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A SEARCH FOR OPTIMAL NESTED ROW-COLUMN DESIGNS

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SUMMARY. In this paper we initiate the study of optimality in nested row column setting and establish, under the usual fixed effects model, optimality of certain *non-binary* nested designs which *have not* hitherto been considered in the literature. Some series of such optimal designs have been constructed. In case of a mixed effects model (with all but the treatment effects random), an analogous study has been made and optimality results indicated.

1. Introduction

The study of nested designs in a general framework was initiated in Srivastava (1978). In Srivastava (1981), a formalised general set up with nested structure was considered and its information matrix derived. Singh and Dey (1979) presented the analysis of designs in a situation where a two-way (row-column) structure is present and it is nested within another (nuisance) factor called block. In Singh and Dey (1979) and in some subsequent papers (Agarwal and Prasad (1982 a, b, c)) various methods of construction of nested row-column designs are presented.

Optimality study in the context of two-way elimination of heterogeneity is comparatively recent. The pioneering work is in Kiefer (1975). The subsequent papers in this area include Cheng (1981), Jacroux (1986), Shah and Eccleston (1986) and Bagchi and Shah (1989). So far, nested row-column set up has not been studied from the point of view of optimality. In the present paper, we attempt to fill up this gap.

To start with, our plan was to prove the optimality property of the binary nested designs constructed by the earlier authors. But surprisingly, we end up proving that a class of non binary designs (termed BN-RC designs in Definition 2.5) perform very well, to the extent of being universally optimal under a fixed effects model. Under a mixed effects model (with all effects random), however, the relation between the variance components plays an important role in the determination of optimal designs.

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The BN-RC designs are new in the literature and they give rise to interesting combinatorial problems. In this paper, we present a few series of them.

2. Definitions and notations

Definition 2.1. Suppose the experimental units can be grouped into b blocks, each of which is a $p \times q$ rectangle where row effects and column effects are orthogonal. Such a setting is called a nested row-column setting.

In such a setting, there are altogether bp rows and bq columns, the rows and columns in the j-th block being numbered from $(j-1)p+1$ to jp and $(j-1)q+1$ to jq respectively.

Suppose we want to compare effects of v treatments in such a setting. Let y_{ijk} denote the observation from the (j, k)-th cell of the i-th block. If treatment h is applied in this cell, then we assume the model

$$y_{ijk} = \beta_i + \alpha_{ij} + \gamma_{ik} + \tau_h + e_{ijk}$$

where β_i = effect of i-th block, α_{ij} = effect of j-th row in i-th block, γ_{ik} = effect of k-th column in i-th block and τ_h = effect of h-th treatment. Regarding the error components, we assume usual homoscedasticity conditions to hold.

Let $L(v \times b)$ denote the treatment$\times$block incidence matrix, $M(v \times bq)$ the treatment$\times$column and $N(v \times bp)$ the treatment$\times$row incidence matrices. Let r_i denote the replication number of the i-th treatment, $1 \leqslant i \leqslant v$. Then from Singh and Dey (1979), the C-matrix of a design in such a setting is given by

$$C = D_r - p^{-1} MM' - q^{-1} NN' + (pq)^{-1} LL' \quad \ldots \quad (2.1)$$

where $D_r = \text{Diag}\ (r_1, r_2, \ldots, r_v)$.

Definition 2.2. If L is a binary matrix with entries 0 and 1 then the design is said to be a binary nested row-column design.

Clearly such a design can exist only when $qp \leqslant v$.

Notation 2.3. $\mathcal{D}_N(p, q, b, v)$ will denote the class of all connected nested designs with v treatments under the set-up considered in Definition 2.1. $\mathcal{D}_B(p, q, b, v)$ will denote the subclass of binary designs of $\mathcal{D}_N(p, q, b, v)$. The parameters p, q, b, v of a nested design will have the meaning in Definition 2.1. $\mathcal{D}_{(b,k,v)}$ will denote the class of all proper and connected block designs with b blocks, each of size k, and v treatments.

We shall denote a BIBD with parameters (v, b, r, k, λ) by BIBD (v, b, k). Also, an Youden square design with block size k and v treatments will be denoted by YSD (v, k).

Definition 2.3. (Agarwal and Prasad (1982a). A nested row-column design which is binary and has its C-matrix completely symmetric is termed a balanced incomplete block row-column (BIB-RC) design.

With the notation of (2.1), let $K = NN' - p^{-1} LL'$.

