On Parallel Summability of Matrices

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ABSTRACT

It is shown that the generalized inverses characterize the parallel sum. The almost positive definite ($a.p.d.$) matrices introduced by Duffin and Morley [2] are of two types, whose intersection is the class of quasi-positive-definite matrices (Mitra and Puri [7]). The $a.p.d.$ matrices of any one type form a "saturated" subclass of pairwise parallel summable $a.p.d.$ matrices.

1. INTRODUCTION

Anderson and Duffin [1] were led to the concept of parallel sum of two hermitian nonnegative definite ($h.n.n.d.$) matrices of the same order $n \times n$ from the parallel connection of two n-port electrical networks involving only resistors. They have established many interesting properties of the parallel sum of a pair of $h.n.n.d.$ matrices. The concept was later extended to arbitrary pairs of matrices of the same order satisfying a "parallel summability" condition, and most of the properties proved by Anderson and Duffin were shown to be true in such a general context (Rao and Mitra [8]). The extension not only works for rectangular matrices but is even seen to be valid for matrices defined on more general fields. In fact Section 2 of this paper is written in the same spirit and requires no explicit specification of the field

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involved. In Section 4 we list two "saturated" subclasses of pairwise parallel summable *a.p.d.* matrices. Here a complex field is assumed. Determining "saturated" classes or subclasses of parallel summable matrices on more general fields is still an open problem.

Matrices are denoted by capital letters, column vectors by lowercase letters. If A is a matrix, $\mathscr{M}(A)$, $\mathscr{N}(A)$, and A' denote the column span, null space, and transpose of A. For a complex matrix A, A^* denotes its complex conjugate transpose. Matrices A and B are said to be disjoint [5] if $\mathscr{M}(A)$ and $\mathscr{M}(B)$ are virtually disjoint—that is, have only the null vector in common—and so are $\mathscr{M}(A')$ and $\mathscr{M}(B')$. A square complex matrix A is said to be an EP matrix if A and A^* have identical column spans or equivalently identical null spaces. A^- denotes a generalized inverse (g-inverse) of A, that is, a solution X of the matrix equation $AXA = A$; $\{A^-\}$ represents the class of all g-inverses of A. Wherever applicable, A_L^{-1} will denote a left inverse of A, that is, matrix satisfying the condition $A_L^{-1}A = I$. The right inverse A_R^{-1} similarly satisfies the condition $AA_R^{-1} = I$. For a complex matrix A, A^+ denotes its Moore–Penrose inverse [8].

The following lemma is well known (see e.g. [8], [9], and [10]). We shall however give here a proof which is valid for any field.

LEMMA 1.1. *If A and B are nonnull matrices, AC^-B is invariant under choice of C^- iff*

$$\mathscr{M}(B) \subset \mathscr{M}(C), \qquad \mathscr{M}(A') \subset \mathscr{M}(C').$$

Proof. The "if" part is trivial. For the "only if" part choose and fix C^- and suppose $\mathscr{M}(B) \not\subset \mathscr{M}(C)$. Here $(I - CC^-)B \neq 0$. This implies the existence of a row vector b' such that $b'(I - CC^-)B \neq \mathbf{0}'$. Also $A \neq 0$ implies the existence of a column vector a such that $Aa \neq \mathbf{0}$. Observe that with a and b so determined

$$Aab'(I - CC^-)B \neq 0.$$

Put $G = C^- + ab'(I - CC^-)$; observe that $G \in \{C^-\}$ and

$$AGB \neq AC^-B.$$

The necessity of $\mathscr{M}(A') \subset \mathscr{M}(C')$ is similarly established.

DEFINITION. Matrices A and B of order $m \times n$ each are said to be parallel summable ($p.s$) if $A(A + B)^-B$ is invariant under the choice of the generalized inverse $(A+B)^-$. If A and B are *p.s.*, $A(A+B)^-B$ is called the parallel sum of A and B and denoted by the symbol $P(A, B)$.

