Journal of the Indian Math. Soc.
46 (1982) 1–29

ON SOME INVOLUTIONS AND RETRACTIONS ARISING IN TEICHMULLER SPACES

By SUBHASHIS NAG*

[Received July 6, 1982]

§ 1. **Introduction.** Let T be the Teichmuller space of Riemann Surfaces of a fixed finite type. The Teichmuller modular group is known to act biholomorphically and isometrically on T (equipped with its Teichmuller metric). The fixed point sets of these mappings, which are non-empty if the mapping is of finite order, are complex submanifolds of T consisting of those Riemann Surfaces which carry a certain group of automorphisms.

Let f be an involutory (i.e. $f \circ f = 1$) element of the modular group. Then we define in § 3 a natural geometrical retraction, H_f, of T onto the fixed point set T^f of f. For example, we can thus associate naturally to any compact Riemann Surface of genus greater than one a hyperelliptic Riemann Surface of the same genus by choosing f appropriately.

In § 3 we study this retraction induced by f; we prove its continuity and we are able to characterize its fibers (i. e. the sets $H_f^{-1}(x)$) in terms of geodesics (or quadratic differentials) determined by the eigenvectors of the action of df (respectivly d^*f) on the holomorphic tangent space (respectively cotangent space) of T at a fixed point x of f.

Using some general results of the author we are able to provide conditions for holomorphy of the maps H_f in § 4.

---

*This research is from the author's 1980 doctoral thesis at Cornell University. The author is very grateful to his thesis advisor, Professor C.J. Earle, for suggesting the problem here and for his support and help.

The author is currently a Visiting Fellow at the Tata Institute of Fundamental Research.

In § 5 we study in more detail one of these retractions H_f on the Teichmuller space $T(1, 2)$ of 'twice punctured tori'. We have an explicit construction for a holomorphic retraction $\widetilde{H}$ from $T(1, 2)$ to the fixed point set T^f of f. Comparing H_f and $\widetilde{H}$ shows that they agree up to first order approximation near their common target set T^f even though globally they are distinct mappings. The proof that H_f and $\widetilde{H}$ are different mappings rests on an interesting result regarding the nature of the fibers of the Bers fiber space over Teichmuller space.

The approximation of H_f by $\widetilde{H}$ leads to the vanishing of a certain integral involving classical elliptic functions as explained in § 5. We substantiate this Teichmuller theoretic result by utilizing some classical function theory involving the evaluation of an unusual double-series.

Besides the new results about these natural retractions the paper also proves some fundamental facts about the geometry of the Teichmuller metric in Teichmuller spaces. For example we establish in Theorem 3.14 that a Teichmuller geodesic is uniquely determined by its tangent direction at any point on it. The idea of that proof is due to C. J. Earle.*

We commence by presenting some necessary material on Teichmuller spaces in § 2.

§ 2. Preliminaries on Teichmuller Spaces. As references for the standard material below and for relevant definitions about quasiconformal mappings we cite Ahlfors [1] and Kravetz [5].

Let U be the upper half-plane and G be a Fuchsian group acting on U. Let $L_\infty(U, G)$ be the closed linear subspace of $L_\infty(U, \mathbf{C})$ consisting of elements μ satisfying

$$\mu(g(z))\cdot\overline{g'(z)}/g'(z) = \mu(z) \text{ a.e. for all } g \in G.$$

Let $M(G) = \{\mu \in L_\infty(U, G) : \| \mu \|_\infty < 1\}$. Given $\mu \in M(G)$ it is classical that there is a unique quasiconformal (q.c.) automorphism w_μ

*A study somewhat similar to the results of the present paper was carried out by Marden and Masur "A foliation of Teichmuller space by twist invariant disks" Math. Scand. 36 (1975) 211–228, 39 (1976) 232–238.

of U which fixes 0, 1, ∞ and whose Beltrami coefficient is μ. (Recall that every q.c. automorphism of U extends continuously to an automorphism of the closure of U in the Riemann sphere). For $\mu \in M(G)$, w_μ is compatible with G in the sense that $G_\mu = w_\mu \circ G \circ w_\mu^{-1}$ is also a group of Mobius transformations of U.

For $\mu, \nu \in M(G)$ we set $\mu \sim \nu$ (and $w_\mu \sim w_\nu$) if and only if

$$g_\mu = w_\mu \circ g \circ w_\mu^{-1} = w_\nu \circ g \circ w_\nu^{-1} = g_\nu, \text{ for all } g \in G.$$

Then $T(G) = M(G)/\sim$ is the *Teichmuller space of the group G which* has a structure of a finite dimensional complex manifold such that the quotient map $M(G) \to T(G)$ is a holomorphic submersion when G is finitely generated of the first kind. We write $[\mu]$ for the equivalence class of μ.

If S_0 is a fixed Riemann surface of type (g, k), (i.e. genus g with k punctures), then we say (S_0, f, S) is a marked Riemann surface when S is another Riemann surface of the same type and $f : S_0 \to S$ (the 'marking map') is a q.c. homeomorphism. Two marked surfaces (S_0, f, S) and (S_0, f', S') are called $\sim$ equivalent if there is a conformal map $h : S \to S'$ such that the q.c. self-map $(f')^{-1} \circ h \circ f$ of S_0 is homotopic to the identity. We set $T(S_0) = T(g, k) = \sim$ equivalence classes of marked surfaces. As usual in Teichmuller theory we assume $2g - 2 + k > 0$ to avoid a few well-known special cases.

When G is a Fuchsian group of the first kind without elliptic elements such that U/G is a surface of type (g, k), say S_0, then $T(G)$ and $T(g, k)$ are *isomorphic by a map* α in the following manner:

Given $[\mu] \in T(G)$ let $\alpha([\mu])$ be the marked Riemann surface $(S_0, f_\mu, U/G_\mu)$ where $G_\mu = w_\mu \circ G \circ w_\mu^{-1}$ as before, and $f_\mu : S_0 \to U/G$ is the map induced by $w_\mu : U \to U$.

$T(G)$ and $T(g, k)$ will be henceforth identified via α, (G will always be a fixed point free Fuchsian group of the first kind on U).

The Modular Group. Let S_0 be a Riemann surface of type (g, k).

Given a q.c. self-homeomorphism $f : S_0 \to S_0$ we get a biholomorphic mapping $f^* : T(S_0) \to T(S_0)$

$$f^*[(S_0, g, S)] = [(S_0, g \circ f, S)].$$

f^* is called the *modular transformation induced by f*. The quotient group: q.c. self-homeomorphisms modulo those homotopic to the identity, is called the *Teichmuller modular group*, Mod(g, k).

If $\mu \in M(S_0)$ represents (S_0, g, S) then the Beltrami differential $\mu \cdot f$ represents $f^*[(S_0, g, S)]$ where if $f(w) = z$ and $\mu = \mu(z)\, d\bar{z}/dz$ in local coordinates on S_0, then

$$\mu \cdot f = \frac{\mu(f(w))\, \theta(w) + \nu(w)}{1 + \bar{\nu}(w)\, \mu(f(w))\, \theta(w)} \frac{d\bar{w}}{dw}, \quad \nu = \frac{f_{\bar{w}}}{f_w}, \quad \theta = \frac{\overline{(f_w)}}{f_w}. \qquad (2.1)$$

TEICHMULLER'S THEOREM AND THE TEICHMULLER METRIC:

The Teichmuller metric τ on $T(G)$ is given by

$$\tau([\mu], [\nu]) = \frac{1}{2} \inf \{\log K(w_\sigma \circ w_\tau^{-1}) : [\sigma] = [\mu] \text{ and } [\tau] = [\nu]\}$$

where $K(f)$ is the maximal dilatation of the q.c. map f.

On $T(g, k)$ the Teichmuller metric is the one induced from τ by the isomorphism $\alpha : T(G) \to T(g, k)$; α becomes an isometry.

The modular group acts as isometries with respect to this complete metric.

The integrable quadratic differentials $Q(G)$ for the group G are the holomorphic functions ϕ in U satisfying

$$\phi(gz)g'(z)^2 = \phi(z), \text{ for all } z \in U, g \in G$$

and

$$\| \phi \| = \iint_{U/G} | \phi(z) |\, dxdy < \infty$$

If $U/G = X$ we set $Q(G) = Q(X)$, and $M(G) = M(X)$.

