Allocation to Strata and Relative Efficiencies of Stratified and Unstratified πPS Sampling Schemes

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[Received December 1972. Final revision December 1973]

SUMMARY

The problem of optimum allocation to strata has been earlier examined in the light of *a priori* distributions. In this context, under the criterion of minimum expected variance, the sampling strategy consisting of an unstratified πPS sampling scheme together with the Horvitz–Thompson (HT) estimator was shown to be inferior to the strategy consisting of a stratified πPS sampling scheme with the corresponding HT estimator with this optimum allocation. In this paper, when stratification is based on the auxiliary information, we study whether a stratified πPS sampling strategy with various non-optimal allocations is likely to be worth while and whether it should be attempted at all. For populations commonly met in practice, we derive sufficient conditions for unstratified πPS sampling to be preferable to non-optimal stratified πPS sampling. An illustrative example is provided towards the end of the paper.

Keywords: SAMPLING WITH INCLUSION PROBABILITIES PROPORTIONAL TO SIZE; HORVITZ–THOMPSON ESTIMATOR; STRATIFICATION BY SIZE; ALLOCATION TO STRATA; SUPER POPULATION MODEL

1. INTRODUCTION

CONSIDER a finite population of size N. Values of $\mathscr{X}$ (an essentially positive auxiliary character closely related to the character $\mathscr{Y}$ under study) are available for all units, and the population is divided into k strata of sizes N_i, $i = 1, 2, \ldots, k$, defined by k non-overlapping ranges of values of $\mathscr{X}$. For convenience we label the strata, and the units within the strata, in ascending order of $\mathscr{X}$, so that if x_{ij}, y_{ij} are the values of $\mathscr{X}$, $\mathscr{Y}$ respectively for the jth unit of stratum i, then

$$0 < x_{11} \leqslant x_{12} \leqslant \ldots \leqslant x_{1N_1} \leqslant x_{21} \leqslant \ldots \leqslant x_{kN_k}.$$

Let a πPS (π_λ, the probability of inclusion of the λth unit in the sample, Proportional to Size) sample of size n_i be taken from the ith stratum (Hanurav, 1967; Vijayan, 1968; and others) such that $\sum_{i=1}^{k} n_i = n$. Let $\pi_{(i)j}$ denote the probability of inclusion of the jth unit of the ith stratum in the sample, given by $\pi_{(i)j} = n_i x_{ij}/X_i$, where $X_i = \sum x_{ij}$ is the total of the $\mathscr{X}$-values of the ith stratum. As an estimator of the population total $Y = \sum\sum y_{ij}$, consider the Horvitz–Thompson estimator (Horvitz and Thompson, 1952)

$$\hat{Y}_S = \sum_i \sum_j{}' y_{ij}/\pi_{(i)j}$$
$$= \sum_i \sum_j{}' y_{ij}/(n_i x_{ij}/X_i), \qquad (1.1)$$

where $\sum_j'$ denotes summation over the sampled units in the ith stratum. Next consider the corresponding Horvitz–Thompson estimator of the population total Y based on a πPS sample of size n from the whole population (unstratified) given by

$$\hat{Y}_U = \sum_i \sum_j{}' y_{ij}/\pi'_{(i)j}, \tag{1.2}$$

where $\sum_j'$ denotes summation over those units out of the sampled n that belong to the ith stratum and $\pi'_{(i)j}$'s are the probabilities of inclusion of the units given by $\pi'_{(i)j} = nx_{ij}/X$, where $X = \sum X_i$ (we assume that n, the values x_{ij} and the stratification adopted are such that values n_i can be chosen so that none of the $\pi_{(i)j}$ or $\pi'_{(i)j}$ exceed unity).

Cochran (1946) showed that whenever auxiliary information on a character $\mathscr{X}$ closely related to the character $\mathscr{Y}$ under study is available, this information can be used to set up a criterion of optimality, by regarding $\mathbf{y} = (y_{11}, y_{12}, \ldots, y_{kN_k})$ as a realization of an N-length random vector with distribution depending on $\mathbf{x} = (x_{11}, x_{12}, \ldots, x_{kN_k})$ and some unknown parameters.

