m}_1$ and $\{\Lambda^{(n)}: n \geqslant 1\}$ determine $\mathbf{m}^{(n)}$ for all $n \geqslant 1$.

Our aim, now, shall be to start with the data $\{m^{(1)}, \Lambda^{(1)}, \Lambda^{(2)}, \ldots\}$ and build a model of an AF-algebra with this data. Specifically, we assume that the following data are given:

(a) a sequence $\{v_n : n \geqslant 1\}$ of positive integers;

(b) a vector $\mathbf{m}^{(1)}$ in $\mathbf{R}^{v_1}$ with positive integral coordinates; and

(c) a sequence $\{\Lambda^{(n)}: n \geqslant 1\}$, where, $\Lambda^{(n)}$ is a non-zero $v_n \times v_{n+1}$ matrix with non-negative integral entries.

As above, we have a sequence $\{\mathbf{m}^{(n)} : n \geqslant 1\}$ defined by $\mathbf{m}^{(n)} = (\Lambda^{(1)} \Lambda^{(2)} \ldots \Lambda^{(n-1)})^t \mathbf{m}^{(1)}$. The starting point for the construction is a certain space of sequences.

DEFINITION 1. With ν_n, $\Lambda^{(n)}$, $\mathbf{m}^{(1)}$ as above, define the *associated sequence-space* Ω as follows:

$$\Omega = \{\alpha \in \mathbf{Z}_+^{\mathbf{Z}_+} : 1 \leqslant \alpha_{2n} \leqslant \nu_n,\ 1 \leqslant \alpha_1 \leqslant m^{(1)}_{\alpha_2},\ 1 \leqslant \alpha_{2n+1} \leqslant \Lambda^{(n)}_{\alpha_{2n},\alpha_{2n+2}} \text{ for all } n \geqslant 1\},$$

where, of course, $\mathbf{Z}_+ = \{1, 2, \ldots\}$. ▨

The following notation will be handy in the future: for any subset I of $\mathbf{Z}_+$, we shall write $\alpha \to \alpha_I$ for the restriction mapping $\Omega \to \mathbf{Z}_+^I$; thus, for instance, $\alpha_{[2,4]} = (\alpha_2, \alpha_3, \alpha_4)$; we shall also write $\alpha_{n]}$ for $\alpha_{[1,n]}$, $\alpha_{n)}$ for $\alpha_{[1,n)}$, $\alpha_{(n}$ for $\alpha_{(n,\infty)}$ and $\alpha_{[n}$ for $\alpha_{[n,\infty)}$. We shall write Ω_I for the set $\{\alpha_I : \alpha \in \Omega\}$. One last bit of notation: if $\{I_1, \ldots, I_k\}$ is a partition of $\mathbf{Z}_+$, and if $\gamma_i \in \Omega_{I_i}$ for $1 \leqslant i \leqslant k$, and if there exists $\alpha \in \Omega$ such that $\alpha_{I_i} = \gamma_i$ for $1 \leqslant i \leqslant k$, we shall write $\gamma_1 * \ldots * \gamma_k$ for α.

Now consider the (in general, non-separable) Hilbert space $l^2(\Omega)$ of square-summable functions on Ω; denote the canonical orthonormal basis by $\{\xi_\beta : \beta \in \Omega\}$. (Thus, $\xi_\beta(\alpha) = \delta_{\alpha,\beta}$, where δ denotes the Kronecker symbol.) Each (bounded) operator x on $l^2(\Omega)$ corresponds uniquely to its matrix $((x(\alpha,\beta)))_{\alpha,\beta \in \Omega}$, where, of course, $x(\alpha,\beta) = \langle x\xi_\beta, \xi_\alpha \rangle$ for every α and β in Ω.

For $n = 1, 2, \ldots$, define M_n to be the set of operators x on $l^2(\Omega)$ whose matrices satisfy the following conditions:

(i) $x(\alpha,\beta) = 0$ unless $\alpha_{[2n} = \beta_{[2n}$; and

(ii) $x(\alpha,\beta) = x(\alpha',\beta')$ whenever $\alpha, \beta, \alpha', \beta' \in \Omega$ satisfy

$$\alpha_{[2n} = \beta_{[2n},\quad \alpha'_{[2n} = \beta'_{[2n},\quad \alpha_{2n]} = \alpha'_{2n]} \quad \text{and} \quad \beta_{2n]} = \beta'_{2n]}.$$

In other words, $x \in M_n$ iff there is a function $x_{2n]} : \Omega_{2n]} \times \Omega_{2n]} \to \mathbf{C}$ satisfying

$$\text{(1)} \qquad x(\alpha,\beta) = \delta_{\alpha_{[2n},\beta_{[2n}}\, x_{2n]}(\alpha_{2n]}, \beta_{2n]}) \quad \forall\, \alpha, \beta \in \Omega.$$

PROPOSITION 2. (a) *Each M_n is a finite-dimensional C^*-algebra of operators;*

(b) *$M_n \subset M_{n+1}$ for all $n \geqslant 1$;*

(c) *if x' is an operator on $l^2(\Omega)$, then $x' \in M_n'$ iff there exists a bounded measurable function $x'_{[2n} : \Omega_{[2n} \times \Omega_{[2n} \to \mathbf{C}$ such that*

$$x'(\alpha,\beta) = \delta_{\alpha_{2n]},\beta_{2n]}}\, x'_{[2n}(\alpha_{[2n}, \beta_{[2n}) \quad \textit{for all } \alpha, \beta \in \Omega;$$