DEFINITION 2.1. Let X be a Riemann surface of type (g, k) and $\phi \in Q(X)$ be an integrable quadratic differential on X. Then, for t in the unit disc Λ, $\mu = t\bar{\phi}/|\phi|$ $(\epsilon M(X))$ is called a *Teichmuller-Beltrami (T-B) differential* on X. (For $\phi \equiv 0$ set $\mu = 0$.) The T-B differentials form a subset $M_{TB}(X)$ of $M(X)$. We shall call $\Phi^X_{TB} : M_{TB}(X) \to T(X)$ the restriction of the quotient map $\Phi : M(X) \to T(X)$.

DEFINITION 2.2. Let f be a homeomorphism of a Riemann surface X (type (g, k)) to another Riemann surface X' of the same type. Then f is called a *Teichmuller mapping* if either f is conformal, or f is q.c. with complex dilatation equal to a T-B differential on X.

THEOREM 2.3 (TEICHMULLER). *(i) Given two Riemann surfaces X and X' of type (g, k) and homeomorphism $h : X \to X'$ then there is a unique Teichmuller map $f : X \to X'$ in the homotopy class of h; f has the property that $K(f) < K(f_0)$ for any other q.c. homeomorphism f_0 in the homotopy class of h.*

(ii) Let $Q_0(X)$ be the open unit ball in $Q(X)$ (equipped with the L^1 norm). Then the map

$$g : Q_0(X) \to T(X)$$

given by

$$g(\phi) = \Phi^X_{TB}(\|\phi\| \bar{\phi}/|\phi|)$$

is a homeomorphism onto $T(X)$.

As a corollary we find that $T(X)$ is homeomorphic to $R^{6g-6+2k}$.

We now study the Teichmuller metric more carefully.

THEOREM 2.4. *The Teichmuller metric τ on $T(g, k)$ is a Finsler metric making $T(g, k)$ a straight space. In particular, given any X and X' in $T(g, k)$ there is a unique geodesic on which they lie. Indeed, if $X' = g(\phi)$ (notation as above) then the geodesic segment between X and X' is*

$$\{g(k\phi) : 0 \leqslant k \leqslant 1\}.$$

For relevant definitions and proofs see Kravetz [5] and Royden [9]. We now define a Teichmuller geodesic disc:

DEFINITION 2.5. Let $X \in T(g, k)$ and $\phi \in Q(X)$. Then the map $e_\phi : \Delta \to T(g, k)$ given by $e_\phi(z) = \Phi^X_{TB}(z\bar{\phi}/|\phi|)$ is a holomorphic isometry from the Poincare disc Δ to $T(g, k)$ with the Teichmuller metric.

The image of such an e_ϕ is called a *Teichmuller geodesic disc*. The images of Poincare lines in Δ are *Teichmuller lines* in $T(g, k)$.

THE BERS FIBER SPACE. Let G be a torsion-free Fuchsian group acting on U with quotient surface $U/G = X$ of type (g, k).

DEFINITION 2.6. (i) The *Bers fiber space* $\pi_0 : F(G) = F(g, k) \to T(g, k)$ is defined by

$$F(G) = \{([\mu], z) \in T(g, k) \times \mathbf{C} : \mu \in M(G), z \in w^\mu(U)\}$$

and

$$\pi_0([\mu], z) = [\mu]$$

Here the map w^μ is the q.c. automorphism of the complex sphere fixing $0, 1, \infty$ which has complex dilatation μ on U and 0 on the lower half-plane.

§ 3. **Involutions and Induced Retractions on T(g, n).** Let X be a Riemann surface of type (g, n), $(2g + n - 2 > 0)$, $\bar{X}$ being the closed surface of genus g in which X is embedded.

Let $f \in \mathrm{Mod}(g, n)$ be an arbitrary involutory (i.e., $f \circ f = 1$) element of the Teichmuller modular group. As usual let $T(X)^f$ = fix-point set of $f (= \{x \in T(X) : f(x) = x\})$. f induces a retraction $H_f : T(X) \to T(X)^f$ as described in the following.

DEFINITION 3.1. For any $x \in T(X)$ define a map H_f by

$$H_f(x) = \begin{cases} \text{midpoint of the Teichmuller geodesic segment joining } x \text{ to} \\ f(x), \text{ if } x \notin T(X)^f; \ x, \text{ if } x \in T(X)^f. \end{cases}$$

We claim that *H_f is a well-defined retraction of $T(X)$ onto $T(X)^f$.*

All we need to show is that $H_f(x) \in T(X)^f$. This is clear since the isometry f carries the geodesic segment from x to $f(x)$ onto itself with the endpoints reversed. Since f is an isometry the midpoint must be a fixed point of f. (See also Kravetz [5, § 6]).

REMARK 3.2. For any involutory $f \in \mathrm{Mod}(g, n)$ the set $T(X)^f$ is a non-empty complex-submanifold of $T(X)$ and is itself analytically isomorphic to a Teichmuller space. This is well known; see Earle [2]. Hence we can ask whether the geometrically induced retraction

$H_f : T(X) \to T(X)^f$ is continuous, or holomorphic, etc. These questions will occupy us in later sections.

EXAMPLE 3.3. Let X be a closed surface of genus $g > 2$. Let $f \in \text{Mod}(g, 0)$ be induced by the hyperelliptic involution on X. Then $T(X)^f$ is one component* of the set of hyperelliptic points in $T(X)$. Thus H_f as defined in 3.1 gives a natural way to associate to any Riemann surface of genus g a hyperelliptic Riemann surface of the same genus.

DESCRIPTION OF THE RETRACTIONS H_f. We will prove that H_f is continuous and then proceed to characterize the fixed point sets T^f and the fibers of these retractions. The description of T^f is known but is included for uniformity and clarity of exposition.

DEFINITION 3.4. (a) Let $b : E \to T(g, n)$ be the cotangent bundle of $T(g, n)$. We know that any $S \in T(g, n)$ represents a Riemann surface of type (g, n) and the cotangent space to $T(g, n)$ at S is the finite-dimensional Banach space $Q(S)$ of integrable holomorphic quadratic differentials on S, with the norm:

$$\| \phi \| = \frac{1}{2} \int_S | \phi |.$$

Let $E_0 \subset E$ be the open subset of cotangent vectors of norm less than 1.

(b) Given $\phi \in E_0$ let $b(\phi) = S$. Then ϕ is a quadratic differential on S, and the Teichmuller-Beltrami differential $\| \phi \| \cdot \bar{\phi} / | \phi |$ on S determines a point $g(\phi)$ (as explained in Theorem 2.3 (ii)) in $T(g, n)$;

$$g(\phi) = \Phi^S_{TB}\left(\| \phi \| \frac{\bar{\phi}}{| \phi |} \right).$$

Whenever $S \in T(g, n)$, $\phi \in Q(S)$ and $\mu = t \dfrac{\bar{\phi}}{| \phi |}$ is a Teichmuller-Beltrami (T-B) differential on S with $t \in \Delta$ (the unit disc), we shall say that μ is a T-B differential formed from ϕ.

We are now ready to prove.

---

*I thank a referee for correcting an inaccuracy here.

THEOREM 3.5. *For any involutory $f \in Mod(g, n)$ the retraction H_f is continuous.*

PROOF. Consider the map $E_0 \to T(g, n) \times T(g, n)$ given by

$$\phi \longmapsto (b(\phi), g(\phi)).$$

By Earle [3] we know that this map is a homeomorphism. Following him we will call the inverse map F,

$$F : T(g, n) \times T(g, n) \to E_0.$$

Notice that $F(S_1, S_2) = Q_1 \in Q(S_1)$ such that the T-B differential $\| \phi_1 \| \cdot \bar{\phi}_1 / | \phi_1 |$ on S_1 represents the point S_2. Now, the Teichmuller geodesic segment joining S_1 to S_2 is $\{g(k\phi_1) : 0 \leqslant k \leqslant 1\}$. A calculation then shows that at the midpoint the value k is

$$k = \frac{1}{\| \phi_1 \|} \left(\frac{\psi - 1}{\psi + 1} \right) \tag{3.1}$$

where $$\psi = \left(\frac{1 + \| \phi_1 \|}{1 - \| \phi_1 \|} \right)^{1/2}$$

Let us define

$$m : E_0 \to E_1$$

by

$$m(\phi_1) = k\phi_1 \text{ with } k \text{ as determined in (3.1).}$$

Setting $1 \times f : T(g, n) \to T(g, n) \times T(g, n)$ to be the map

$$S \longmapsto (S, f(S))$$

we see that

$$H_f = g \circ m \circ F \circ (1 \times f).$$

Each member of this composition is readily seen to be continuous. Hence the composite map H_f is continuous, as claimed. □

We aim to describe exactly the fibers $H_f^{-1}(S)$ for any $S \in T(X)^f$. Let $\hat{f}$ be an involutory holomorphic automorphism of S whose residue-class in Mod(g, n) is f.