A null matrix is clearly *p.s.* with an arbitrary matrix of the same order. The following theorem is a simple consequence of Lemma 1.1.

THEOREM 1.1. *Nonnull matrices A and B are p.s. iff*

$$\mathcal{M}(A)\subset\mathcal{M}(A+B), \qquad \mathcal{M}(A')\subset\mathcal{M}(A'+B'), \tag{1.1a}$$

or equivalently

$$\mathcal{M}(B)\subset\mathcal{M}(A+B), \qquad \mathcal{M}(B')\subset\mathcal{M}(A'+B'). \tag{1.1b}$$

Theorem 1.2 lists some known properties of the parallel sum [8].

THEOREM 1.2. *If A and B are p.s. matrices of order* $m\times n$ *each, then*

(a) $P(A, B)=P(B, A)$;

(b) *A' and B' are p.s. and* $P(A', B')=[P(A, B)]'$ *(for complex matrices A^*, B^* are also p.s. and* $P(A^*, B^*)=[P(A, B)]^*$*)*;

(c) *for a matrix C of order* $p\times m$ *and rank m, CA and CB are p.s. and* $P(CA, CB)=CP(A, B)$;

(d) $\{[P(A, B)]^-\}=\{A^-+B^-\}$;

(e) $\mathcal{M}[P(A, B)]=\mathcal{M}(A)\cap\mathcal{M}(B)$;

(f) $P[P(A, B), C]=P[A, P(B, C)]$ *when all the parallel sum operations involved are permissible.*

THEOREM 1.3. *Let A, B be p.s. matrices of order* $m\times n$ *each and* $P(A, B)=C$. *Then*

(a) *either of* (i) $\mathcal{M}(B)\subset\mathcal{M}(A)$ *or* (ii) $\mathcal{M}(B')\subset\mathcal{M}(A')$ *implies the other and*

$$\text{Rank}(A-C)=\text{Rank } A;$$

(b) $\text{Rank}(A-C)=\text{Rank } A \Rightarrow$ *A and* $-C$ *are p.s. and*

$$B=-P(A, -C)+W,$$

where A and W are disjoint matrices;

(c) *in general*

$$\text{Rank}(A - C) \geqslant 2\,\text{Rank}\,A - \text{Rank}(A + B).$$

Further, if A *and* C *are matrices of order* $m \times n$ *each such that* (i) $\mathcal{M}(C) \subset \mathcal{M}(A)$, $\mathcal{M}(C') \subset \mathcal{M}(A')$ *and* (ii) $\text{Rank}(A - C) \geqslant 2\,\text{Rank}\,A - \min(m, n)$, *then there exists a matrix* X *of order* $m \times n$ *such that* A *and* X *are* $p.s.$ *and*

$$P(A, X) = C.$$

Theorem 1.3 is proved in Mitra and Puri [6].

A pair of $h.n.n.d.$ matrices A, B of order $n \times n$ each are always $p.s.$, and $P(A, B)$ is $h.n.n.d.$ This was shown by Anderson and Duffin [1] along with theorem 1.2(a), (e), (f) for this special case. They further showed that if P_A and P_B are the orthogonal projectors onto $\mathcal{M}(A)$ and $\mathcal{M}(B)$ under the norm induced by the inner product $(x, y) = y^*x$, then $2P(P_A, P_B)$ is the orthogonal projector onto $\mathcal{M}(A) \cap \mathcal{M}(B)$, and that the Moore-Penrose inverse of $P(A, B)$ is given by $P(A^+ + B^+)P$, where P is the orthogonal projector onto $\mathcal{M}(A) \cap \mathcal{M}(B)$.

In the present paper we show that the property in Theorem 1.2(d) characterizes in a way the parallel sum (Section 2).

Lemma 1.2 is well known. The "if" part is now folklore. The "only if" part was proved for the first time in Mitra [5]. The proof given here makes an interesting use of the parallel sum concept.