(d) *if $x \in \mathcal{L}(l^2(\Omega))$, and if $n \leqslant m$, then $x \in M_m \cap M_n'$ iff there exists a function $x_{[2n,2m]} : \Omega_{[2n,2m]} \times \Omega_{[2n,2m]} \to \mathbf{C}$ such that*

$$x(\alpha,\beta) = \delta_{\alpha_{2n]},\beta_{2n]}}\, \delta_{\alpha_{[2m},\beta_{[2m}}\, x_{[2n,2m]}(\alpha_{[2n,2m]}, \beta_{[2n,2m]}).$$

for all α, β in Ω; in particular, $x \in Z(M_n)$ iff there exists a function $x_{[2n]} : \Omega_{[2n]} \times \Omega_{[2n]} \to \mathbf{C}$ such that

$$x(\alpha,\beta) = \delta_{\alpha,\beta}\, x_{[2n]}(\alpha_{2n},\beta_{2n}) \quad \forall\ \alpha,\beta \text{ in } \Omega$$

(and consequently, $Z(M_n)$ is v_n-dimensional);

(c) *for each $n \geq 1$ and $1 \leqslant j \leqslant v_n$, define projections $p_j^{(n)}$ in $Z(M_n)$ by $p_j^{(n)}(\alpha, \beta) = \delta_{\alpha,\beta}\delta_{j,\alpha_{2n}}$; then, $\{p_1^{(n)}, \ldots, p_{v_n}^{(n)}\}$ is a partition of 1 into minimal central projections of M_n;*

(f) *with respect to $\{p_1^{(n)}, \ldots, p_{v_n}^{(n)}\}$ and $\{p_1^{(n+1)}, \ldots, p_{v_{n+1}}^{(n+1)}\}$, the inclusion matrix $\Lambda_{M_n}^{M_{n+1}}$ is precisely the matrix $\Lambda^{(n)}$.*

Proof. (a) & (b). It is clear from the definition that $M_n \subset M_{n+1}$ and that M_n is a self-adjoint vector space of operators: to verify that M_n is an algebra, if $z = xy$, with $x, y \in M_n$ and if $\alpha, \beta \in \Omega$, we have

$$z(\alpha, \beta) = \sum_{\gamma \in \Omega} x(\alpha, \gamma)\, y(\gamma, \beta) =$$

$$= \sum_{\gamma \in \Omega} \delta_{\alpha_{[2n},\gamma_{[2n}}\, \delta_{\gamma_{[2n},\beta_{[2n}}\, x_{2n]}(\alpha_{2n]}, \gamma_{2n]})\, y_{2n]}(\gamma_{2n]}, \beta_{2n]}) =$$

$$= \delta_{\alpha_{[2n},\beta_{[2n}} \sum_{\{\gamma \in \Omega : \gamma_{[2n} = \alpha_{[2n}\}} x_{2n]}(\alpha_{2n]}, \gamma_{2n]})\, y_{2n]}(\gamma_{2n]}, \beta_{2n]});$$

notice now that the sum, although seeming to depend upon $\alpha_{[2n}$, actually does not, since

$$\sum_{\{\gamma \in \Omega : \gamma_{[2n} = \alpha_{[2n}\}} f(\gamma_{2n]}) = \sum_{\{\theta \in \Omega_{2n]} : \theta_{2n} = \alpha_{2n}\}} f(\theta),$$

for any function f defined on $\Omega_{2n]}$.

Finally, M_n is finite-dimensional, since it has a finite basis given by $\{u_{\gamma,\varkappa} : \gamma, \varkappa \in \Omega_{2n]},\ \gamma_{2n} = \varkappa_{2n}\}$, where

$$(2) \qquad u_{\gamma,\varkappa}(\alpha, \beta) = \delta_{\alpha_{[2n},\beta_{[2n}}\, \delta_{\gamma,\alpha_{2n]}}\, \delta_{\varkappa,\beta_{2n]}}$$

(c) Let $x' \in \mathscr{L}(\ell^2(\Omega))$, and let $\{u_{\gamma,\varkappa} : \gamma, \varkappa \in \Omega_{2n]}\}$ be as in (2) above. Then, or any α, β in Ω, we have

$$(x'u_{\gamma,\varkappa})(\alpha, \beta) = \sum_{\theta \in \Omega} x'(\alpha, \theta)\delta_{\theta_{[2n},\beta_{[2n}}\, \delta_{\gamma,\theta_{2n]}}\, \delta_{\varkappa,\beta_{2n]}} =$$

$$= \delta_{\varkappa,\beta_{2n]}}\, x'(\alpha, \gamma \circ \beta_{[2n});$$

(although the concatenation $\gamma * \beta_{(2n}$ may be inadmissible if $\gamma_{2n} \neq \beta_{2n}$, note that the right side is non-zero only when $x = \beta_{2n]}$, in which case, we have $\gamma_{2n} = x_{2n} = \beta_{2n}$ and there is no problem). A similar computation shows that

$$(u_{\gamma,x} x')(\alpha, \beta) = \delta_{\alpha_{2n]},\gamma} x'(x * \alpha_{(2n}, \beta).$$

Hence, $x' \in M_n'$ iff x' commutes with $u_{\gamma,x}$ for each γ, x in $\Omega_{2n]}$ satisfying $\gamma_{2n} = x_{2n}$, which happens iff $\delta_{\alpha_{2n]},\gamma} x'(x * \alpha_{(2n}, \beta) = \delta_{x,\beta_{2n]}} x'(\alpha, \gamma * \beta_{(2n})$ for every α, β in Ω and for every γ, x as above; it is not very hard now to deduce (c).