Given $\mathbf{x}$ or equivalently $\mathbf{p} = (p_{11}, p_{12}, \ldots, p_{kN_k})$ where $p_{ij} = x_{ij}/X$, we explicitly formulate our model $\theta(g)$ thus:

$$\left.\begin{aligned} \mathscr{E}_{\theta(g)}(y_{ij}|p_{ij}) &= ap_{ij}, \\ \mathscr{V}_{\theta(g)}(y_{ij}|p_{ij}) &= \sigma^2 p_{ij}^g, \\ \mathscr{C}_{\theta(g)}(y_{ij}, y_{rh}|p_{ij}, p_{rh}) &= 0, \end{aligned}\right\} \tag{1.3}$$

where the script letters $\mathscr{E}$, $\mathscr{V}$ and $\mathscr{C}$ denote the conditional expectation, variance and covariance given p_{ij}'s. In this realistic model of practical interest, while there are no theoretical limitations on the value of g, it is observed that g is non-negative and in most of the practical situations it is found to lie between 1 and 2. This has been borne out by the empirical studies of Smith (1938), Jessen (1942) and Mahalanobis (1944). Under this model (1.3) we now compare the two strategies, the stratified πPS sampling scheme together with the estimator $\hat{Y}_S$ and unstratified πPS sampling scheme together with the estimator $\hat{Y}_U$ in the expected variance sense.

Considering the Horvitz–Thompson estimator $\hat{Y}_S$ defined in (1.1) we have

$$\begin{aligned} \operatorname{var}(\hat{Y}_S) &= E(\hat{Y}_S^2) - Y^2 \\ &= \sum_{i=1}^{k} \left\{ \sum_{j=1}^{N_i} (\pi_{(i)j}^{-1} - 1) y_{ij}^2 + \sum_{j \neq h}^{N_i} \sum^{N_i} (\pi_{(i)jh} \pi_{(i)j}^{-1} \pi_{(i)h}^{-1} - 1) y_{ij} y_{ih} \right\}, \end{aligned}$$

where $\pi_{(i)jh}$ is the probability of joint inclusion of the jth and hth unit of the ith stratum in the sample. Further under the model $\theta(g)$ of (1.3) we have

$$\begin{aligned} \mathscr{E}_{\theta(g)} \operatorname{var}(\hat{Y}_S) &= \sum_{i=1}^{k} \left\{ \sum_{j=1}^{N_i} (\pi_{(i)j}^{-1} - 1)(\sigma^2 p_{ij}^g + a^2 p_{ij}^2) \right. \\ &\qquad \left. + \sum_{j \neq h}^{N_i} \sum^{N_i} (\pi_{(i)jh} \pi_{(i)j}^{-1} \pi_{(i)h}^{-1} - 1) a^2 p_{ij} p_{ih} \right\} \\ &= \sigma^2 \sum_{i=1}^{k} \sum_{j=1}^{N_i} (\pi_{(i)j}^{-1} - 1) p_{ij}^g + a^2 \operatorname{var}(\hat{P}_S), \end{aligned}$$

where $\hat{P}_g$ is obtained by replacing y_{ij} in (1.1) by p_{ij}. Clearly $\text{var}(\hat{P}_S) = 0$, so that

$$\mathscr{E}_{\theta(g)} \text{var}(\hat{Y}_S) = \sigma^2 \sum_{i=1}^{k} \sum_{j=1}^{N_i} (\pi_{(i)j}^{-1} - 1) p_{ij}^g, \tag{1.4}$$

where

$$\pi_{(i)j} = n_i x_{ij}/X_i = n_i p_{ij}/P_i, \; P_i = \textstyle\sum_j p_{ij}.$$