LEMMA 3.6. *If $\mu = k \dfrac{\bar{\phi}}{|\phi|} \in M_{TB}(S)$, $(0 \neq)\phi \in Q(S)$, then $\mu \cdot \hat{f} \in M_{TB}(S)$ also. Indeed, if $f^* : Q(S) \to Q(S)$ is the involutory linear co-differential map induced by f on the cotangent space to $T(g, n)$ at S, then*

$$\mu \cdot \hat{f} = k \frac{\overline{f^*\phi}}{|f^*\phi|}$$

Proof. In terms of local coordinates w and z on S let the holomorphic map $\hat{f}$ be given from the w-patch to the z-patch by $\hat{f}(w) = z$. Then in these local coordinates the mapping $(f^*)^{-1} : Q(S) \to Q(S)$ is given by $\phi(z)dz^2 \longmapsto (\phi \circ \hat{f})\cdot\left(\frac{d\hat{f}}{dw}\right)^2 dw^2$, at w. Since f^* is involutory, $f^* = (f^*)^{-1}$. Now if $\phi = \phi(z)dz^2$ in the z-patch, then $\mu = k\,\frac{\overline{\phi(z)}}{|\phi(z)|}\,\frac{d\bar{z}}{dz}$ in this local coordinate. Hence, calculating in local coordinates,

$$\begin{aligned}
\mu\cdot f(w) &= k\,\frac{\overline{\phi(\hat{f}(w))}}{|\phi(\hat{f}(w))|}\cdot\frac{\overline{\hat{f}'(w)}}{\hat{f}'(w)}\cdot\frac{d\bar{w}}{dw} \quad \text{(See equation (2.1))}\\
&= k\,\frac{\overline{\phi(\hat{f}(w))\,\hat{f}'(w)^2}}{|\phi(\hat{f}(w))|\,|\hat{f}'(w)|^2}\,\frac{d\bar{w}}{dw}\\
&= k\,\frac{\overline{f^*\phi}}{|f^*\phi|}\,(w) \text{ as claimed.}
\end{aligned}$$

Since $f^*\phi \in Q(S)$ we see $\mu\cdot\hat{f} \in M_{TB}(S)$, and we are done. □

Definition 3.7. (i) Since f is involutory on $T(g, n)$ and fixes S ($\epsilon\, T(g, n)^f$) the codifferential f^* is involutory and linear on $Q(S)$, and hence has exactly two eigenvalues: $+1$ and -1. Let the corresponding eigenspaces be called $E^S(+)$ and $E^S(-)$ respectively. f^* on $Q(S)$ is an isomorphism since it is its own inverse, hence $Q(S) = E^S(+) \oplus E^S(-)$.

(ii) For any $(0 \neq)\, \phi \in Q(S)$ let λ_ϕ: Unit disc $\Delta \to M_{TB}(S)$ be given by $\lambda_\phi(t) = t\,\frac{\bar{\phi}}{|\phi|}$ (T-B differential on S formed from ϕ).

Notational Remark. When the base-point $S \in T(g, n)$ is kept fixed we will write Φ and Φ_{TB} for Φ^S and Φ^S_{TB} and similarly drop the superscript S from $E^S(+)$ and $E^S(-)$.

Theorem 3.8. *The manifold of fixed points of f, $T(X)^f$, is precisely the union of the geodesic discs given by images of e_ϕ, $(0 \neq)\phi \in E^S(+)$ (for any $S \in T(X)^f$ we start with).*

Proof. $\phi \in E(+)$ implies $f^*\phi = \phi$, so if $\mu = k\,\frac{\bar{\phi}}{|\phi|}$, $(k \in \Delta)$, then

$\mu \cdot \hat{f} = \mu$ by Lemma 3.6. Now, the modular group acts via

$$f(\Phi_{TB}(\mu)) = \Phi_{TB}(\mu \cdot \hat{f})$$

Thus $\mu \cdot \hat{f} = \mu$ implies that f fixes $\Phi_{TB}(\mu) = e_\phi(k)$, any $k \in \Delta$, any $\phi \in E(+)$. Thus the union of the images of e_ϕ, $\phi \in E(+)$, is in $T(X)^f$. For the reverse inclusion notice that for any $S_1 \in T(g, n)$ we have unique $\mu = k \dfrac{\bar{\phi}}{|\phi|} \in M_{TB}(S)$ such that $S_1 = \Phi^S_{TB}(\mu)$ (see Teichmuller's theorem (Theorem (2.3)). If $S_1 \in T(g, n)^f$ then

$f(S_1) = S_1$, so $\Phi_{TB}(\mu \cdot \hat{f}) = \Phi_{TB}(\mu)$ (By Lemma 3.6 $\mu \cdot \hat{f} \in M_{TB}(S)$)

Hence $\mu \cdot \hat{f} = \mu$ by uniqueness of T-B differential representing S_1.

Thus:

$$k \frac{\overline{f^*\phi}}{|f^*\phi|} = k \frac{\bar{\phi}}{|\phi|}$$

Now it is easily seen that two T-B differentials $k \dfrac{\bar{\phi}}{|\phi|}$ and $k \dfrac{\bar{\psi}}{|\psi|}$ are equal only if ϕ is a positive multiple of ψ. So

$$f^*\phi = (\text{some positive constant}) \cdot \phi.$$

But this implies ϕ is an eigenvector of f^* with positive eigenvalue. The only positive eigenvalue of f^* is $+1$, hence $\phi \in E^S(+)$ as was required to prove. □

Thus we see that the $+1$ eigenvectors at $S \in T(X)^f$ cull out the fixpoint set $T(X)^f$. We will now see that in the same sense, the -1 eigenvectors at $S \in T(X)^f$ cull out the fiber $H_f^{-1}(S)$ at S as a union of geodesic discs.

THEOREM 3.9. *The fiber of H_f at $S \in T(g, n)^f$ is precisely the union of the geodesic discs given by the images of e_ϕ, $(0 \neq) \phi \in E^S(-)$.*

COROLLARY 3.10. *The fibers of H_f are all connected topological submanifolds of $T(g, n)$ of (real) dimension = twice the complex dimension of $E^S(-)$ (for any $S \in T(g, n)^f$). And dim $E^S(-)$ = codimension of $T(g, n)^f$ in $T(g, n)$.*

The proofs require a sequence of Lemmas.

Lemma 3.11. *Teichmuller distances from any point S on $T(g, n)^f$ to any pair of f-related points, Z and $f(Z)$, are equal.*

Proof. Consider the geodesic segments joining S to Z (${}_Sl_Z$ say) and the segment joining S to $f(Z)$ (${}_Sl_{f(Z)}$). Since f is an isometry fixing S and carrying Z to $f(Z)$, it carrries ${}_Sl_Z$ to ${}_Sl_{f(Z)}$ isometrically. Hence the result. □

Lemma 3.12. *The only point z on the Poincare unit disc, Δ, equidistant from every pair k and $-k$, for all $k \in \Delta$, is $z = 0$.*

This is a simple fact whose proof is a calculation left to the reader.