(d) and (e) are fairly easy consequences of (c).

(f) With $\{p_i^{(n)} : 1 \leqslant i \leqslant \nu_n\}$ and $\{p_j^{(n+1)} : 1 \leqslant j \leqslant \nu_{n+1}\}$ as in (e), note that $\Lambda_{M_n}^{M_{n+1}}(i, j)$ is the maximum number of pairwise orthogonal non-zero projections in $(M_{n+1} \cap M_n')p_i^{(n)}p_j^{(n+1)}$; it is easily seen (using the description of $M_{n+1} \cap M_n'$ given by (d)) that such a collection is given by $\{q_k : 1 \leqslant k \leqslant \Lambda_{ij}^{(n)}\}$, where

$$q_k(\alpha, \beta) = \delta_{\alpha,\beta}\delta_{i,x_{2n}}\delta_{k,x_{2n+1}}\delta_{j,x_{2n+2}}. \qquad ■$$

Let $M_1 \subset M_2 \subset \ldots$ be as above, and let us write M_∞ for $\bigcup M_n$. We shall denote by C the collection of operators in M_∞ which have a diagonal matrix with respect to the canonical basis of $\ell^2(\Omega)$; thus, $C = \{x \in M_\infty : x(\alpha, \beta) = \delta_{\alpha,\beta}\varphi(\alpha)$ for some bounded function φ on $\Omega\}$. It is fairly clear that C is an abelian $*$-subalgebra of M_∞; in fact, if we let $C_n = C \cap M_n$, then C_n is a maximal abelian C^*-subalgebra of M_n and there is a natural identification: $C_n \cong \ell^\infty(\Omega_{2n]})$. It is also clear that the map $E : M_\infty \to C$ given by $(Ex)(\alpha, \beta) = \delta_{\alpha,\beta} x(\alpha, \alpha)$ defines a conditional expectation of M_∞ onto C.

PROPOSITOIN 3. (a) *Let φ be a state on M_∞. Then there is a unique probability measure μ defined on the Borel sets of Ω such that*

$$\varphi(x) = \int x(\alpha, \alpha)\, d\mu(\alpha) \quad \textit{for all } x \textit{ in } C. \tag{3}$$

(b) *If μ is a probability measure defined on the Borel sets of Ω, there is a unique state φ on M_∞ which satisfies both* (3) *and the condition $\varphi = \varphi \circ E$. (Thus, equation* (3) *sets up a bijection between probability measures μ on Ω and states φ which satisfy $\varphi = \varphi \circ E$.)*

Proof. Since $C_n \cong \ell^\infty(\Omega_{2n]})$, it follows – by considering φ/C_n – that for each n, there is a unique probability measure μ_n defined on the subsets of $\Omega_{2n]}$ such that $\varphi(x) = \int_{\Omega_{2n]}} x_{2n]}(\gamma, \gamma)\, d\mu_n(\gamma)$ for all x in C_n. Since $(\varphi/C_{n+1})/C_n = \varphi/C_n$, it follows that the sequence of measures $\{\mu_n\}$ is consistent in the sense that if $F \subset \Omega_{[2n]}$, and if $F^{\sim} = \{\alpha \in \Omega_{[2n+2]} : \alpha_{[2n]} \in F\}$, then $\mu_{n+1}(F^{\sim}) = \mu_n(F)$. It follows now from Kolmogorov's consistency theorem that there is a unique probability measure μ on Ω such that for each $n \geq 1$, and for every $F \subset \Omega_{[2n]}$, $\mu(\{\alpha \in \Omega : \alpha_{[2n]} \in F\}) = \mu_n(F)$; it follows easily that this μ satisfies (3).

(b) Any probability measure μ on Ω defines a state φ_0 on C via equation (3); ust let $\varphi = \varphi_0 \circ E$. ☐

We shall now consider the GNS-representation π_φ associated with a state φ on M_∞ which satisfies $\varphi = \varphi \circ E$. Let μ be the probability measure on Ω which is associated with φ as in Proposition 3. We shall see that $\pi_\varphi(M_\infty)''$ may be naturally identified with the groupoid-von Neumann algebra associated with $(R, \mu^{\sim})$, where R is the "tail-equivalence relation" on Ω and $\mu^{\sim}$ is a measure on R obtained using μ and counting measure on the orbits.

To be precise, let us define

$$R = \{(\alpha, \beta) \in \Omega \times \Omega : \exists\, n \geq 1 \text{ such that } \alpha_{(2n} = \beta_{(2n}\}.$$

Clearly R defines an equivalence relationon Ω which is Borel — in fact, R is an F_σ subspace of the Polish space $\Omega \times \Omega$. Let $\mu^{\sim}$ be the measure defined on the Borel subsets of R by

$$\mu^{\sim}(F) = \int \Big(\sum_{\beta \in \Omega} 1_F(\beta, \alpha)\Big)\, d\mu(\alpha).$$

(Here and elsewhere, the symbol 1_F will denote the indicator- or characteristic function of F. Notice that since R-equivalence classes are countable, there are no measurability problems.) The measure $\mu^{\sim}$ is a positive σ-finite measure, since R is exhausted by the increasing sequence $\{F_n\}$ of sets of finite measure, given by $F_n = \{(\alpha, \beta) \in R : \alpha_{(2n} = \beta_{(2n}\}$.