Definition 2.4. (a) If a nested row-column design d in $\mathcal{D}_N\ (p,\ q,\ b,\ v)$ is such that $K = 0$ and M is the incidence matrix of a BBD, then we call it a balanced nested row-column design and denote it by BN–RC design $(p,\ q,\ b,\ v)$.

A BN–RC design need not be binary.

(b) A nested row column design with $K = 0$ and M the incidence matrix of a group divisible design with parameters $(v,\ b,\ r,\ k,\ \lambda_1,\ \lambda_2,\ m,\ n)$ is called a group divisible nested row-column design and is denoted by GDN–RC design $(p,\ q,\ b,\ v,\ \lambda_1,\ \lambda_2,\ m,\ n)$. When $m = 2$ and $\lambda_2 = \lambda_1 + 1$, it is called a most balanced group divisible nested row-column design and is denoted by MBGDN-RC design $(p,\ q,\ b,\ v,\ \lambda_1,\ \lambda_2)$ in conformity with the definition of most balanced group divisible design (Coniffe and Stone (1974)) in the one-way set-up. This design was termed an extreme regular graph design of type 1 and was proved to be ψ_f-optimal of type 1 in Cheng (1978).

3. The search for optimal designs

3.0. *Two illustrative examples* : Normally we expect binary balanced designs to perform better than non-binary designs in a given set-up. But in the present context, we find the truth to be the contrary. To illustrate the point, we consider two examples below.

Example 3.0.1. Suppose an experiment involves three 2×2 arrays and four experimental treatments are to be compared. A *natural* choice for a design would be

d :

1	3	1	4	1	2
4	2	2	3	3	4

As against d, let us consider the highly non-binary design

d_0 :

1	2	1	3	1	4
2	1	3	1	4	1

Elementary computations yield the following regarding variances of BLUE's of elementary treatment contrasts :

$$\left.\begin{aligned} V(\widehat{\tau_1-\tau_i}) &= 2\sigma^2 \text{ for } d \\ &= \sigma^2 \text{ for } d_0 \end{aligned}\right\} \quad i = 2, 3, 4$$

$$V(\widehat{\tau_i-\tau_j}) = 2\sigma^2 \text{ for } d \text{ or } d_0,\ 2 \leqslant i \neq j \leqslant 4.$$

Strangely enough, the performance of the design d_0 is at least as good as or better than that of the design d in respect of the elementary treatment contrasts.

Example 3.0.2. This time suppose there are six 2×2 arrays and once again we have to administer four treatments. Consider the rival designs

$d' = 2$ replications of d

d_0' :

1	2
2	1

1	3
3	1

1	4
4	1

2	3
3	2

2	4
4	2

3	4
4	3

This time it can be easily verified that $V(\widehat{\tau_i-\tau_j}) = \sigma^2/2$ using the design d_0' as against $V(\widehat{\tau_i-\tau_j}) = \sigma^2$ using the design d' and this is true for *any* pair of treatments.

Clearly, d' is the design one would naturally suggest as it is 'balanced' in *all* senses. However, d_0' is, as a matter of fact, universally optimal in the entire class of competing designs. This is indicated later in Corollary 3.1.1(a). The designs of type d or d' were studied by Agarwal and Prasad (1982a).

3.1. *Optimality results* : Let ψ be a nonincreasing criterion (defined over the class of all C-matrices of order v) in the sense that $\psi(b\ C) \leqslant \psi(C)$ whenever the scalar b is $\geqslant 1$. Then we have the following main theorem of this paper.

Theorem 3.1.1 : *Suppose there is a design d^* in $\mathcal{D}_N\ (p, q, b, v)$ such that*

(i) *K is null and* (ii) *M_{d*} is the incidence matrix of a design which is ψ-optimal in $\mathcal{D}\ (bq, p, v)$.*

Then d^* is ψ-optimal in $\mathcal{D}_N$ (p, q, b, v).

Proof: Follows immediately from the observation that

$$C_d = D_r - p^{-1} MM' - q^{-1} K \qquad \dots \quad (3.1.1)$$

whenever $d \in \mathcal{D}_N$ (p, q, b, v).

Corollary 3.1.1 : (a) *A BN-RC design, when it exists, is universally optimal within $\mathcal{D}_N$ (p, q, b, v).*

(b) *An MBGDN-RC design, when it exists, is optimal with respect to all generalized criteria of type* 1.

Proof follows from Theorem 3.1.1 and the well known optimality properties of BBD and MBGDDs or ERGD's of type 1. Recall the numbering of the columns as stated after Definition 2.1.