Lemma 3.13. *$S \in T(X)^f$ and $(0 \neq)\phi \in E^S(-)$ implies*

$$(Im\ e_\phi) \cap (T(X)^f) = \{S\}.$$

Proof. Suppose there is a point $S' \neq S$ also in Im $e_\phi \cap T(X)^f$. Let $z \in \Delta$ so that $e_\phi(z) = S'$. Now $e_\phi(0) = S$ and $S' \neq S$ means $z \neq 0$ since e_ϕ is an injective embedding. Now notice that if

$$\mu = \lambda_\phi(k) = k\frac{\bar{\phi}}{|\phi|},\ k \in \Delta \text{ then}$$

$$\mu \cdot \hat{f} = k\frac{\overline{f^*\phi}}{|f^*\phi|} \text{ by Lemma 3.6}$$

$$= k\frac{\overline{(-\phi)}}{|-\phi|} \text{ since } \phi \in E^S(-)$$

$$= -k\frac{\bar{\phi}}{|\phi|} = \lambda_\phi(-k).$$

Now $f(\Phi_{TB}(\mu)) = \Phi_{TB}(\mu \cdot \hat{f})$, and $\Phi_{TB} \circ \lambda_\phi = e_\phi$ so

$$f(e_\phi(k)) = e_\phi(-k). \tag{3.2}$$

Now $e_\phi(z) = S' \in T(X)^f$, so $e_\phi(z)$ is equidistant from every pair $e_\phi(k)$ and $e_\phi(-k)$, for all $k \in \Delta$ by equation (3.2) and Lemma 3.11. But $e_\phi : \Delta \to T(X)$ is an isometric embedding; therefore z is equidistant from every pair k and $-k$ in the Poincare metric on Δ, for all $k \in \Delta$. By Lemma 3.12 we conclude $z = 0$ which contradicts $S' \neq S$. □

Proof of Theorem 3.9. Given $(0 \neq)\phi \in E^S(-)$ we will first show Im $e_\phi \subset H_f^{-1}(S)$.

For any $k \in \Delta$, $f(e_\phi(k)) = e_\phi(-k)$ by equation (3.2). So $Z \in \operatorname{Im} e_\phi$ means $f(Z) \in \operatorname{Im} e_\phi$.

But $\operatorname{Im} e_\phi$ is a Teichmuller geodesic disc, so if it contains two points it contains the geodesic segment joining the two points. Thus $H_f(Z) \in \operatorname{Im} e_\phi$ if $Z \in \operatorname{Im} e_\phi$. But $H_f(Z) \in T(X)^f$, therefore

$$H_f(Z) \in (\operatorname{Im} e_\phi) \cap (T(X)^f).$$

By Lemma 3.13 we see $H_f(Z) = S$, for all $Z \in \operatorname{Im} e_\phi$ as claimed.

We are left to show that any $S_1 \in H_f^{-1}(S)$ lies in some $\operatorname{Im} e_\phi$, $\phi \in E^S(-)$, $(S_1 \neq S)$. This follows by an argument similar to the latter part of the proof of Theorem 3.8. □

We have now obtained a complete description of $T(g, n)^f$ and the fibers of H_f in terms of the action of the codifferential f^* on the holomorphic cotangent space of $T(g, n)$. We obtain a similar description in terms of the differential of $f(= df$ or $f_*)$ acting on the tangent space to $T(g, n)$. We need the interesting result on Teichmuller geodesics given below.

THEOREM 3.14. *If two Teichmuller geodesics through a point $X \in T(g, n)$ have the same tangent direction at X then the geodesics coincide.*

PROOF. With X as base-point of $T(g, n)$ consider a tangent vector v at X. Choose $\mu \in L_\infty(X)$ to represent v, i.e.

$d\Phi_X(\mu) = \mathrm{v}$, where $\Phi : M(X) \to T(g, n)$ is the quotient map. By 'Teichmuller's Lemma', μ and $\nu \in L_\infty(X)$ represent the same tangent vector if and only if

$$\iint_X (\mu - \nu)\phi = 0 \text{ for all } \phi \in Q(X)$$

So

$$\frac{1}{2}\iint_X \mu\phi = \frac{1}{2}\iint \nu\phi \text{ for all } \phi \in Q(X).$$

So associated to v we have a uniquely determined linear functional l_{v} on $Q(X)$ defined by

$$l_v(\phi) = \frac{1}{2}\iint_X \mu\phi \text{ where } d\Phi_X(\mu) = v. \tag{3.3}$$

Now any geodesic-ray through X is the set $\left\{\Phi\left(t\frac{\bar\psi}{|\psi|}\right); 0 \leqslant t < 1\right\}$ for some $\psi \in Q(X)$, $\psi \neq 0$. We can always normalize ψ so that $\|\psi\| = 1$. The tangent direction to the above geodesic at X is of course

$$v = d\Phi_X\left(\frac{\bar\psi}{|\psi|}\right), \ \mu = \frac{\bar\psi}{|\psi|} \in L_\infty(X).$$

We now *claim* that if l_v is the linear functional determined by $v = d\Phi_X(\mu)$ as in (3.3), then ψ is the *unique* unit-vector in $Q(X)$ with $l_v(\psi) = 1$. (That $l_v(\psi) = 1$ is clear). Indeed, suppose $\phi \in Q(X)$ of unit norm and $l_v(\phi) = 1$. Then

$$1 = \frac{1}{2}\iint_X \frac{\bar\psi}{|\psi|}\phi = \frac{1}{2} Re \iint_X \frac{\bar\psi}{|\psi|}\phi \leqslant \frac{1}{2}\iint_X \left|\frac{\bar\psi}{|\psi|}\phi\right|$$

$$= \frac{1}{2}\iint_X |\phi| = \|\phi\| = 1. \tag{3.4}$$

Hence equality holds throughout the above.

Thus, from (3.4) we see

$$\iint_X Re\left\{\frac{\bar\psi}{|\psi|}\phi\right\} = \iint_X \left|\frac{\bar\psi}{|\psi|}\phi\right|$$

so that

$$\iint_X \left(\left|\frac{\bar\psi}{|\psi|}\cdot\phi\right| - Re\left\{\frac{\bar\psi}{|\psi|}\cdot\phi\right\}\right) = 0.$$

But $\left|\frac{\bar\psi}{|\psi|}\cdot\phi\right| \geqslant Re\left\{\frac{\bar\psi}{|\psi|}\cdot\phi\right\}$ shows that the integerand above is zero almost everywhere on X. Hence

$$|\phi| = Re\left\{\frac{\bar\psi}{|\psi|}\cdot\phi\right\} \text{ almost everywhere on } X$$

So,

$$Re\left\{\frac{\bar\psi(z)}{|\psi(z)|}\cdot\phi(z)\right\} \geqslant 0 \text{ for almost every } z \text{ (local coordinates)}$$

So we get

$$Re\left\{\frac{\phi(z)}{\psi(z)}\right\} \geqslant 0 \text{ almost every } z \tag{3.5}$$

(dividing by $|\psi(z)|$, which is > 0).

Now, $\psi, \phi \in Q(X)$ implies that ψ and ϕ are meromorphic quadratic differentials on the compact surface $\bar{X}$ (in which X is embedded) with at most simple poles at the distinguished point-set $\bar{X} - X$. Hence $\phi/\psi : \bar{X} \to \mathbf{P}^1$ is a meromorphic function on $\bar{X}$. By (3.5), $Re\{\phi/\psi\} \geqslant 0$ everywhere on $\bar{X}$ because a single negative value would force an open set of negative values.

But this implies $\phi/\psi =$ a constant $c(\geqslant 0)$ since the image of a non-constant meromorphic function on a closed Riemann surface is all of CP^1. The constant c can only be 1 as $\| \phi \| = \| \psi \| = 1$. So $\phi = \psi$ as claimed.

To complete the proof of Theorem 3.14 let us suppose that the Teichmuller geodesic-rays determined by $\phi, \psi \in Q(X)$ through X have the same tangent direction v at X. We normalize ϕ, ψ to be of unit norm. Then

$$\mathrm{v} = d\Phi_X\left(\frac{\bar{\phi}}{|\phi|}\right) = d\Phi_X\left(\frac{\bar{\psi}}{|\psi|}\right)$$

So

$$l_{\mathrm{v}}(\phi) = \frac{1}{2}\iint\limits_X \frac{\bar{\phi}}{|\phi|}\phi = \frac{1}{2}\iint\limits_X |\phi| = 1$$

$$\left(\text{since } \mu = \frac{\bar{\phi}}{|\phi|} \text{ repesents v}\right)$$

But by what we proved, $l_{\mathrm{v}}(\psi) = 1$ and ψ is the *unique* unit vector with this property, So $\phi = \psi$, and the geodesics coincide, as claimed. □

Let $S \in T(X)^f$. Then the differential of f, called f_*, maps $T_ST(X)$ (= tangent space at S to $T(X)$), to itself linearly. As f is involutory $(f_*)^2 =$ identity and $T_ST(X) = E_*(+) \oplus E_*(-)$ where $E_*(+)$ and $E_*(-)$ are the eigenspaces of the action corresponding to the only two eigenvalues $+1$ and -1 respectively. We can now state tangent-space analogs of Theorems 3.8 and 3.9.