For each x in M_∞, denote by $\eta(x)$ the function defined on R by $\eta(x)(\alpha, \beta) = \langle x\xi_\beta, \xi_\alpha\rangle$. It follows from the definition of M_n in terms of matrix-entries that if $x \in M_n$, then $\eta(x)$ is supported on the set F_n defined in the last paragraph and that $\eta(x)$ is a bounded function. It is obvious that η is an injective linear map from M_∞ onto $\mathscr{U} = \eta(M_\infty) \subset L^2(R, \mu^{\sim})$; hence $\mathscr{U}$ becomes an associative algebra with involution, with respect to the operations defined by $(\xi \cdot \eta)(\alpha, \beta) = \sum_\gamma \xi(\alpha, \gamma)\eta(\gamma, \beta)$ and $\xi^*(\alpha, \beta) = \overline{\xi(\beta, \alpha)}$ for all ξ, η in $\mathscr{U}$.

Proposition 4. (a) *$\mathscr{U}$ is a left Hilbert algebra with respect to the above algebra structure and the inner product coming from $L^2(R, \mu^{\sim})$;*

(b) *the equation $\pi(x)\xi = \eta(x)\cdot\xi$, $\xi \in L^2(R, \mu^{\sim})$, defines a representation π of M_∞ in $L^2(R, \mu^{\sim})$;*

(c) *$\pi(M_\infty)''$ is the left von Neumann algebra of $\mathscr{U}$;*

(d) *let* ξ_0 *be the unit vector given by* $\xi_0(\alpha, \beta) = \delta_{\alpha,\beta}$; *then* ξ_0 *is a cyclic and separating vector for* $\pi(M_\infty)$ *such that* $\varphi(x) = \langle \pi(x)\xi_0, \xi_0\rangle$ *for all* x *in* M_∞ *— so that this* π *is the* GNS *representation of* M_∞ *associated with* φ.

Proof. Since $\varphi = \varphi \circ E$, it follows that for x in M_∞,

$$\varphi(x) = \varphi(Ex) = \int_{\Omega} x(\alpha, \alpha)\, d\mu(\alpha)$$

and consequently, for any x, y in M_∞,

$$\varphi(y^*x) = \int_{\Omega} (y^*x)(\alpha, \alpha)\, d\mu(\alpha) = \int_{\Omega} (\sum_{\beta \in \Omega} \overline{y(\beta, \alpha)}\, x(\beta, \alpha))\, d\mu(\alpha) =$$

$$= \int_{R} \eta(x)\, \overline{\eta(y)}\, d\mu^{\sim} = \langle \eta(x), \eta(y)\rangle;$$

further, for any x, y in M_∞ and $(\alpha, \beta) \in R$,

$$(\eta(xy))(\alpha, \beta) = \sum_{\{\gamma \in \Omega : (\alpha, \gamma) \in R\}} x(\alpha, \gamma) y(\gamma, \beta) = (\pi(x)\eta(y))(\alpha, \beta)$$

and hence $\eta(xy) = \pi(x)\eta(y)$.

Finally, for each $n \geq 1$, let $\mathscr{F}_n$ be the σ-algebra of sets in Ω that is generated by the maps $\{\alpha \to \alpha_j : 1 \leq j \leq 2n\}$: then the Borel σ-algebra $\mathscr{F}$ is generated by $\bigcup \mathscr{F}_n$ so that also the Borel σ-algebra of $\Omega \times \Omega$ — which is just $\mathscr{F} \otimes \mathscr{F}$ — is generated by $\bigcup (\mathscr{F}_n \otimes \mathscr{F}_n)$; it follows that if K is any Borel set in $\Omega \times \Omega$, the reduced σ-algebra $(\mathscr{F} \otimes \mathscr{F})/K$ $(= \{F \cap K : F \in \mathscr{F} \otimes \mathscr{F}\})$ is generated by $\bigcup (\mathscr{F}_n \otimes \otimes \mathscr{F}_n)/K$; hence if $F_n = \{(\alpha, \beta) \in R : \alpha_{[2n} = \beta_{[2n}\}$ as before, it is not hard to deduce that $\bigcup_{m,n=1}^{\infty} L^2(F_n, (\mathscr{F}_m \otimes \mathscr{F}_m)/F_n, \mu^{\sim})$ is dense in $L^2(R, \mu^{\sim})$. Notice now that if $k = \max\{m, n\}$, then $L^2(F_n, (\mathscr{F}_m \otimes \mathscr{F}_m)/F_n, \mu^{\sim}) \subset \eta(M_k) \subset \mathscr{U}$ and so $\mathscr{U}$ is dense in $L^2(R, \mu^{\sim})$. (In fact, $\eta(M_n) = L^2(F_n, (\mathscr{F}_n \otimes \mathscr{F}_n)/F_n, \mu^{\sim})$ and hence the above double-union is exactly equal to $\mathscr{U}$.)