LEMMA 1.2. $\{A^-\} \subset \{B^-\}$ *iff* $A = B + D$ *where* B *and* D *are disjoint matrices.*

Proof. If B and D are disjoint matrices, clearly $\mathcal{M}(B) \subset \mathcal{M}(B + D)$, $\mathcal{M}(B') \subset \mathcal{M}(B' + D')$. Hence B and D are $p.s$ and $P(B, D) = 0$. Further

$$B(B + D)^-(B + D) = B \quad \Rightarrow \quad B(B + D)^- B$$
$$= B \quad \text{for every choice of } (B + D)^-,$$

since $B(B + D)^- D = P(B, D) = 0$. Conversely, if $B(B + D)^- B = B$ for every choice of $(B + D)^-$, Lemma 1.1 would imply $\mathcal{M}(B) \subset \mathcal{M}(B + D)$, $\mathcal{M}(B') \subset \mathcal{M}(B' + D')$. Hence B and D are $p.s.$ Further this would also imply $P(B, D) = B(B + D)^-(B + D) - B(B + D)^- B = B - B = 0$. Hence by Theorem 1.2(b) and (e), B and D are disjoint matrices.

The $\Leftarrow$ part of Lemma 1.3 below is trivial. The $\Rightarrow$ part was proved for the first time in Rao and Mitra [8, Theorem 2.4.2]. Lemma 1.3 is in fact a simple consequence of Lemma 1.2:

LEMMA 1.3. $\{A^-\} = \{B^-\} \Leftrightarrow A = B$.

2. GENERALIZED INVERSES CHARACTERIZE THE PARALLEL SUM

We shall prove here the following theorem.

THEOREM 2.1. *Let A and B be matrices of order $m \times n$ each, and let there exist a matrix C such that*

$$\{C^-\} = \{A^- + B^-\}. \tag{2.1}$$

Then A and B are p.s. and

$$C = P(A, B). \tag{2.2}$$

Note that the crucial part in the proof is to establish the parallel summability of A and B, since on account of the one-to-one correspondence between a matrix and its class of generalized inverses (Lemma 1.3) and in view of Theorem 1.2(d), the rest of the theorem will follow as a simple consequence once this is established.

Proof. Let C_1 be a matrix of full column rank such that

$$\mathcal{M}(C_1) = \mathcal{M}(A) \cap \mathcal{M}(B),$$

and D_1' be a matrix of full column rank such that

$$\mathcal{M}(D_1') = \mathcal{M}(A') \cap \mathcal{M}(B').$$

Let $(A^-)_0$ and $(B^-)_0$ be particular choices of A^- and B^- respectively. A typical member of $\{A^- + B^-\}$ is therefore

$$(A^-)_0 + (B^-)_0 + X,$$

where X is an arbitrary solution of

$$D_1 X C_1 = 0,$$

while if (2.1) holds, a typical member of $\{C^-\}$ is

$$(A^-)_0 + (B^-)_0 + Y,$$

where Y is an arbitrary solution of

$$CYC = 0.$$

Hence (2.1) implies

$$\mathcal{M}(C) = \mathcal{M}(C_1) = \mathcal{M}(A) \cap \mathcal{M}(B),$$
$$\mathcal{M}(C') = \mathcal{M}(D_1') = \mathcal{M}(A') \cap \mathcal{M}(B').$$

Note that this implies in particular that

$$\dim[\mathcal{M}(A) \cap \mathcal{M}(B)] = \dim[\mathcal{M}(A') \cap \mathcal{M}(B')] = r \quad \text{(say)}.$$

Let A and B be matrices of rank s and t respectively, and C_2 and D_2' be matrices with s-r columns each such that

$$\mathcal{M}(A) = \mathcal{M}(C_1 : C_2), \qquad \mathcal{M}(A') = \mathcal{M}(D_1' : D_2').$$

Similarly, let C_3 and D_3' be matrices with t-r columns each such that

$$\mathcal{M}(B) = \mathcal{M}(C_1 : C_3), \qquad \mathcal{M}(B') = \mathcal{M}(D_1' : D_3').$$