THEOREM 3.15. *For $S \in T(X)^f$, the union of the geodesics through S whose tangent directions are in $E_*(+)$ is $T(X)^f$.*

THEOREM 3.16. *For $S \in T(X)^f$, the union of the geodesics through S whose tangent directions are in $E_*(-)$ is the fiber $H_f^{-1}(S)$.*

PROOF OF 3.15. Let $S_1 \in T(X)^f$, $S_1 \neq S$. The geodesic segment ${}_S l_{S_1}$ joining S to S_1 must remain fixed by f. Hence its tangent at S is fixed under f_*, so the tangent was in $E_*(+)$.

Conversely, if a geodesic through S has tangent at S lying in $E_*(+)$, then under f the geodesic goes to a geodesic with same tangent vector at S. By Theorem 3.14 this guarantees that the geodesic went to itself, and since it must fall on itself isometrically, it must be pointwise fixed under f. This completes the proof. □

The proof of 3.16 is similar, so we omit it.

§ 4. **Criteria for holomorphy of the retractions** H_f: Some general theorems of the author (Nag [8]) give answers to the following problem: Given a map $f: X \to Y$, X a complex manifold and Y a real manifold, find conditions that will allow Y to have a complex structure with respect to which f is holomorphic. It turns out that these theorems provide conditions for holomorphy of retractions. We notice the following crucial lemma.

LEMMA 4.1. *Let $f: X \to Y$ be a retraction of a complex-manifold X onto a complex-submanifold, Y, of X. Then the only possible complex-structure on (the topological manifold) Y which can make f holomorphic is the original structure, σ, induced on Y as a submanifold.*

PROOF. Let σ' be another complex-structure on Y-making f holomorphic. Then $f: X \to Y_{\sigma'}$, holomorphic implies that f restricted to any submanifold of X is holomorphic to $Y_{\sigma'}$. In particular

$$f|_{Y_\sigma}: Y_\sigma \to Y_{\sigma'} \text{ is holomorphic.}$$

But f is a retraction, so $f|_{Y_\sigma}$ = the identity map, 1. So $1: Y_\sigma \to Y_{\sigma'}$ is a holomorphic homeomorphism, hence a biholomorphism, showing $\sigma' = \sigma$ as required. □

To quote some general theorems we need some discussion and definitions.

Let X^m be a m dimensional complex manifold. We let $\mathrm{Gr}_{(m-d)}(TX)$

denote the Grassmann bundle of $(m-d)$ dimensional complex subspaces in the tangent bundle TX of X. The total space $\mathrm{Gr}_{(m-d)}(TX)$ inherits a natural complex structure from X.

DEFINITION 4.2(A). A $(m-d)$ dimensional *distribution* on X is a section of the $\mathrm{Gr}_{(m-d)}(TX)$ bundle over X. We say the *distribution is analytic* if the section is an analytic function.

REMARK. Note that the distribution is analytic if and only if it can be spanned locally by $(m-d)$ linearly independent analytic vector fields.

Let $f: X \to Y$ be a C^1-submersion from a complex manifold X onto a real C^1-manifold Y. Then if $y \in Y$ and $x \in f^{-1}(y)$, the differential of f, $d_xf: T_xX \to T_yY$, is a surjective linear map. If ker d_xf is a complex subspace of T_xX then (and only then) does T_yY inherit a complex vector-space structure such that d_xf is C-linear. (Indeed, $d_xf: T_xX/\ker d_xf \to T_yY$ is then a C-linear isomorphism). This leads to the following definition:

DEFINITION 4.2(B). If for all $y \in Y$, T_yY inherits via d_xf a unique complex structure independent of the choice of $x \in f^{-1}(y)$, then we say that f *induces a well-defined almost complex structure* on Y. In this definition we allow X and Y to be complex Banach manifolds.

DEFINITION 4.2(C). Let Y be a C^1-manifold. A complex structure σ on Y will be said to be *compatible* with the C^1-structure if the C^1-structure underlying σ coincides with the original C^1-structure on Y.

The following theorems are proved by the author in [8].

THEOREM 4.3. *Let $f: X \to Y$ be a surjective C^1-submersion from a m-dimensional complex manifold to a 2d-dimensional C^1-manifold with fibers $f^{-1}(y)$ connected for all $y \in Y$. Then there is a complex-structure on Y, compatible with the C^1-structure, making f holomorphic, if and only if*

(1) *The fibers are $(m-d)$ dimensional complex submanifolds in x and*

(2) *the distribution on* X *given by* $\Delta(x)$ = *tangent space to the fiber through* x, *is analytic.*

The complex-structure, when it exists, is unique.

Another set of necessary and sufficient conditions are below:

THEOREM 4.4. *Let* $f: X \to Y$ *be a surjective* C^1-*submersion from a complex Banach manifold* X *(modelled on a complex Banach space* B*) to a* $2d$-*dimensional* C^1-*manifold* Y. *Then there is a complex structure on* Y, *compatible with its* C^1-*structure, making* f *holomorphic if and only if* f *induces a well-defined almost-complex structure on* Y.

The complex structure, when it exists, is unique.

In view of Lemma 4.1 we immediately obtain the following theorem giving criteria for holomorphy of H_f by using Theorem 4.3 and 4.4.

THEOREM 4.5. (*a*) *If* $H_f: T(X) \to T(X)^f$ *is holomorphic then it is also a holomorphic submersion on an open neighbourhood of* $T(X)^f$ *whose complement is contained in* H_f^{-1} *of a measure-zero set in* $T(X)^f$.

(*b*) $H_f: T(X) \to T(X)^f$ *is a holomorphic submersion if and only if*

(1) H_f *is a* C^1-*submersion,*

(2) *The fibers of* H_f *are complex submanifolds of* $T(X)$,

(3) *The distribution* Δ *in* $T(X)$ *given by the tangent spaces to the fibers is analytic;*

or again, if and only if,

(1′) H_f *is a* C^1-*submersion,*

(2′) H_f *induces a well-defined almost complex structure on* $T(X)^f$.

REMARK. Part (a) of the theorem tells us that the conditions of part (b) must hold at least in an open neighbourhood of $T(X)^f$ for H_f to be holomorphic in the neighbourhood. It is a simple application of Sard's theorem on critical values.

Lastly, we note the following proposition.

PROPOSITION 4.6. If $T(X)^f$ is of codimension 1 in $T(X)$ then the condition (2) of Theorem 4.5 (b) is automatically satisfied.

PROOF. Indeed, we see by Theorem 3.9 that the fibers of H_f are complex discs—in fact Teichmuller geodesic discs. □

Therefore in this case if H_f is a C^1-submersion with the distribution Δ analytic then H_f is holomorphic.

EXAMPLE 4.7. Let X = compact surface of genus 3, and f be the element of the modular group on $T(X)$ corresponding to the hyperelliptic involution. Then $\dim_{\mathbf{C}} T(X) = 6$ and $T(X)^f$ = one component of the hyperelliptic Riemann surfaces in $T(X)$, has codimension 1 in $T(X)$. Hence the fibers of H_f are geodesic discs so Proposition 4.6 applies. However, we are unable to decide whether H_f is holomorphic or not.

REMARK. It seems likely that if $T(X)^f$ has codimension greater than one in $T(X)$ then H_f cannot be a holomorphic retraction. This is because in view of the description of the fibers obtained in Theorem 3.9 one cannot expect the fibers to be complex submanifolds any more.

Another example of H_f with $T(X)^f$ of codimension one is studied in detail in the next section.

§ 5. **Some Special Retractions in T(1, 2).** *An Involutory $f \in Mod(1, 2)$ and Associated Retraction H_f.* We let $\mathcal{L}(1, \tau)$ denote the integer lattice in $\mathbf{C}$ generated by 1 and τ, $\tau \in$ upper half-plane, U. Then $X_\tau = \mathbf{C}/\mathcal{L}(1, \tau)$ is the closed 'τ-torus' and we let $\pi : \mathbf{C} \to X_\tau$ be the quotient map, which is also the holomorphic universal covering map. We will write $\pi(z) = [z]$.