All the assertions of the proposition may now be easily deduced from what has been established so far. ▨

We shall now consider traces on M_∞. Suppose that φ is a faithful tracial state on M_∞. Let $t^{(n)}$ be the positive vector in $\mathbf{R}_+^{V_n}$ which corresponds to the trace φ/M_n; thus, if $x \in M_n$

$$(4) \qquad \varphi(x) = \sum_{\gamma \in \Omega_{2n]}} t^{(n)}_{\gamma_{2n}} x_{2n]}(\gamma_{2n]}, \gamma_{2n]});$$

this equation shows that $\varphi = \varphi_0 \circ E$ and so φ corresponds to a unique probability measure μ as in Proposition 3. Further, we also know that $\mathbf{t}^{(n)} = \Lambda^{(n)}\mathbf{t}^{(n+1)}$.

In the converse direction, it is clear that if $\{\mathbf{t}^{(n)}\}$ is a sequence satisfying

(i) $\mathbf{t}^{(n)}$ is a strictly positive vector in $\mathbf{R}_+^{\nu_n}$, and

(ii) $\Lambda^{(n)}\mathbf{t}^{(n+1)} = \mathbf{t}^{(n)}$, for all $n \geqslant 1$,

then there is a uniquely defined faithful tracial state φ on M_∞ such that φ/M_n corresponds to $\mathbf{t}^{(n)}$. For convenience of reference, we include the following fairly well-known result.

LEMMA 5. *Let Λ be a $\nu \times \nu$ matrix with non-negative integral entries, and such that Λ is primitive in the sense that Λ^k has strictly positive entries for some $k \geqslant 1$. Let M_∞ be an* AF-*algebra for which $\Lambda^{(n)} = \Lambda$ for every $n \geqslant 1$. Then there is a unique tracial state φ on M_∞; further φ is faithful. In particular, $(\pi_\varphi(M))''$ is the hyperfinite* II$_1$ *factor, where of course π_φ denotes the* GNS *representation of M_∞ associated with φ.*

Proof. It follows from the standard Perron-Frobenius theory that if λ is the spectral radius of Λ, there is a strictly positive vector $\mathbf{t}^{(1)}$ in $\mathbf{R}^\nu$ such that $\Lambda\mathbf{t}^{(1)} = \lambda\mathbf{t}^{(1)}$. Now define $\mathbf{t}^{(n)} = \lambda^{1-n}\,\mathbf{t}^{(1)}$ and note that $\Lambda\mathbf{t}^{(n+1)} = \mathbf{t}^{(n)}$ for all n. Let $\mathbf{m}^{(1)} \in \mathbf{Z}_+^\nu$ be arbitrary. Assume that $\mathbf{t}^{(1)}$ has been so normalised that $\sum_{j=1}^{\nu} t_j^{(1)} m_j^{(1)} = 1$; this ensures that the trace φ on M_∞ that is induced by the sequence $\{\mathbf{t}^{(n)}\}$ is a state. Further the strict positivity of $\mathbf{t}^{(n)}$ for each n implies that φ is faithful.

If $\varphi^\sim$ is another tracial state and if $\mathbf{t}^{\sim(n)}$ is the vector in $\mathbf{R}_+^\nu$ which corresponds to $\varphi^\sim/M_n$, it follows that $\mathbf{t}^{\sim(n)} \in \bigcap_{k>0} \Lambda^k\mathbf{R}_+^\nu$, since $\mathbf{t}^{\sim(n)} = \Lambda^k\mathbf{t}^{\sim(n+k)}$ for every n and k; on the other hand, it is a consequence of the primitivity of Λ that $\bigcap_{k>0} \Lambda^k\mathbf{R}_+^\nu = \mathbf{R}_+\mathbf{t}^{(1)}$; deduce that $\mathbf{t}^{\sim(n)} = \alpha_n\mathbf{t}^{(n)}$ for some positive scalar α_n; since $\Lambda\mathbf{t}^{(n)} = \mathbf{t}^{(n-1)}$ and $\Lambda\mathbf{t}^{\sim(n)} = \mathbf{t}^{\sim(n-1)}$, conclude that all the α_n are equal and therefore $\varphi^\sim = \varphi$. The fact that there is a unique tracial state on M_∞ clearly implies that $\pi_\varphi(M_\infty)''$ is a factor of finite type; the primitivity of Λ guarantees the infinite-dimensionality of M_∞ and the proof is complete. ∎

NOTE. (a) There is an obvious minor generalisation of the preceding lemma: if M_∞ is built out of the data $(\mathbf{m}^{(1)}, \Lambda^{(n)} : n \geqslant 1)$, if the sequence $\{\Lambda^{(n)}\}$ is periodic – i.e., there is a $k \geqslant 1$ such that $\nu_{n+k} = \nu_n$ and $\Lambda^{(n+k)} = \Lambda^{(n)}$ for every n – and if $(\Lambda^{(1)} \ldots \Lambda^{(k)})$ is primitive in the sense of the lemma, then M_∞ admits a unique tracial state which is automatically faithful. (Reason: $M_\infty = \bigcup M_n^\sim$ where $M_n^\sim = M_{kn}$ and the lemma applies.)