Then

$$A = (C_1 : C_2) F_a \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}, \qquad B = (C_1 : C_3) F_b \begin{pmatrix} D_1 \\ D_3 \end{pmatrix},$$

where

$$F_a = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \quad \text{and} \quad F_b = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}$$

are seen to be invertible matrices of order $s \times s$ and $t \times t$ respectively. The partitioning of F_a and F_b should be clear from the context. Let us write

$$F_a^{-1} = \begin{pmatrix} U^{11} & U^{12} \\ U^{21} & U^{22} \end{pmatrix}, \qquad F_b^{-1} = \begin{pmatrix} V^{11} & V^{12} \\ V^{21} & V^{22} \end{pmatrix},$$

$$\begin{pmatrix} D_1 \\ D_2 \end{pmatrix}_R^{-1} F_a^{-1}(C_1 : C_2)_L^{-1} \in \{A^-\},$$

$$\begin{pmatrix} D_1 \\ D_3 \end{pmatrix}_R^{-1} F_b^{-1}(C_1 : C_3)_L^{-1} \in \{B^-\}.$$

Hence if (2.1) is true,

$$G = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}_R^{-1} F_a^{-1}(C_1 : C_2)_L^{-1} + \begin{pmatrix} D_1 \\ D_3 \end{pmatrix}_R^{-1} F_b^{-1}(C_1 : C_3)_L^{-1} \in \{C^-\},$$

and

$$CGC = C \quad \Rightarrow \quad D_1GC_1 \text{ is nonsingular.} \tag{2.3}$$

However,

$$\begin{aligned} D_1GC_1 &= (I:0)\begin{pmatrix} U^{11} & U^{12} \\ U^{21} & U^{22} \end{pmatrix}\begin{pmatrix} I \\ 0 \end{pmatrix} + (I:0)\begin{pmatrix} V^{11} & V^{12} \\ V^{21} & V^{22} \end{pmatrix}\begin{pmatrix} I \\ 0 \end{pmatrix} \\ &= U^{11} + V^{11}. \end{aligned}$$

We now show

$$\det(U^{11} + V^{11}) \neq 0 \quad \Rightarrow \quad \det H \neq 0, \tag{2.4}$$

where

$$H = \begin{pmatrix} U_{11} + V_{11} & U_{12} & V_{12} \\ U_{21} & U_{22} & 0 \\ V_{21} & 0 & V_{22} \end{pmatrix}.$$

Since

$$A + B = (C_1 : C_2 : C_3) H \begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix},$$

the parallel summability of A and B would then follow from (2.4).

To establish (2.4) one merely checks that

$$LHR = E,$$

where

$$L = \begin{pmatrix} U^{11} & U^{12} & 0 \\ U^{21} & U^{22} & 0 \\ 0 & 0 & I \end{pmatrix}, \qquad R = \begin{pmatrix} V^{11} & 0 & V^{12} \\ 0 & I & 0 \\ V^{21} & 0 & V^{22} \end{pmatrix}$$

and

$$E = \begin{pmatrix} U^{11} + V^{11} & 0 & V^{12} \\ U^{21} & I & 0 \\ 0 & 0 & I \end{pmatrix},$$

and that $\det L = (\det F_a)^{-1} \neq 0$, $\det R = (\det F_b)^{-1} \neq 0$, and $\det E = \det(U^{11} + V^{11})$.

3. ALMOST POSITIVE DEFINITE MATRICES AND TWO SUBTYPES

A complex matrix A is said to be almost definite ($a.d.$) if $x^*Ax = 0 \Rightarrow Ax = 0$ (Duffin and Morely [2]). A is positive semidefinite ($p.s.d.$) if $\operatorname{Re}(x^*Ax) \geq 0$ (Lewis and Newman [4]). A is almost positive definite ($a.p.d.$) if it is both $a.d.$ and $p.s.d.$ [2]. A is quasidefinite ($q.d.$) if $\operatorname{Re}(x^*Ax) = 0 \Rightarrow Ax = 0$. A is quasi-positive-definite if it is both $q.d.$ and $p.s.d.$ (Mitra and Puri [7]). The $a.p.d.$ matrices are of two types, I and II. This classification we shall introduce at the appropriate place. It was shown in Mitra and Puri [7] that a pair of $a.p.d.$ matrices of order $n \times n$ are not necessarily $p.s.$, while a pair of $q.p.d.$ matrices of the same order are always so.