Similarly, considering the corresponding expression for the variance of $\hat{Y}_U$ we have under the model $\theta(g)$ of (1.3)

$$\mathscr{E}_{\theta(g)} \text{var}(\hat{Y}_U) = \sigma^2 \sum_{i=1}^{k} \sum_{j=1}^{N_i} (\pi_{(i)j}'^{-1} - 1) p_{ij}^g, \tag{1.5}$$

where

$$\pi'_{(i)j} = n x_{ij}/X = n p_{ij}.$$

Now consider

$$\begin{aligned} f(g) &= \mathscr{E}_{\theta(g)} \{\text{var}(\hat{Y}_S) - \text{var}(\hat{Y}_U)\}/\sigma^2 \\ &= \sum_{i=1}^{k} \sum_{j=1}^{N_i} p_{ij}^g (\pi_{(i)j}^{-1} - \pi_{(i)j}'^{-1}) \\ &= \sum_{i=1}^{k} \sum_{j=1}^{N_i} p_{ij}^{g-1} (n_i^{-1} P_i - n^{-1}) \\ &= \sum_{i=1}^{k} \sum_{j=1}^{N_i} a_i p_{ij}^{g-1}, \end{aligned} \tag{1.6}$$

where $a_i = n_i^{-1} P_i - n^{-1}$.

The problem of allocation to strata has been earlier examined in the light of *a priori* distributions by Hanurav (1965) and Rao (1968). It was also shown in Rao (1968) that under the model $\theta(g)$ of (1.3), allocation to strata which minimizes (1.4) is given by the "$\theta(g)$-optimum allocation"

$$n_i = n\left(X_i \sum_{j=1}^{N_i} x_{ij}^{g-1}\right)^{\frac{1}{2}} \Bigg/ \sum_{i=1}^{k} \left(X_i \sum_{j=1}^{N_i} x_{ij}^{g-1}\right)^{\frac{1}{2}}.$$

From this it is easy to see that when $g = 2$, $\theta(2)$-optimum allocation reduces to allocation proportional to the stratum totals of the x_{ij}'s. We next have Theorem 1.1.

Theorem 1.1 (Rao, 1968). In the sense of expected variance, under $\theta(g)$, unstratified πPS sampling is *inferior* to stratified πPS sampling with $\theta(g)$-optimum allocation except that for $g = 2$, both the schemes are equivalent.

It is, however, not known under what conditions unstratified πPS sampling is still inferior to stratified πPS sampling when one deviates from the $\theta(g)$-optimum allocation. With this in mind, we consider whether stratified πPS sampling with various non-optimal allocations is likely to be worth while and whether it should at all be attempted.

2. MAIN RESULTS

Theorem 2.1. Let $0 < p_{11} \leqslant p_{12} \leqslant \ldots \leqslant p_{1N_1} \leqslant p_{21} \leqslant \ldots \leqslant p_{kN_k}$ and the allocation $\mathbf{n} = (n_1, n_2, \ldots, n_k)$ be such that $a_i = n^{i-1} P_i - n^{-1}$, $i = 1, 2, \ldots, k$, are non-decreasing and not all equal where $P_i = \sum p_{ij}$. Further, let $a_i \leqslant 0$ for $i \leqslant t$ and $a_i > 0$ for $i > t$ and not all p_{ij}'s for $i > t$ are equal to p_{iN_i}. Let $f(g) = \sum\sum a_i p_{ij}^{g-1}$. Then

(a) if $f(1) < 0$, there exists a unique g_0 in the interval $(1,2]$ such that $f(g) \leqslant 0$ or >0 according as $g \leqslant g_0$ or $> g$,

(b) if $f(1) \geqslant 0$, $f(g) > 0$ for all g in $(1,2]$.

Proof. Let $h(g) = \sum\sum a_i Z_{ij}^{g-1}$ where $Z_{ij} = p_{ij}/p_{iN_i}$. Note that

$$\begin{aligned} h'(g) &= \sum_{i=1}^{k} \sum_{j=1}^{N_i} a_i Z_{ij}^{g-1} \log Z_{ij} \\ &= \sum_{i=1}^{t} \sum_{j=1}^{N_i} a_i Z_{ij}^{g-1} \log Z_{ij} + \sum_{i=t+1}^{k} \sum_{j=1}^{N_i} a_i Z_{ij}^{g-1} \log Z_{ij} \\ &\geqslant 0 \quad \text{(since all terms are } \geqslant 0) \end{aligned}$$

with strict inequality when not all p_{ij}'s for $i > t$ are equal to p_{iN_i}. Thus $h(g)$ is increasing with g provided not all p_{ij}'s for $i > t$ are equal to p_{iN_i}.