Fix $\tau_0 \in U$. Let $X'_{\tau_0} = X_{\tau_0} - \{[0], [\frac{1}{2}]\}$ be taken as base point for the Teichmuller space $T(1, 2)$ of twice punctured tori.

PROPOSITION 5.1. *The map $f_0 : \mathbf{C} \to \mathbf{C}$ given by $f_0(z) = z + \frac{1}{2}$ induces an involutory automorphism, say f, on the Riemann surface X'_{τ_0}. f may be identified with the* 180° *rotation of X'_{τ_0} depicted in Figure* 1, *f interchanges the two punctures of X_{τ_0}.*

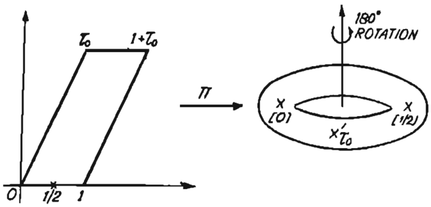


Figure 1. The involution $\hat{f}$.

The proof of this is clear.

DEFINITION 5.2. The element of the modular group Mod (1, 2) induced from the mapping $\hat{f}$ of Proposition 5.1 will be named f. f is thus an involutory automorphism of $T(1, 2)$ having X'_{τ_0} as a fixed point ($\tau_0 \in U$ is fixed but arbitrary).

Let us proceed to identify the fixed point set $T(1, 2)^{f}$ ($= T^f$) of f. We know from general principles that T^f must be a connected complex submanifold of $T(1, 2)$ isomorphic to the Teichmuller space of X'_{τ_0} with symmetries $H = \{\hat{f}, 1\}$. Since $X'_{\tau_0}/H =$ a torus with one puncture we expect that $T^f = T(1, 1) \cong U$. Indeed we have the following theorem.

THEOREM 5.3. *Let $X_\tau = \mathbf{C}/\mathcal{L}(1, \tau)$ and $X'_\tau = X_\tau - \{[0], [\frac{1}{2}]\}$, for any $\tau \in U$. Then there is a natural (affine) marking map $\phi_\tau : X'_{\tau_0} \to X'_\tau$ given by*

$$\phi_\tau([x + y\tau_0]) = [x + y\tau] \textit{ for } x, y \in \mathbf{R}.$$

The Teichmuller points $\xi_\tau = [(X'_{\tau_0}, \phi_\tau, X'_\tau)] \in T(1, 2)$ for $\tau \in U$ form the fixed point set T^f, ϕ_τ is in fact the Teichmuller map (in its homotopy class) between X'_{τ_0} and X'_τ. So $T^f \cong U$.

For the proof we need two Lemmas.

LEMMA 5.4. *The integrable quadratic differentials $Q(X'_{\tau_0})$ on X'_{τ_0} are a 2-dimensional* **C**-*vector space spanned by the basis*

$$\{dz^2, sn(z + \tau_0/2)dz^2\}$$

where sn is the classical Jacobian elliptic function with periods '$4K' = 1$ and '$2iK' = \tau_0$ (see Whittaker-Watson [10, p. 504]).

Here z is the uniformising parameter on the universal cover C of X_{τ_0} and the above quadratic differentials on C induce differentials on X'_{τ_0}. This is not hard to see from the properties of the sn function.

LEMMA 5.5. *The codifferential f^* of the involution f acting on $Q(X'_{\tau_0})$ decomposes the latter into the direct sum of eigen-spaces $E(+)$ and $E(-)$, (as in Definition 3.7). Indeed here*

$$E(+) = \text{subspace of } Q(X'_{\tau_0}) \text{ spanned by } `dz^2' \tag{5.1}$$

$$E(-) = \text{subspace of } Q(X'_{\tau_0}) \text{ spanned by } `sn(z + \tau_0/2)dz^2'. \tag{5.2}$$

PROOF. The involution $f \in \text{Mod}(1, 2)$ at $X'_{\tau_0} \in T(1, 2)$ was induced from $f_0(z) = z + \frac{1}{2}$ on the complex plane (Definition 5.2). Now, by a crucial property of the sn function:

$$sn(u \mp \tfrac{1}{2}) = -sn\, u \text{ (see Whittaker-Watson [10, p. 500])}. \tag{5.3}$$

So

$$f_0^*(sn(z + \tau_0/2)dz^2) = -(sn(w + \tau_0/2)dw^2). \tag{5.4}$$

Thus we see $dz^2 \in E(+)$, and $sn(z + \tau_0/2)dz^2 \in E(-)$. The lemma follows easily. □

PROOF OF THEOREM 5.3. Knowing $E(+)$ from Lemma 5.5 we can apply Theorem 3.8 to assert that

$$T^f = \{S \in T(1, 2) : S \text{ lies on the geodesic disc given by } \operatorname{Im} e_\phi \text{ (through } X'_{\tau_0}),\ \phi = dz^2\}$$

We claim that the marking map $\phi_\tau : X_0' \to X'$ has complex dilatation given by a T-B differential $t\bar{\phi}/|\phi|$ with $\phi \in E(+)$ and $t \in \Delta$. This follows by a calculation. Hence indeed ϕ_τ is a Teichmuller mapping corresponding to a quadratic differential in $E(+)$ and therefore the point ξ_τ of Theorem 5.3 is in T^f.

Conversely, it is easy to see that the whole geodesic disc $\operatorname{Im} e_\phi$, $\phi = dz^2$ consists of the ξ_τ, $\tau \in U$. Hence we have $T^f = \{X'_\tau$ with marking map $\phi_\tau : \tau \in U\} \cong U$ proving our result. □

THEOREM 5.6. *Let H_f (Definition 3.1) be the retraction of $T(1,2)$ to $T(1,2)^f (= T^f)$ induced by the involutory $f \in Mod(1,2)$ (of Definition 5.2). If X'_{τ_0} ($T_0 \in U$) is an arbitrary point of T^f then the fiber $H_f^{-1}(X'_{\tau_0})$ is the geodesic disc through X'_{τ_0} given by Im e_ϕ where*

$$\phi = sn(z + \tau_0/2)dz^2 \; (\in E(-)).$$

PROOF. This is a restatement of Theorem 3.9 and equation (5.2) of Lemma 5.5. □

A deeper study of H_f will follow after we have made some other considerations on $T(1,2)$.

AN EXPLICIT HOLOMORPHIC RETRACTION $\tilde{H}$ OF $T(1,2)$ ONTO T^f. We saw in Theorem 5.3 that T^f is essentially a $T(1,1)$ ($\cong U$) embedded in $T(1,2)$. There is a natural holomorphic projection $\pi: T(1,2) \to T(1,1)$ obtained by 'forgetting a puncture' (see for example Kra [4]). We want to define a holomorphic retraction $\tilde{H}: T(1,2) \to T^f$ by identifying it essentially with this 'forgetful map' π. The simplest way to do this is as follows.

Let X'_{τ_0} as defined at the beginning of Section 5 be kept as our base-point for $T(1,2)$. Define.

$$\psi: T(1,2) \to U$$

by $\psi([\mu]) = w^\mu(\tau_0)$ (w^μ as in Definition 2.6). Notice that, by the classical Bers isomorphism theorem, the Bers fiber space $F(1,1) \cong T(1,2)$ and the map ψ becomes identifiable with the Bers fiber projection τ_0 (Definition 2.6).

Now $\sigma: U \to T^f$ defined by $\sigma(\tau) = \xi_\tau$, where ξ_τ is the element of T^f defined in Theorem 5.3, is a holomorphic isomorphism. We define

$$\tilde{H}: T(1,2) \to T^f \text{ as the composition } \sigma \circ \psi.$$

Clearly we have the following:

THEOREM 5.7. *$\tilde{H} = \sigma \circ \psi$ defined above is a holomorphic retraction of*

$T(1, 2)$ onto the fixed point set T^f of the involution f. The fibers of H are the fibers of the natural projection $\pi: T(1, 2) \to T(1, 1)$.

Now the following was proved in the author's paper [6].