We shall now prove a lemma.

LEMMA 3.1. *A pair of a.p.d. matrices are p.s. if one of them is q.p.d.*

Proof. Let V and W be *a.p.d.* matrices and V in addition be *q.p.d.* Then

$$\begin{aligned}(V+W)x=0 \quad &\Rightarrow \quad x^*(V+W)x=0 \quad \Rightarrow \quad \operatorname{Re} x^*(V+W)x=0\\ &\Rightarrow \quad \operatorname{Re} x^*Vx=0 \quad \Rightarrow \quad Vx=0\\ &\Rightarrow \quad \mathscr{M}(V')\subset \mathscr{M}(V'+W').\end{aligned}$$

Similarly $(V^*+W^*)x=0 \Rightarrow V^*x=0 \Rightarrow \mathscr{M}(V)\subset \mathscr{M}(V+W)$. Using Theorem 1.1, V and W are seen to be *p.s.*

Every complex matrix V of order $n\times n$ can be written as

$$V_{re}+iV_{im}, \tag{3.1}$$

where V_{re} and V_{im} are Hermitian matrices. Put for example

$$V_{re}=\frac{V+V^*}{2},$$

$$V_{im}=\frac{i(V^*-V)}{2}.$$

One can further use the spectral representation to split

$$V_{re}=V_{re}^{+}-V_{re}^{-} \tag{3.2}$$

where V_{re}^{+} and V_{re}^{-} are *h.n.n.d* matrices and further

$$V_{re}^{+}V_{re}^{-}=0. \tag{3.3}$$

We similarly split V_{im} as

$$V_{im}=V_{im}^{+}-V_{im}^{-}. \tag{3.4}$$

In (3.2)–(3.4) and in what follows the notation used should not be confused with that for the Moore–Penrose and generalized inverses of the relevant matrices [which if used would be denoted respectively by $(V_{re})^+$ and $(V_{re})^-$ rather than V_{re}^+ and V_{re}^-, etc.]. Similarly the real parts of V^+ or V^- would be denoted by $(V^+)_{re}$ and $(V^-)_{re}$ respectively. The following lemma characterizes the *a.p.d.* matrices in terms of its four components just enumerated.

Let P denote the orthogonal projector onto $\mathscr{M}(V_{re}^+)$ under the usual Euclidean inner product $(x, u) = u^*x$. Put $Q = I - P$.

LEMMA 3.2. *The following two statements are equivalent*:

(1) V *is a.p.d.*

(2) (a) $V_{re}^- = 0$, *and*
(b) $Q(V_{im}^+ - V_{im}^-)Q$ *is either h.n.n.d. or hermitian nonpositive definite (h.n.p.d.), and*

$$\operatorname{Rank}[Q(V_{im}^+ - V_{im}^-)Q] = \operatorname{Rank}(V_{im}^+ - V_{im}^-)Q. \tag{3.5}$$

Proof. Observe that

$$(2)(a) \quad \Leftrightarrow \quad V \text{ is p.s.d.} \tag{3.6}$$

(2) ⇒ (1):

$$\begin{aligned} x^*Vx = 0 \quad &\Rightarrow \quad \operatorname{Re} x^*Vx = 0 \quad \Rightarrow \quad x^*V_{re}^+x = 0 \quad \Rightarrow \quad V_{re}^+x = 0 \\ &\Rightarrow \quad x = Qy \quad \text{for some } y \\ &\Rightarrow \quad y^*Q(V_{im}^+ - V_{im}^-)Qy = 0 \\ &\Rightarrow \quad (V_{im}^+ - V_{im}^-)Qy = 0 \quad \text{on account of (2)(b)} \\ &\Rightarrow \quad (V_{im}^+ - V_{im}^-)x = 0 \quad \Rightarrow \quad Vx = 0 \\ &\Rightarrow \quad V \text{ is a.d.} \quad \Rightarrow \quad V \text{ is a.p.d.} \end{aligned}$$