Theorem 3.1.2 : (i) *A set of necessary and sufficient conditions for a design in $\mathcal{D}_N$ (p, q, b, v) to have $K = 0$ is the following*

(a) $n_{it} = u_{ij}, (j-1)p+1 \leqslant t \leqslant jp$

$l_{ij} = p.\, u_{ij},\ 1 \leqslant j \leqslant b,\ 1 \leqslant i \leqslant v,$

where the u_{ij}'s are non-negative integers satisfying

(b) $\sum_{i=1}^{v} \sum_{j=1}^{b} u_{ij} = q.b.$

(ii) *In order that a design in $\mathcal{D}_N$ (p, q, b, v) satisfying the condition in* (i) *also has the entries of M as nearly equal as possible, the u_{ij}'s must also satisfy*

(c) $(q/p)\,[p/v] \leqslant u_{ij} \leqslant (q/p)\,([p/v]+1)$

where $[x]$ denotes the integral part of x.

Proof: Easy.

Before we go to the actual constructions of optimal designs let us examine the conditions (b) and (c) of Theorem 3.1.2 more carefully.

Lemma 3.1.3 : *If $q < p$ and there exist integers u_{ij} satisfying (b) and (c) of Theorem* 3.1.2 *then v divides q.*

Proof: If $q < p$, then there can be at most one integer i_0 satisfying

$$(q/p)\,[p/v] \leqslant i_0 \leqslant (q/p)\,([p/v]+1).$$

So (c) implies $u_{ij} = i_0 = [(q/p)\ ([p/v]+1)]$, $1 \leqslant j \leqslant b$, $1 \leqslant i \leqslant v$. But this together with (b) of Theorem 3.1.2 implies $i_0 = q/v$, i.e. v divides q. Hence the result.

Lemma 3.1.4 : *When v does not divide q, a necessary and sufficient condition for the existence of a set of integers u_{ij} satisfying the conditions* (*b*) *and* (*c*) *of Theorem* 3.1.2 *is given below :*

(i) $(q/p)\,[p/v] \leqslant [q/v]$

(ii) $(q/p)\,([p/v]+1) \geqslant [q/v]+1$.

Proof : Easy.

Remark 3.1.1. Theorem 3.1.2 shows that a nested design with $K = 0$ is necessarily non-binary. But in view of Theorem 3.1.1, such designs are really the appropriate candidates for being optimal, not the binary ones. In particular, if p, q, v, b are such that both a BN-RC design (d_1) and BIB-RC design (d_2) exist, then

$$C_{d_1} = \frac{q}{q-1}\, C_{d_2}$$

so that the former is uniformly better than the latter. Examples of nested settings when both a BIB-RC and a BN-RC design exist are given in Section 3.2 after Theorems 3.2.2 and 3.2.4.

Remark 3.1.2. When v divides p and v does not divide q, the conditions of Lemma 3.1.4 do not hold. But the set of conditions obtained by interchanging the roles of p and q in Theorem 3.1.2 is satisfied by the design with the following incidence matrices :

$M = (p/v)\ J(v \times bq)$, $L = (pq/v)\ J(v \times b)$, and the entries of N are $[q/v]$ and $[q/v]+1$.

In view of this remark and Lemma 3.1.4, we make the following assumption without loss of generality.

Assumption 3.1.5. If v divides one but not both of p and q, we refer to the one divisible by v as q.

If v divides neither of p, q we refer to the smaller one as p.

3.2. *Methods of construction of optimal nested designs.* This section presents a few methods of construction for BN-RC designs and one method of construction of MBGDN-RC designs.

Case 1. $p < q < v$:

Theorem 3.2.1 : *Suppose a BIBD* (v, b, q) *and a* $YSD(q, p)$ *(see Notation 2.3) exist. Then a BN-RC design with parameters* p, q, b, v *exists.*

Proof : For each j, $1 \leqslant j \leqslant b$, the q treatments in the j-th block of the BIBD are used in forming the YSD arranged in a $p \times q$ array. That this constitutes the j-th block of the required nested row-column design is easy to see.

For our next result, we will utilize the notion of *cyclic difference* matrix defined below.