(b) The argument in the lemma also shows how to construct AF-algebras which do not admit any faithful tracial state; for instance, let $\nu_n = 2$ for every n and let $\Lambda^{(n)} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, and note that $\bigcap_{n>1} \Lambda^n\mathbf{R}_+^2 = \mathbf{R}_+ \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and so if a trace φ on M_∞

corresponds to the sequence $\{t^{(n)}\}$, then $t^{(n)} = \begin{bmatrix} a_n \\ 0 \end{bmatrix}$ for some $a_n \geqslant 0$ for all n, so that φ is not faithful. ∎

Henceforth, we shall assume that:

(i) $\nu_{2n+1} = \nu_1$ and $\nu_{2n} = \nu_2$ for all n;

(ii) $\Lambda^{(2n+1)} = \Lambda$ and $\Lambda^{(2n)} = \Lambda^t$ where Λ is a fixed $\nu_1 \times \nu_2$ matrix with non-negative integral entries such that $\Lambda\Lambda^t$ is primitive in the sense of the preceding lemma with Perron eigenvalue and eigenvector denoted by λ and $t^{(1)}$ respectively:

(iii) $t^{(2n+1)} = \lambda^{-n} t^{(1)}$ and $t^{(2n)} = \Lambda^t t^{(2n+1)}$ for all n;

(iv) φ is the faithful trace on M_∞ associated with $\{t^{(n)}\}$;

(v) Ω is the associated sequence space;

(vi) μ is the measure on Ω associated with φ; and

(vii) $R \subset \Omega \times \Omega$ as in Proposition 4.

Hence, by the last lemma and Proposition 4, the left von Neumann algebra associated with $\mathscr{U}$ as in Proposition 4 is the hyperfinite II_1 factor. The reason for our interest in this special case is that this is precisely the situation that is encountered when one applies Jones' "basic construction" to the inclusion $M_1 \overset{\Lambda}{\subset} M_2$. In the next proposition, we give explicit formulae for the resulting sequence $\{e_n : n \geqslant 1\}$ of projections in M_∞ which satisfy the relations

$$e_i e_j = e_j e_i \quad \text{if } |i-j| > 1, \quad \text{and} \quad e_i e_{i\pm 1} e_i = \lambda^{-1} e_i \quad \text{for all } i.$$

PROPOSITION 6. *For $n = 1, 2, \ldots,$ define the elements e_n in M_∞ — by their matrix-coefficients $e_n(\alpha, \beta)$ — as follows:*

$$e_n(\alpha, \beta) = \delta_{\alpha_{[1,2n]}, \beta_{[1,2n]}} \delta_{\alpha_{[2n+4,\infty)}, \beta_{[2n+4,\infty)}} \delta_{\alpha_{2n}, \alpha_{2n+4}} \delta_{\alpha_{2n+1}, \alpha_{2n+3}} \delta_{\beta_{2n+1}, \beta_{2n+3}} \times$$

$$\times (t^{(n+1)}_{\alpha_{2n+2}} t^{(n+1)}_{\beta_{2n+2}})^{1/2} / t^{(n)}_{\alpha_{2n}};$$

(of course, it is assumed we that are in the situation described by (i)—(vii) *above) then $\{e_n\}$ is a sequence of projections in M_∞ which satisfy the following:*

(a) $e_m e_n = e_n e_m$ *if* $|m-n| > 1$;

(b) $e_n e_{n\pm 1} e_n = \lambda^{-1} e_n$ $\forall n \geqslant 1$;

(c) $\varphi(e_n x) = \lambda^{-1} \varphi(x)$ *whenever* $x \in M_{n+1}$.

Proof. To start with, note from the definition of e_n and from Proposition 2(d) that $e_n \in M_{n+2} \cap M_n'$ for each n and so (i) is immediate. Also, $e_n(\alpha, \beta) = e_n(\beta, \alpha) \in \mathbf{R}$

so that $e_n = e_n^*$. Now compute:

$$e_n^2(\alpha,\beta) = \sum_{\gamma\in\Omega} e_n(\alpha,\gamma)e_n(\gamma,\beta) =$$

$$= \delta_{\alpha_{2n},\alpha_{2n+4}}\delta_{\alpha_{2n+1},\alpha_{2n+3}}\delta_{\beta_{2n+1},\beta_{2n+3}}\delta_{\alpha_{2n]},\beta_{2n]}}\delta_{\alpha_{[2n+4},\beta_{[2n+4}} \times$$

$$\times \sum_{\substack{\{\gamma\in\Omega:\gamma_{2n]}=\alpha_{2n]},\gamma_{[2n+4}=\alpha_{[2n+4}\}\\ \gamma_{2n+1}=\gamma_{2n+3}}} (t^{(n+1)}_{\alpha_{2n+2}} t^{(n+1)}_{\beta_{2n+2}})^{1/2}\cdot\frac{t^{(n+1)}_{\gamma_{2n+2}}}{t^{(n)}_{\alpha_{2n}} t^{(n)}_{\beta_{2n}}} =$$

$$= \delta_{\alpha_{2n},\alpha_{2n+4}}\delta_{\alpha_{2n+1},\alpha_{2n+3}}\delta_{\beta_{2n+1},\beta_{2n+3}}\delta_{\alpha_{2n]},\beta_{2n]}}\delta_{\alpha_{[2n+4},\beta_{[2n+4}} \times$$

$$\times \frac{(t^{(n+1)}_{\alpha_{2n+2}} t^{(n+1)}_{\beta_{2n+2}})^{1/2}}{t^{(n)}_{\alpha_{2n}} t^{(n)}_{\beta_{2n}}} \sum_{j=1}^{v_{n+1}} \sum_{k=1}^{\Lambda^{(n)}_{\alpha_{2n},j}} t_j^{(n+1)} = e_n(\alpha,\beta),$$

since

$$\sum_{j=1}^{v_{n+1}} \sum_{k=1}^{\Lambda^{(n)}_{\alpha_{2n},j}} t_j^{(n+1)} = (\Lambda^{(n)}t^{(n+1)})_{\alpha_{2n}} = t^{(n)}_{\alpha_{2n}},$$

thus establishing that each e_n is a projection.