Next we have

$$\begin{aligned} f(2) &= \sum_{i=1}^{k} \sum_{j=1}^{N_i} p_{ij}(n_i^{-1} P_i - n^{-1}) \\ &= \sum_{i=1}^{k} P_i^2 n_i^{-1} - \left(\sum_{1}^{k} P_i\right)^2 \Big/ n \\ &\geqslant 0 \end{aligned}$$

(by the Cauchy–Schwarz inequality), equality occurring when and only when P_i/n_i, for $i = 1, 2, \ldots, k$, are all equal.

Now observing that $h(g) = p_{iN_i}^{1-g} f(g)$, it follows that $h(2) \geqslant 0$: also $h(1) = f(1)$. Hence it follows that when $f(1) < 0$, there exists a unique g_0 in the interval $(1,2]$ such that $h(g) \leqslant 0$ or >0, and so $f(g) \leqslant 0$ or >0 according as $g \leqslant g_0$ or $> g_0$, and when $f(1) \geqslant 0$, $h(g) > 0$ and so, $f(g) > 0$ for all g in $(1,2]$.

Corollary 2.1. If $f(2) = 0$, then $f(g) = 0$ for all g.

Corollary 2.2. When $g = 1$,

$$f(1) = \sum_{i=1}^{k} \left(\frac{N_i P_i}{n_i}\right) - \frac{N}{n} = \begin{cases} 0 \text{ for } n_i \propto N_i, \text{ i.e. proportional allocation,} \\ 0 \text{ for } n_i \propto X_i, \text{ i.e. allocation proportional to stratum totals,} \\ (k^2/n)\operatorname{cov}(N_i, P_i) \text{ for equal allocation or for } n_i \propto N_i X_i, \\ \quad \text{which is } < 0 \text{ when } N_i \text{ decreases as } X_i \text{ increases,} \\ n^{-1}\left\{\sum_{1}^{k} \sqrt{(N_i P_i)}\right\}^2 - \dfrac{N}{n} \text{ for } \theta(1)\text{–optimum allocation which} \\ \quad \text{is } < 0. \end{cases}$$

When $g = 2$,

$$f(2) = \sum_{i=1}^{k} P_i^2 n_i^{-1} - n^{-1} = \begin{cases} 0 \text{ for } \theta(2)\text{–optimum allocation,} \\ >0 \text{ for any other allocation, provided not all } P_i/n_i, \\ \quad i = 1, 2, \ldots, k \text{ are equal.} \end{cases}$$

Remark 2.1. The uniqueness of g_0 in Vijayan (1966) can be established on similar lines as in Theorem 2.1 above.

3. INTERPRETATION OF THE RESULTS

Consider a stratification of the population based on the auxiliary information $\mathscr{X}$ such that

$$0 < x_{11} \leqslant x_{12} \leqslant \ldots \leqslant x_{kN_1} \leqslant x_{21} \leqslant \ldots \leqslant x_{kN_k}$$

(cf. Introduction). It is then possible to consider an allocation $\mathbf{n} = (n_1, n_2, \ldots, n_k)$ for which X_i/n_i (equivalently a_i's) for $i = 1, 2, \ldots, k$ are non-decreasing and not all equal, where X_i is the total of the $\mathscr{X}$-values of the ith stratum. For example, if the stratification is such that a large number of units with small $\mathscr{X}$-values are grouped in the former strata and a small number of units with large $\mathscr{X}$-values are grouped in the latter strata and X_i's are non-decreasing, then this nature of the stratification might suggest an allocation away from the optimum with n_i's decreasing, thereby implying that X_i/n_i's are non-decreasing and not all equal. Interpretation of the results of Section 2 would now enable us to study the efficiency of unstratified sampling as compared to stratification with such non-optimal allocations for which a_i's are non-decreasing and not all equal.