Definition 3.2.1. Let $G = (X, +)$ be a group. Let $A = ((a_{ij}))_{1 \leqslant i \leqslant t, 1 \leqslant j \leqslant n}$ be a matrix with elements of the group as its entries. Let D_{ij} denote the collection of all cyclic differences of the pairs of elements in the i-th row of A at distance j. More specifically,

$$D_{ij} = \{(a_{iu} - a_{iu'}),\ u - u' \equiv j (mod\ n)\},\ 1 \leqslant j \leqslant n-1.$$

For any element $g \neq 0$, $g \in G$, let $x_{ij}(g)$ denote the number of times g appears in D_{ij}. Let $\sum_{i=1}^{t} x_{ij}(g) = \lambda_j(g)$. If $\lambda_j(g) = \lambda_j$, a constant for all $g \neq 0$, $g \epsilon G$, $1 \leqslant j \leqslant n-1$, then A is said to be a cyclic difference matrix.

We shall denote a cyclic difference matrix A by $A\ (v, t, n, \lambda_j, 1 \leqslant j \leqslant n-1)$ where v is the size of the group, t, n are the numbers of rows and columns respectively, and λ_j is as above.

Theorem 3.2.2 : *Let there exist a cyclic difference matrix* $A(v, t, q, \lambda_j, 1 \leqslant j \leqslant q)$. *Then there exists a BN-RC design with parameters* $p, q, b = tv, v$ *for any integer* p *satisfying* $2 \leqslant p \leqslant q-1$.

Proof : Let G be the underlying group of the cyclic difference matrix.

Let $T_{ig} = A_i + \rho_q'(g)$, $g \in G$, $1 \leqslant i \leqslant t$, where A_i is the i-th row of A and $\rho_q\ (g)$ is a column vector of length q with all elements equal to g.

Let P be the permutation matrix of order q, as follows :

$$P = \begin{bmatrix} 0 & 1 \\ I_{q-1} & 0 \end{bmatrix}.$$

Let R_{ig} denote the $p \times q$ rectangle, the h-th row of which is

$$T_{ig}\ P^{h-1},\ 1 \leqslant h \leqslant p.$$

The collection of the rectangles $\{R_{ig},\ g \in G,\ 1 \leqslant i \leqslant t\}$ constitutes our required design.

To prove that this is a BN–RC design it is enough to show that the columns constitute the blocks of a BIBD. To show this, let α, $\beta \in G$, $\alpha \neq \beta$. From the assumptions, there are λ_j sequences in the family $\{T_{ig}, g \in G, 1 \leqslant i \leqslant t\}$ in which α and β appear at distance j. (The meaning of the term 'distance' is as in Definition 3.2.1.) In the corresponding rectangles R_{ig} the number of columns in which both α and β appear is p_j, where p_j is given in Lemma 3.2.3 below. Thus the pair α, β appear together in $\lambda = \sum_{j=1}^{q-1} p_j\lambda_j$ columns of the nested design. Since λ is independent of α, β the result follows.

Lemma 3.2.3 : *Let* a_i, $1 \leqslant i \leqslant q$ *be* q *symbols. Let* $A_{p\times q} = ((a_{ij}))$ *be the cyclic matrix given by* $a_{ij} = a_{i+j-1\ (\mathrm{mod}\, q)}$, $1 \leqslant p$, $1 \leqslant j \leqslant q$. *Let* k, m *be integers such that* $1 \leqslant k \leqslant m \leqslant q$ *and* $m-k = j$. *Then the number* p_j *of columns of* A *in which both* a_k *and* a_m *appear is given by*

$$p_j = \begin{cases} p-j & \text{if } 1 \leqslant j \leqslant \min(p-1,\, q-p) \\ p+j-q & \text{if } q-1 \geqslant j \geqslant \max(p,\, q-p+1) \\ 2p-q & \text{if } q-p+1 \leqslant j \leqslant p-1 \\ 0 & \text{if } p-1 < j < q-p+1 \end{cases}$$

Proof: By simple enumeration.

Example 3.2.1. If $v = qt+1$ is a prime power, let α denote a primitive element of $GF(v)$. Then the matrix $A = ((a_{ij}))$ with $a_{ij} = \alpha^{i+jt}$, $0 \leqslant i \leqslant j \leqslant q-1$ is a cyclic diffference matrix, with $\lambda_j = 1$ for each j, $1 \leqslant j \leqslant q-1$.

Corollary 3.2.2. *A BN-RC design* $(p,\ q,\ b = tv,\ v)$ *exists whenever* $v = qt+1$ *is a prime power and* $2 \leqslant p \leqslant q$.