Not (2) ⇒ not (1): Consider a matrix $V = V_{re}^+ + i(V_{im}^+ - V_{im}^-)$ which does not satisfy (2)(b). That is, here $Q(V_{im}^+ - V_{im}^-)Q$ is neither *h.n.n.d.* nor *h.n.p.d.*, or else

$$\operatorname{Rank}[Q(V_{im}^+ - V_{im}^-)Q] < \operatorname{Rank}(V_{im}^+ - V_{im}^-)Q.$$

In any of these cases, $\exists y$ such that $y^*Q(V_{\text{im}}^+ - V_{\text{im}}^-)Qy = 0$, but

$$(V_{\text{im}}^+ - V_{\text{im}}^-)Qy \neq 0.$$

Observe that for such a choice of y if $x = Qy$

$$x^*Vx = 0 \quad \text{but} \quad Vx = i(V_{\text{im}}^+ - V_{\text{im}}^-)Qy \neq 0.$$

Thus V is not *a.d.*

DEFINITION. The matrix V is said to be *a.p.d.* of type I if V is *a.p.d.* and $Q[V_{\text{im}}^+ - V_{\text{im}}^-]Q$ is *h.n.n.d.* and the rank condition (3.5) is satisfied. It is *a.p.d.* of type II if $Q[V_{\text{im}}^+ - V_{\text{im}}^-]Q$ is *h.n.p.d.* and the rank condition (3.5) is satisfied.

Let $\mathscr{C}_1$ and $\mathscr{C}_2$ denote the classes of *a.p.d.* matrices of type I and type II respectively, and $\mathscr{C}$ denote the class of *q.p.d.* matrices all of the same order. The following lemma is easily established.

LEMMA 3.3. *If* $V \in \mathscr{C}_1$ *then* $V^* \in \mathscr{C}_2$ *and vice versa.*

LEMMA 3.4. $\mathscr{C}_1 \cap \mathscr{C}_2 = \mathscr{C}$.

Proof. Let $V = V_{\text{re}}^+ + i(V_{\text{im}}^+ - V_{\text{im}}^-) \in \mathscr{C}_1$, and Q be defined as before. $Q[V_{\text{im}}^+ - V_{\text{im}}^-]Q$ is *h.n.n.d.*, and the rank condition (3.5) is satisfied. If also $V \in \mathscr{C}_2$,

$$\begin{aligned} Q[V_{\text{im}}^- - V_{\text{im}}^+]Q \text{ is h.n.n.d.} \quad &\Rightarrow \quad Q[V_{\text{im}}^+ - V_{\text{im}}^-]Q = 0 \\ &\Rightarrow \quad (V_{\text{im}}^+ - V_{\text{im}}^-)Q = 0 \quad \text{on account of (3.5).} \end{aligned}$$

Here $\operatorname{Re} x^*Vx = 0 \Rightarrow Vx = 0 \Rightarrow V \in \mathscr{C}$.