Part (a) of the above theorem implies that whenever $\sum N_i P_i n_i^{-1} - Nn^{-1}$ (equivalently $\sum N_i X_i n_i^{-1} - NXn^{-1}) < 0$, there exists a value g_0 of the super-population parameter g such that stratified πPS sampling with the given allocation $\mathbf{n}$ is better (i.e. $f(g) < 0$) or worse (i.e. $f(g) > 0$) than unstratified πPS sampling of size n according as $g < g_0$ or $> g_0$. At this value $g = g_0$, stratification is as good as unstratified sampling. Furthermore, whenever $\sum N_i P_i n_i^{-1} - Nn^{-1}$ (equivalently $\sum N_i X_i n_i^{-1} - NXn^{-1}) \geqslant 0$, by part (b) of the theorem, stratified πPS sampling with the given allocation $\mathbf{n}$ is worse than unstratified πPS sampling (of size n).

Corollary 2.1 implies that when the allocation is $\theta(2)$-optimum for all g (i.e. the allocation proportional to the stratum totals the X_i's), then stratified πPS sampling and unstratified πPS sampling are equivalent for all g. Provided the required conditions of the theorem are satisfied which reduce to the ordering (non-decreasing) of the $\mathscr{X}$-values and the ordering (non-decreasing and not all equal) of the stratum means, Corollary 2.2 implies that stratified πPS sampling with allocation proportional to the stratum sizes is worse than unstratified πPS sampling.

However, with equal allocation to the strata (or allocation proportional to $N_i X_i$), in practice, we do come across stratified populations with ordered (non-decreasing) $\mathscr{X}$-values for which as N_i decreases X_i increases so that $\text{cov}(N_i, X_i)$ is negative and the conditions of the theorem are automatically satisfied, thereby implying that stratified πPS sampling is better than unstratified πPS sampling for values of the super-population parameter g near 1. Moreover, as mentioned for the case of proportional allocation when we have the ordering (non-decreasing) of the $\mathscr{X}$-values and the ordering (non-decreasing and not all equal) of the stratum means, stratified πPS sampling with $\theta(1)$-optimum allocation is better than unstratified πPS sampling for values of g close to 1.

4. ILLUSTRATIONS AND REMARKS

In this section we illustrate the above results using real data on crops and grass acreage given by Sampford (1962, p. 61) which relates to 35 farms in Orkney. The population was divided into three strata (Sampford, p. 72) containing farms 1–12, farms 13–24 and farms 25–35. Here the stratum sizes are $N_1 = 12$, $N_2 = 12$ and $N_3 = 11$ and the stratum totals of the crops and grass acreage are $X_1 = 735$, $X_2 = 1537$ and $X_3 = 3487$ respectively. An overall sample of size $n = 9$ is taken for illustration and various feasible allocations (with the restriction that at least two units be selected from each stratum for the estimability of the variance of the estimator) are considered.

We present Table 1 showing the efficiency of unstratified πPS sampling as compared to stratified πPS sampling for these allocations.

TABLE 1

The efficiency of unstratified πPS sampling compared to stratified πPS sampling for all feasible allocations for a fixed sample size $n = 9$ for $g = 1{\cdot}0$ to $1{\cdot}9$