THEOREM 5.8. *The fibers of the natural projection $\pi: T(g, n+1) \to T(g, n)$ are never Teichmuller geodesic discs (except for the trivial case $(g, n) = (0, 3)$) although they are properly and holomorphically embedded complex discs in $T(g, n+1)$.*

We saw in Theorem 5.6 that the geometrically induced retraction H_f had geodesic discs as fibers. So from Theorems 5.7 and 5.8 we get:

THEOREM 5.9. *H_f and $\tilde{H}$ are distinct maps from $T(1, 2)$ onto T^f.*

This result is especially interesting in contrast with the following result we will establish: the maps H_f and $\tilde{H}$ *do* coincide upto first order approximation near their common target set T^f.

To prove this we need local coordinates for $T(1, 2)$ in the neighbourhood of a point of T^f to compare the positions of the fibers of H_f and $\tilde{H}$ above the point. Let $X'_{\tau_0} \in T^f$ be the point of T^f we have used as base point, (it is the τ_0-torus with punctures at [0] and [1/2]). Then define

$$\psi_1: T(1, 2) \to U \times \mathbf{C} \quad \text{as}$$

$$\psi_1: ([\mu]) = (w^\mu(\tau_0), w^\mu(1/2)) = (\tau, z) \text{ say.}$$

It has been proved by the author in his paper [7] that this map is a holomorphic universal covering map of its image set $D_{1,2}$ (which is actually the Torelli space for twice-punctured tori). Hence we may use ψ_1 to give local coordinates to $T(1, 2)$ near $X'_{\tau_0} \in T^f$.

Now the map $\psi_1: T(1, 2) \to$ Torelli space $D_{1,2}$ is the universal covering, so the modular transformation f on $T(1, 2)$ induces a transformation f_* on $D_{1,2}$. The definition of ψ_1 and f shows that

$$f_*(\tau, z) = (\tau, 1 - z) \qquad (5.5)$$

(This follows also from the general results about automorphisms on Torelli spaces in the author's paper [7]).

REMARK. The fixed point set of f_* on Torelli space is of course the projection by ψ_1 of the fixed point set T^f of f. Notice that the fixed point set of f_* is the set $\{(\tau, z) \in D_{1,2} : z = 1/2\}$. This is in exact agreement with the description of T^f given in Theorem 5.3.

Now recall that the fiber $H_f^{-1}(\xi_\tau)$ is a geodesic disc Im e_j where $j \in E^{X_\tau'}(-) (\subset Q(X_\tau'))$. Consider the function $g: \Delta \to U$ given by

$$g = \pi_1 \circ \psi_1 \circ e_j.$$

(Δ is the unit disc, e_j is as in Definition 2.5, and π_1 is projection to the first factor).

LEMMA 5.10. *g is an even analytic function on Δ, and therefore* $g'(0) = 0$.

PROOF. By Lemma 5.5 we know that $j \in E(-)$ on X_τ' is induced from the quadratic differential $sn(z + \tau/2)\, dz^2$. Of course g is holomorphic since each component map e_j, ψ_1, π_1 is holomorphic.

Now, equation (3.2) of Section 3 says (since $j \in E(-)$) that

$$f(e_j(t)) = e_j(-t); \quad \text{let } \psi_1 \circ e_j(t) = (\tau, z).$$

So

$$\begin{aligned} g(-t) &= \pi_1 \circ \psi_1 \circ e_j(-t) \\ &= \pi_1 \circ \psi_1 \circ f \circ e_j(t) \\ &= \pi_1 \circ f_* \circ \psi_1 \circ e_j(t) \\ &= \pi_1 \circ f_*(\tau, z) \\ &= \pi_1(\tau, 1 - z) \\ &= \tau = \pi_1 \circ \psi_1 \circ e_j(t) = g(t), \end{aligned}$$

as claimed. □

The function g gives the values of the first coordinate function (τ) (obtained from the ψ_1-given coordinate system) on the fiber $H_f^{-1}(\xi_\tau)$.

Now, the fiber $\tilde{H}^{-1}(\xi_\tau)$ was seen (Theorem 5.7) to be the fiber of the natural projection $\pi: T(1,2) \to T(1,1)$. Hence the first coordinate function $\tilde{g}$(via ψ_1) of the fiber $\tilde{H}^{-1}(\xi_\tau)$ is the constant function $\tilde{g}: \Delta \to U$, $\tilde{g}(t) = \tau$, $t \in \Delta$. Indeed, in a local coordinate system given by ψ_1 around ξ_τ the fiber $\tilde{H}^{-1}(\xi_\tau)$ is described by $\{(\tau, z): \tau \text{ fixed}, z \text{ varies}\}$. (For recall that $\tilde{H} = \sigma \circ \psi = \sigma \circ \pi_1 \circ \psi_1$). This motivates the following

Definition 5.11. Let $\gamma = g - \tilde{g}$ be the difference of the first coordinate functions on the fibers above ξ_τ of H_f and $\tilde{H}$ respectively. We will say H_f and $\tilde{H}$ coincide to nth order approximation at ξ_τ if

$$\gamma(0) = \gamma'(0) = \gamma''(0) = \ldots = \gamma^{(n)}(0) = 0.$$

This is justified because $\gamma: \Delta \to U$ is a holomorphic function measuring the proximity of the fiber of $\tilde{H}_f$ to the fiber of $\tilde{H}$ at ξ_τ.

Theorem 5.12. *H_f and $\tilde{H}$ coincide up to first order approximation all along their common target set T'.*

This follows from Lemma 5.10.

Remark. The analytic second coordinate function $h = \pi_2 \circ \psi_1 \circ e_f$ on the fiber $H_f^{-1}(\xi_\tau)$ satisfies the equation $h(-t) = 1 - h(t)$ in a manner similar to Lemma 5.10.

We now want to develop the function-theoretic consequences of the comparison between H_f and $\tilde{H}$ as announced before.

Theorem 5.13. *With $g: \Delta \to U$ as in Proposition 5.10 we have*

$$g'(0) = -\frac{1}{\pi}\tau_0(\tau_0 - 1)\iint_{\mathbf{C}} \frac{\overline{sn(z+\tau_0/2)}}{|sn(z+\tau_0/2)|} \cdot \frac{dx\,dy}{z(z-1)(z-\tau_0)} \qquad (5.6)$$

(here $\tau_0 \in U$, sn is the classical Jacobian elliptic function with periods 1 and τ_0). Hence as $g'(0) = 0$, we get

$$\iint_{\mathbf{C}} e^{-i\arg(sn(z+\tau_0/2))}/z(z-1)(z-\tau_0)\,dx\,dy = 0 \quad (\text{any } \tau_0 \in U). \qquad (5.7)$$

Proof. Recall the perturbation formula for the solution of Beltrami equations (see Ahlfors [1]).

If $\mu(z, t)$ is a Beltrami differential for all values of the (real or complex) parameter t near $t = 0$ and

$$\mu(z, t) = t\nu(z) + t\varepsilon(z, t)$$

where ν and ε are in L_∞ and $\|\varepsilon(z, t)\|_\infty \to 0$ as $t \to 0$ then the q.c. solution $w^{\mu(z,t)}(\zeta)$ has a t-derivative at $t = 0$ given by

$$w^{\nu}(\zeta) = -\frac{1}{\pi}\zeta(\zeta - 1)\iint_{\mathbf{C}} \nu(z)\cdot\frac{1}{z(z-1)(z-\zeta)}\,dx\,dy, \; (z = x + iy). \quad (5.8)$$

For our application set

$$\mu(z, t) = t\frac{\bar{j}}{|j|} = t\nu(z)$$

where $j = sn(z + \tau_0/2)$. Then recall $\pi_1 \circ \psi_1([\mu]) = w^{\mu}(\tau_0)$; so we get

$$g(t) = w^{\mu(z,t)}(\tau_0)$$

and hence the results follow since $w^{\nu}(\tau_0) = g'(0) = 0$. □

We wish to point out the interesting fact that the vanishing integral identity (5.7) has been obtained as a consequence of our Teichmüller theoretic results involving H_f, etc. We will do some classical analysis below to derive (5.7) in a less mysterious fashion.

PROPOSITION 5.14. Let $w_{m,n} = m + n\tau$, $(m, n) \in \mathbf{Z} \times \mathbf{Z}$ and define

$$\theta(z, \tau) = \sum_{(m,n)\in \mathbf{Z}^2} \frac{1}{(z + w_{m,n})(z - 1 + w_{m,n})(z - \tau + w_{m,n})}$$

This double series converges uniformly and absolutely in compact z-sets (any fixed $\tau \in U$) to a value $\theta(\tau)$ *independent of* z.