Remark 3.2.1. If in particular, $p \mid t$, then a BIB–RC design exists (see 3.4 of Agarwal and Prasad (1982b)). The design constructed here is uniformly better than this BIB-RC design as explained in Remark 3.1.1.

Case 2. $p \mid q$.

Theorem 3.2.4 : *A BN–RC design* $(p,\ q = tp,\ b,\ v)$ *exists, whenever a BIBD* $(v,\ b,\ p)$ *exists.*

Proof: The j-th block of the design is obtained by the juxtaposition of t latin squares of order p with the p treatments occurring in the j-th block of the BIBD as symbols, $1 \leqslant j \leqslant b$.

Corollary 3.2.4 : *A BN–RC design* $\left(p = q = 2,\, b = \binom{v}{2},\, v\right)$ *always exists.*

Remark 3.2.2. The design d_0' of Example 3.0.2 is a member of this series with $v = 4$.

Remark 3.2.3. If in particular $v = 4\,t+1$ and v is a prime power, then a BIB-RC design with the same parameters exists by 3.3 of Agarwal and Prasad (1982b).

Case 3. $p < v < q$.

Theorem 3.2.5 : *If there exists a BN–RC design* $(p,\, q^*,\, v,\, b)$ *where* q^* *divides* q, *then a BN–RC design* $(p,\, q,\, v\, b)$ *also exists.*

Proof : Let $i = q/q^*$. Then the j-th block of the new design is obtained by the row-wise juxtaposition of the j-th block of the given BN–RC design i times, $1 \leqslant j \leqslant b$.

Theorem 3.2.6 : *If* $q \equiv p \pmod{v}$ *and a BIBD* (v, b, p) *exists, then a BN-RC design* $(p,\, q,\, b,\, v)$ *exists.*

Proof : Let G_v denote the group of residues mod v. Let the treatments be identified with the elements of G_v. Let B_i denote the $p\times 1$ vector representing the i-th block of the given *BIBD* $(v,\, b,\, p)$.

Let $B_{ij} = B_i+\rho_p(j)$, $j \in G_v$, where $\rho_p(j)$ is a column vector of length p with each element j. Let M_i denote the $p\times v$ rectangle of which the j-th column is B_{ij}, $j \in G_v$. Let $\delta = (q-p)/v$. (δ is integral from the assumption.)

Let L_i denote a latin square of order p with the entries of B_i as symbols. Then the row-wise juxtaposition of δ copies of M_i and one copy of L_i is the i-th block of the required BN-RC design, $1 \leqslant i \leqslant b$.

Case 4. $v < p < q$.

Theorem 3.2.7 : *If* $q \equiv p \pmod{v}$ *and* $p \equiv p_1 \bmod v$, $0 < p_1 < v$, *and a BIBD* $(v,\, b,\, p_1)$ *exists, then a BN-RC design* $(p,\, q,\, b,\, v)$ *exists.*

Proof : Let $p=p_1+\delta_1 v$, $p_1<v$ and $q=p+\delta_2\, v$, δ_1, δ_2 being positive integers. Starting with the i-th block of the given BIBD we construct the $p_1\times v$ array M_i in the same way as in Theorem 3.2.6. Let R be a $\delta_1\times\delta_2$ block matrix with entries as latin squares of order v with the v treatments as symbols. Let $\overline{M}_i$ be the $p_1\times\delta_2\, v$ array obtained by row-wise juxtaposition of δ_2 copies of M_i. Then

$$S_i = \left[\begin{array}{c|c} R & \\ \hline \overline{M}_i & L_i \end{array}\right]$$

is the i-th block of the BN-RC design, where L_i is a latin square of order p with symbols same as in the first column of S_i.

Construction of MBGDN-RC designs.

Theorem 3.2.7 : *An MBGDN-RC design* ($p = 2$, $q = 4$, $b = t^2$, $v = 4t$, λ_1, 0, $\lambda_2 = 1$) *exists for all positive integers* t.

Proof : Let a_i, $1 \leqslant i \leqslant 2t$ and b_i, $1 \leqslant i \leqslant 2t$ denote the treatments belonging to the two groups respectively. Let us index the blocks by the ordered pairs (i, j) $1 \leqslant i, j \leqslant t$. Then the (i, j)-th block is as follows.

$$\begin{bmatrix} a_{2i-1} & b_{2j-1} & a_{2i} & b_{2j} \\ b_{2j-1} & a_{2i} & b_{2j} & a_{2i-1} \end{bmatrix}, \; 1 \leqslant i, j \leqslant t.$$

That this is actually an MBGDN-RC design is easy to see.