	Allocation							
g	(2, 3, 4)	(2, 4, 3)	(3, 2, 4)	(3, 4, 2)	(3, 3, 3)	(4, 2, 3)	(4, 3, 2)	(2, 2, 5)
1·0	0·8648	0·9646	0·9612	1·2604	0·9686	1·1092	1·3086	0·9343
1·1	0·8921	1·0118	0·9955	1·3543	1·0219	1·1668	1·4061	0·9510
1·2	0·9208	1·0620	1·0306	1·4540	1·0779	1·2266	1·5089	0·9675
1·3	0·9508	1·1151	1·0662	1·5593	1·1365	1·2882	1·6167	0·9839
1·4	0·9820	1·1708	1·1023	1·6689	1·1972	1·3514	1·7291	1·0002
1·5	1·0138	1·2283	1·1384	1·7821	1·2594	1·4153	1·8444	1·0160
1·6	1·0469	1·2889	1·1748	1·9010	1·3242	1·4809	1·9649	1·0315
1·7	1·0800	1·3500	1·2106	2·0206	1·3889	1·5459	2·0858	1·0464
1·8	1·1141	1·4135	1·2463	2·1445	1·4555	1·6119	2·2107	1·0607
1·9	1·1477	1·4765	1·2809	2·2674	1·5213	1·6764	2·3341	1·0742

In Table 1 corresponding to the allocations (3, 4, 2), (4, 2, 3) and (4, 3, 2) for which $f(1) = \sum_{i=1}^{3} N_i n_i^{-1} P_i - Nn^{-1}$ is positive, stratified πPS sampling is not recommended. If $f(1)$ is negative, which corresponds to the allocations (2, 4, 3), (3, 2, 4) and (3, 3, 3), if the value of g, the super-population parameter, is not far away from unity, then stratification might be used. Also note that for the allocation (2, 3, 4) the value of g_0 is between 1·4 and 1·5 and for values of $g \geqslant 1{\cdot}6$ the efficiency is nearly 1, which shows that stratification is, as can be expected, better than unstratified sampling since in this case the $\theta(g)$-optimum allocation is very close to (2, 3, 4) for all values of g (cf. Theorem 1.1).

For $g = 2$ the optimum allocation (by chance effect of rounding off) reduces to (2, 2, 5) which does not satisfy the conditions on a_i. In view of the fact that the conditions on a_i are sufficient, but not necessary for the theorem to hold, the efficiency corresponding to this allocation (2, 2, 5) as well is given in the last column of Table 1. It is interesting to observe that, for this allocation, the efficiencies for $g > 1{\cdot}3$ are very close to 1.

Remark 4.1. Instead of πPS sampling within each stratum, one can think of using $G\pi PS$ (Generalized πPS) sampling for which $\pi_{(i)j}$ is proportional to $x_{ij}^{g/2}$ and $\sum_j' x_{ij}^{1-(g/2)}$ is constant (Rao, 1971) within each stratum (the symbol $\sum_j'$ stands for summation over the sample units from the ith stratum). Because of the homogeneity of the $\mathscr{X}$-values within each stratum, the latter condition is mostly satisfied. Now from (1.4) and (1.5) we have

$$\begin{aligned} K(g) &= \mathscr{E}_{\theta(g)}\{\operatorname{var}(\hat{Y}'_S) - \operatorname{var}(\hat{Y}_U)\}/\sigma^2 \\ &= \sum_i \Big(\sum_j p_{ij}^{g/2}\Big)^2 \Big/ n_i - \sum_i \sum_j p_{ij}^{g-1}/n \\ &\leqslant f(g) \quad \text{for all } g \qquad (4.1) \end{aligned}$$

(by the Cauchy–Schwarz inequality), where $\hat{Y}'_g$ is the Horvitz–Thompson estimator with

$$\pi_{(i)j} = n_i x_{ij}^{g/2} \Big/ \sum_{j=1}^{N} x_{ij}^{g/2}.$$

Hence, whenever $f(1) < 0$ in which case there is a g_0 (by Theorem 2.1) below which stratified πPS sampling is better than unstratified πPS sampling, it automatically follows from (4.1) that with a stratified $G\pi PS$ sampling, one is better off for values of g at least up to this g_0. On the other hand, if $f(1) \geqslant 0$, while stratified πPS sampling is not recommended, one might expect that with a stratified $G\pi PS$ sampling within each stratum, one might still do better for values of g close to unity.

ACKNOWLEDGEMENTS

The authors are grateful to the referee for the critical comments and helpful suggestions on an earlier version of this paper.

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