PROOF. Let z vary in some compact subset K in $\mathbf{C}$. We will, for the present, keep K disjoint from the lattice points $w_{m,n}$. It is easy to see by estimating the terms that the series converges uniformly and absolutely in K.

Therefore $\theta(z, \tau)$ is a holomorphic function of z on

$$\mathbf{C} - \{w_{m,n} : (m, n) \in \mathbf{Z}^2\}.$$

Now, at one of the isolated singularities, say $z = w_{m_0,n_0}$, there are only three terms $t_{m,n}$ with $(z - w_{m_0,n_0})$ in the denominator, and in each of these terms $1/(z - w_{m_0, n_0})$ occurs only to the first power. Since, all the rest of the terms have a finite holomorphic sum near $z = w_{m_0, n_0}$, we see that the singularity of $\theta(z, \tau)$ at w_{m_0, n_0}, can at most be a simple pole.

But, by rearrangement of terms, we see

$$\theta(z + 1, \tau) = \theta(z, \tau)$$

and

$$\theta(z + \tau, \tau) = \theta(z, \tau)$$

Thus $\theta(z, \tau)$ is meromorphic doubly-periodic (1 and τ) function of z with at most one simple pole in each period parallelogram. From elliptic function theory this implies the proposition since no nonconstant elliptic function can have order less than two. □

Now we will use a perturbation argument similar to the proof of Theorem 5.13 to actually evaluate the double-series.

THEOREM 5.15. *The value* $\theta(\tau) = \dfrac{2i\pi}{\tau(\tau - 1)}$; *hence we get the identity*

$$\sum_{(m, n) \in Z^2} \frac{1}{(z + w_{m,n})(z - 1 + w_{m,n})(z - \tau + w_{m,n})}$$

$$= \frac{2i\pi}{\tau(\tau - 1)}, \quad (w_{m,n} = m + n\tau),$$

for all $z \in \mathbf{C}$ *and* $\tau \in U$.

PROOF. Define a one-parameter (t) family of Beltrami differentials on $\mathbf{C}$ by

$$\mu(z, t) = t; \text{ (thus } \mu \text{ is constant for each } t).$$

Then notice

$$w^{\mu}(z) = (z + t\bar{z})/(1 + t),$$

Direct calculation shows

$$\dot{w}(\tau) = \frac{d}{dt} w^{\mu(t)}(\tau) \Big|_{t=0} = (\bar{\tau} - \tau).$$

But, via the perturbation equation (5.8) we have

$$\dot{w}^{\mu}(\tau) = -\frac{\tau(\tau - 1)}{\pi} \iint_{\mathbf{C}} \frac{1}{z(z - 1)(z - \tau)} \, dx \, dy.$$

Let $P_{m,n}$ be the 'period parallelogram' with vertices $w_{m,n}$, $w_{m+1,n}$, $w_{m+1,n+1}$ and $w_{m,n+1}$. Then, breaking up $\iint_C$ to a sum of $\iint_{P_{m,n}}$ we get, after interchanging sum and integral that

$$\iint_C \frac{1}{z(z-1)(z-\tau)}\,dx\,dy = \theta(\tau)\cdot(\text{area of } P_{0,0})\ (\text{using Proposition 5.14})$$

Using the two equations for $w^{\mu}(\tau)$ we get then

$$\theta_0(\tau)\cdot(\text{area of } P_{0,0})\left(-\frac{\tau(\tau-1)}{\pi}\right) = (\bar{\tau}-\tau)$$

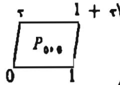
implying

$$\theta(\tau) = \frac{2i\pi}{\tau(\tau-\tau)} \text{ since } (\text{area of } P_{0,0}) = \frac{1}{2}(\bar{\tau}-\tau). \qquad \square$$

LEMMA 5.16.

$$\iint_C exp\,(-\,i\,arg\,sn(z+\tau/2)\cdot\frac{dxdy}{z(z-1)(z-1)}\ (sn \text{ with periods } 1 \text{ and } \tau)$$

$$= \frac{2i\pi}{\tau(\tau-1)}\iint_{P_{0,0}} exp\,(-\,i\,arg\,sn\,(z+\tau/2))\,dx\,dy$$

$\Bigg(P_{0,0}$ is  $\Bigg)$

PROOF. Notice that since sn is elliptic with periods 1 and τ we can break up the integral on the left-hand side into a sum of integrals over the period parallelograms $P_{m,n}$ (as in the previous proof). The result then follows immediately. $\square$

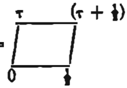
To give a direct proof of the identity (5.7) we are now reduced to showing the following.

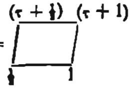
THEOREM 5.17. *Let sn be the Jacobian elliptic function with periods 1 and τ and $P_{0,0}$ be the fundamental period parallelogram. Then*

$$\iint_{P_0} exp\,(-\,i\,arg\,sn\,(z+\tau/2))\,dxdy = 0.$$

Equivalently, $\iint_{P_{0,0}} \cos(\arg sn\, z)\, dxdy = \iint_{P_{0,0}} \sin(\arg sn\, z)\, dxdy = 0.$

PROOF. Recall $sn\,(z + \frac{1}{2}) = -\, sn\, z$ (see equation (5.3)). Break up $P_{0,0}$ into the two parallelograms

$P_1 =$ 

and $P_2 =$ 

Since $\iint_{P_{0,0}} = \iint_{P_1} + \iint_{P_2}$, the result follows as the last two integrals cancel. □

In view of Lemma 5.16 and the above we have established the identity (5.7) by classical methods.

REMARK. Every complex torus with two punctures admits an involution which interchanges the punctures and has four fix-points on the surface. The quotient surface is a sphere with five distinguished points and indeed the Teichmuller spaces $T(1, 2)$ and $T(0, 5)$ are isomorphic. Using elliptic functions to uniformize the torus one can study this isomorphism and carry over the involutory f and associated results from $T(1, 2)$ on to $T(0, 5)$.

We only mention here the interesting vanishing integral that corresponds to identity (5.7) from this study. We have deduced:

$$\iint_{\mathbf{C}} \frac{\xi}{|\xi|} \frac{d\xi \wedge d\bar{\xi}}{|(1-\xi^2)(k^2-\xi^2)|} = 0, \text{ for any } k \in \mathbf{C} - \{0, \pm 1\}$$

and other similar results.

REFERENCES

1. AHLFORS, L. *Lectures on quasiconformal mappings*, Van Nostrand Co., Inc., Princeton, N.J., 1966.

2. EARLE, C.J. On moduli of closed Riemann surfaces with symmetries, in *Advances in the theory of Riemann Surfaces*, Ann. of Math. Studies, Princeton University Press, Princeton, N.J., No. 66 (1971).

3. EARLE, C.J. The Teichmuller distance is differentiable, *Duke Math. J.*, Vol. 44, No. 2 (1977), 389–397.

4. KRA, I. Canonical mappings between Teichmuller spaces, *Bull. of AMS*, Vol. 4, No. 2 (March 1981), 143–179.

5. KRAVETZ, S. On the geometry of Teichmuller spaces and the structure of their modular groups, *Ann. Acad. Sci. Fenn.*, 278 (1959), 1–35.

6. NAG, S. Non-geodesic discs embedded in Teichmuller Spaces, *Amer. J. of Math.*, Vol. 104, No. 2, (1982) 399–408.

7. NAG, S. The Torelli Spaces of punctured tori and spheres, *Duke Math. J.*, Vol. 48, No. 2, (June 1981), 359–388.

8. NAG, S. On the holomorphy of maps from a complex to a real manifold, *Pacific J. Math.* 10, 1 (1984). 191–201. Errata 1985.

9. ROYDEN, H. Automorphisms and isometries of Teichmuller space, in *Advances in the theory of Riemann surfaces*, Ann. of Math. Studies, Princeton Univ. Press, Princeton, NJ, No. 66 (1971).

10. WHITTAKER, E.T. and G.N. WATSON, *A course of modern analysis*, 4th edition, Cambridge University Press (1973 Reprint).

Indian Statistical Institute
B.T. Road
Calcutta 700 035