As for (c), if $x\in M_{n+1}$, then $(e_nx)(\alpha,\alpha) = \sum_{\gamma\in\Omega} e_n(\alpha,\gamma)x(\gamma,\alpha)$; it follows from the definitions of M_{n+1} and e_n that $e_n(\alpha,\gamma)x(\gamma,\alpha)$ can be non-zero only if $\alpha_{[1,2n]\cup[2n+2,\infty)} = \gamma_{[1,2n]\cup[2n+2,\infty)}$, $\alpha_{2n}=\alpha_{2n+4}$, $\alpha_{2n+1}=\alpha_{2n+3}$, $\gamma_{2n}=\gamma_{2n+4}$ and $\gamma_{2n+1} = \gamma_{2n+3}$; this implies that

$$(e_nx)(\alpha,\alpha) = \delta_{\alpha_{2n},\alpha_{2n+4}}\delta_{\alpha_{2n+1},\alpha_{2n+3}}(t^{(n+1)}_{\alpha_{2n+2}}/t^{(n)}_{\alpha_{2n}})x_{2n+2]}(\alpha_{2n+3]},\alpha_{2n+2]})$$

and since $e_nx\in M_{n+2}$, it follows from equation (4) that

$$\varphi(e_nx) = \sum_{\alpha\in\Omega_{2n+4]}} (e_nx_{2n})_{+4]}(\alpha,\alpha)t^{(n+2)}_{\alpha_{2n+4}} =$$

$$= \sum_{\gamma\in\Omega_{2n+2]}} (t^{(n+1)}_{\gamma_{2n+2}}/t^{(n)}_{\gamma_{2n}})x_{2n+2]}(\gamma,\gamma)t^{(n+2)}_{\gamma_{2n}} =$$

$$= \sum_{\gamma\in\Omega_{2n+2]}} t^{(n+1)}_{\gamma_{2n+2}}x_{2n+2]}(\gamma,\gamma)\,\lambda^{-1} = \lambda^{-1}\varphi(x) \qquad (\text{since } t^{(k+2)} = \lambda^{-1}t^{(k)}).$$

We come finally to (b). It is a consequence of the definition of the e_k's that if $\alpha, \beta, \gamma, x \in \Omega$, then the only way that $(e_n(\alpha, \gamma)e_{n+1}(\gamma, x)e_n(x, \beta))$ can be non-zero is if

$$\alpha_{2n]} = \gamma_{2n]},\ \alpha_{[2n+4} = \gamma_{[2n+4},\ \alpha_{2n} = \alpha_{2n+4},\ \alpha_{2n+1} = \alpha_{2n+3},\ \gamma_{2n+1} = \gamma_{2n+3},$$

$$\beta_{2n]} = x_{2n]},\ \beta_{[2n+4} = x_{[2n+4},\ \beta_{2n} = \beta_{2n+4},\ \beta_{2n+1} = \beta_{2n+3},\ x_{2n+1} = x_{2n+3},$$

$$\gamma_{2n+2]} = x_{2n+2]},\ \gamma_{[2n+6} = x_{[2n+6},\ \gamma_{2n+4} = \gamma_{2n+6},\ \gamma_{2n+3} = \gamma_{2n+5},$$

and

$$x_{2n+3} = x_{2n+5},$$

which happens precisely when

$$\alpha_{2n]} = \beta_{2n]},\ \alpha_{[2n+4} = \beta_{[2n+4},\ \alpha_{2n} = \alpha_{2n+4},\ \alpha_{2n+1} = \alpha_{2n+3},\ \beta_{2n+1} = \beta_{2n+3},$$

$$\gamma = (\alpha_1, \ldots, \alpha_{2n}, \alpha_{2n+5}, \alpha_{2n+6}, \alpha_{2n+5}, \alpha_{2n}, \alpha_{2n+5}, \alpha_{2n+6}, \alpha_{2n+7}, \ldots)$$

and

$$x = (\beta_1, \ldots, \beta_{2n}, \beta_{2n+5}, \beta_{2n+6}, \beta_{2n+5}, \beta_{2n}, \beta_{2n+5}, \beta_{2n+6}, \beta_{2n+7}, \ldots).$$