By Corollary 3.1.1 (b), this design is ψ_f-optimal of type 1 within $\mathcal{D}_N(2, 4, t^2, 4t)$ whereas by Corollary 3.1.1 (a), the other designs constructed in this section are universally optimal within the respective classes.

4. Investigation under mixed effects

By a mixed effects model, we mean that all effects except treatment effects in the linear model are random with expectation zero.

In this situation, we find, optimality results are very sensitive to the relation between the unknown variance components. In other words the designs do not behave uniformly (with respect to a given optimality criterion) over the feasible range of variation of the variance σ_B^2, σ_R^2, σ_C^2, respectively of the block effects, row within block effects and column within block effects.

Let

$$\left.\begin{aligned} w &= (\sigma^2)^{-1} \\ w_1 &= (\sigma^2+q\sigma_R^2)^{-1} \\ w_2 &= (\sigma^2+p\sigma_C^2)^{-1} \\ w_3 &= (\sigma^2+q\sigma_R^2+p\sigma_C^2+pq\sigma_B^2)^{-1} \\ u &= w-w_1-w_2+w_3 \\ n &= pqb \end{aligned}\right\} \quad \ldots \quad (4.1)$$

where the parameters have the same meaning as in Definition 2.1. Then the C-matrix of a nested row-column design under a mixed effects model with the above parameters is given by

$$C_d(M) = w\, D_r-(w-w_1)p^{-1}\, MM'-(w-w_2)\, q^{-1}\, NN' +(w-w_1-w_2+w_3)\, (pq)^{-1}\, LL'-w_3\, n^{-1}\, rr' \quad \ldots \quad (4.2)$$

Notation as in (2.1).

We observe the following :

Theorem 4.1 : (a) *If* $u < 0$, *(see* (4.1)) *then a BIB-RC design, if it exists, is university optimal within* $\mathcal{D}_N$ (p, q, b, v).

(b) *If* $u > 0$, *then a BN-RC design is universally optimal within* $\mathcal{D}_N$ $(p, ,q\ b,v)$ *provided it satisfies the additional condition that the rows constitute the blocks of a BBD.*

Proof: Let us first note that

$$w > w_1,\ w_2 > w_3. \quad \dots \quad (4.3)$$

Hence the result (a) follows immediately from the expression in (4.2), in view of the fact that a BIB-RC minimises the traces of each of the matrices MM', NN', LL' and rr'.

To prove (b), we rewrite (4.2) as

$$C_d^{(M)} = u\ C_d^{(F)} + E$$

where

$$E = (w_1 + w_2 - w_3)\ D_r - p^{-1}\ (w_2 - w_3)\ MM'$$
$$-q^{-1}\ (w_1 - w_3)\ NN' - n^{-1}\ w_3\ rr'.$$

Now it is enough to verify the sufficient conditions for universal optimality for the matrix E, in view of corollary 3.1.1.(a) since the coefficient of $C_d^{(F)}$ is assumed to be positive. But those conditions are immediate from definition of a BN–RC design and the additional property assumed.

Remark 4.1. The BN-RC designs constructed in Theorems 3.2.1, 3.2.2, 3.2.4 (with $t = 1$), 3.2.6 and 3.2.7 satisfy the additional property mentioned in Theorem 4.1(b) and hence are universally optimal in the relevant class under a mixed effects model satisfying $u > 0$.

5. Concluding remarks

The findings in this nested row-column setting are interesting in some respects.

(i) In this setting, we have an example of a nonbinary design which performs uniformly better than a binary design (when they co-exist) under the fixed effects model (see Remark 3.1.1).

(ii) The optimality property of designs is very much model sensitive under a mixed effects model with all (nuisance) factors random. When $u < 0$

(see 4.1), the relative performance of the competing BIB-RC and BN-RC designs is just the opposite of that under a fixed effects model. Again when $u > 0$, their behaviour is similar to that under the fixed effects model, provided the BN-RC designs satisfy the additional row-property (see Theorem 4.1(b)). Whether a BN-RC design without this row-property has any importance under a mixed effects model is yet to be seen.

In case BN-RC designs do *not* exist, it may be possible to obtain high efficiency by making use of BIB-RC designs, whenever they exist. From the relation $C_{d_1} = \frac{q}{q-1} C_{d_2}$ stated in Remark 3.1.1, it is evident that this later type of designs will be quite efficient even when q is moderately large.

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