It can now be deduced that

$$(e_n e_{n+1} e_n)(\alpha, \beta) = \sum_{\gamma, x \in \Omega} e_n(\alpha, \gamma)e_{n+1}(\gamma, x)e_n(x, \beta) =$$

$$= \delta_{\alpha_{2n]}, \beta_{2n]}} \delta_{\alpha_{2n}, \alpha_{2n+4}} \delta_{\alpha_{2n+1}, \alpha_{2n+3}} \delta_{\beta_{2n+1}, \beta_{2n+3}} \delta_{\alpha_{[2n+4}, \beta_{[2n+4}} \times$$

$$\times \frac{(t^{(n+1)}_{\alpha_{2n+2}} t^{(n+1)}_{\alpha_{2n+4}} t^{(n+2)}_{\alpha_{2n+3}} t^{(n+2)}_{\beta_{2n+3}} t^{(n+1)}_{\beta_{2n+4}} t^{(n+1)}_{\beta_{2n+2}})^{1/2}}{t^{(n)}_{\alpha_{2n}} (t^{(n+1)}_{\alpha_{2n+4}} t^{(n+1)}_{\beta_{2n+4}})^{1/2} t^{(n)}_{\beta_{2n}}} = \lambda^{-1} e_n(\alpha, \beta),$$

since $t^{(n+2)}_{\alpha_{2n+4}} = \lambda^{-1} t^{(n)}_{\alpha_{2n+4}}$ and $t^{(n+2)}_{\beta_{2n+4}} = \lambda^{-1} t^{(n)}_{\beta_{2n+4}}$ and since $\alpha_{2n} = \alpha_{2n+4} = \beta_{2n} = \beta_{2n+4}$ for any (α, β) for which either $e_n(\alpha, \beta) \neq 0$ or $(e_n e_{n+1} e_n)(\alpha, \beta) \neq 0$.

A similar argument shows that $e_n e_{n-1} e_n = \lambda^{-1} e_n$ and the proof of the proposition is complete. ∎

NOTE. It must be remarked here that e_n, as above, is precisely the projection in M_{n+2} that implements the conditional expectation of M_{n+1} onto M_n in the sense that $e_n x e_n = E_n(x)e_n$ $\forall x \in M_{n+1}$ where E_n is the unique conditional expectation of M_{n+1} onto M_n which is compatible with the trace τ/M_{n+1}; this is fairly easily established using the (also easily established) formula for the conditional expectation $E_{n,m}$ of M_n onto M_m (where $m < n$) given by

$$(E_{n,m}x)_{2m]}(\alpha, \beta) = \sum_{\substack{\theta \in \Omega_{[2m,2n]} \\ \theta_{2m} = \alpha_{2m}}} \frac{t^{(n)}_{\theta_{2n}}}{t^{(m)}_{\theta_{2m}}} \cdot x_{2n]}(\alpha * \theta, \beta * \theta)$$

whenever $x \in M_n$, and $\alpha, \beta \in \Omega_{2m]}$ satisfy $\alpha_{2m} = \beta_{2m}$. ∎

The next proposition identifies the range of each e_n, where of course we are assuming that the underlying Hilbert space is $L^2(R, \mu^\sim)$.

PROPOSITION 7. *Let $\xi \in L^2(R, \mu^\sim)$ and $n \geqslant 1$; then, ξ belongs to the range of e_n if and only if there is a function f defined on $\Omega_{Z_+ \setminus (2n, 2n+4)} \times \Omega$ such that for $\mu^\sim$-a.e. (α, β) in R, we have*

$$\xi(\alpha, \beta) = \delta_{\alpha_{2n}, \alpha_{2n+4}} \delta_{\alpha_{2n+1}, \alpha_{2n+3}} (t^{(n+1)}_{\alpha_{2n+2}})^{1/2} f(\alpha_{Z_+ \setminus (2n, 2n+4)}, \beta).$$

Proof. $e_n\xi = \xi$ iff $(e_n\xi)(\alpha, \beta) = \xi(\alpha, \beta)$ a.e.$(\mu^\sim)$; now compute:

$$(e_n\xi)(\alpha, \beta) = \sum_\gamma e_n(\alpha, \gamma)\xi(\gamma, \beta) = \delta_{\alpha_{2n}\alpha_{2n+4}} \delta_{\alpha_{2n+1}, \alpha_{2n+3}} \times$$

$$\times \left(\sum_{j=1}^{v_{n+1}} \sum_{k=1}^{\Lambda^{(n)}_{2n,j}} \frac{(t_j^{(n+1)} t^{(n+1)}_{\alpha_{2n+3}})^{1/2}}{t^{(n)}_{\alpha_{2n}}} x(\alpha_{2n]} * (jkj) * \alpha_{[2n+4}, \beta) \right).$$

This shows that if $\xi = e_n\xi$, then $\xi(\alpha, \beta)$ has the prescribed form. Conversely, if $\xi(\alpha, \beta)$ has the prescribed form, it is not too hard to verify that $e_n\xi = \xi$. ∎

REMARK. The author became aware, after the preparation of this paper, that A. Ocneanu has obtained (cf. [3]) essentially identical formulae for the projections e_n that arise when one iterates Jones' basic construction in the case of the inclusion $N \subset M$ of a general pair of hyperfinite II_1 factors which satisfy $M \cap N' = \mathbf{C}1$; he does this by considering the AF-algebra generated by the increasing sequence $\{A_n : n \geqslant 0\}$ of finite-dimensional C^*-algebras defined by $A_n = M_n \cap N'$, where $M_0 = N$, $M_1 = M$, and $M_0 \subset M_1 \subset M_2 \subset \ldots$ is the tower obtained by iterating Jones' basic construction in the case of the inclusion $N \subset M$.

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V. S. SUNDER
Indian Statistical Institute,
R.V. College Post, Bangalore 560059,
India.

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