Conversely, if V is *p.s.d.*, then $\operatorname{Re} x^*Vx = 0 \Rightarrow x^*V_{\text{re}}^+x = 0 \Rightarrow V_{\text{re}}^+x = 0 \Rightarrow x = Qy$ for some y. Hence

$$\begin{aligned} V \in C \quad &\Rightarrow \quad \text{for arbitrary } y, \quad [V_{\text{im}}^+ - V_{\text{im}}^-]Qy = 0 \\ &\Rightarrow \quad [V_{\text{im}}^+ - V_{\text{im}}^-]Q = 0 \quad \Rightarrow \quad V \in \mathscr{C}_1 \cap \mathscr{C}_2. \end{aligned}$$

LEMMA 3.5. *If $V \in \mathscr{C}_i$, then $B^*VB \in \mathscr{C}_i$, $i = 1,2$.*

Proof. Let V be an *a.p.d.* type I matrix, P_b the orthogonal projector onto $\mathscr{M}(B^*V_{re}^+B)$, and $Q_b = I - P_b$. Then

$$(B^*VB)_{re} = B^*(V_{re}^+)B = (B^*VB)_{re}^+.$$

If Q is the orthogonal projector onto the orthogonal complement of $\mathscr{M}(V_{re}^+)$, it follows that

$$BQ_b = QK$$

for some matrix K. Then

$$Q_b(B^*VB)_{im}Q_b = K^*QV_{im}QK$$

is h.n.n.d., and Rank $Q_b(B^*VB)_{im}Q_b =$ Rank $K^*QV_{im}QK$

$$= \text{Rank}\, QV_{im}QK = \text{Rank}\, V_{im}QK = \text{Rank}\,(B^*VB)_{im}Q_b.$$

This shows B^*VB is an *a.p.d.* type I matrix. The case when V is an *a.p.d.* type II matrix is dealt with in a similar manner.

LEMMA 3.6. *If $V, W \in \mathscr{C}_i$, then $V + W \in \mathscr{C}_i$, $i = 1,2$.*

Proof. Let V and $W \in \mathscr{C}_1$, and the corresponding Q matrices be denoted by Q_1 and Q_2 respectively. Then

$$(V+W)_{re} = V_{re}^+ + W_{re}^+ = (V+W)_{re}^+,$$

and the corresponding Q matrix is given by $2P(Q_1, Q_2) = Q_0$ (say) using Anderson and Duffin's theorem on the minimum of two projections reported in Section 1 of this paper. The matrix

$$Q_0(V+W)_{im}Q_0 = 4Q_2(Q_1+Q_2)^+Q_1V_{im}Q_1(Q_1+Q_2)^+Q_2$$
$$+4Q_1(Q_1+Q_2)^+Q_2W_{im}Q_2(Q_1+Q_2)^+Q_1$$

is thus clearly *h.n.n.d.*

Further,

$$\begin{aligned} Q_0(V+W)_{im}Q_0x=0 \quad &\Rightarrow \quad x^*Q_0(V+W)_{im}Q_0x=0 \\ &\Rightarrow \quad x^*Q_0V_{im}Q_0x=0, \quad x^*Q_0W_{im}Q_0x=0 \\ &\Rightarrow \quad Q_1V_{im}Q_1(Q_1+Q_2)^+Q_2x=0, \\ &\qquad\quad Q_2W_{im}Q_2(Q_1+Q_2)^+Q_1x=0 \\ &\Rightarrow \quad V_{im}Q_0x=0, W_{im}Q_0x=0 \quad \Rightarrow \quad (V+W)_{im}Q_0x=0. \end{aligned}$$

Since $(V+W)_{im}Q_0x=0 \Rightarrow Q_0(V+W)_{im}Q_0x=0$, it is seen that

$$\text{Rank}\, Q_0(V+W)_{im}Q_0 = \text{Rank}\,(V+W)_{im}Q_0.$$

Hence $(V+W) \in \mathscr{C}_1$.

Let $V, W \in \mathscr{C}_2$. By Lemma 3.1, $V^*, W^* \in \mathscr{C}_1 \Rightarrow (V+W)^* \in \mathscr{C}_1 \Rightarrow (V+W) \in \mathscr{C}_2$.

LEMMA 3.7. *If* $V \in \mathscr{C}_1$ *then* $V^+ \in \mathscr{C}_2$ *and vice versa.*

Proof. From Lemmas 2.5 and 2.1 of Mitra and Puri [7] it follows respectively that since V is *a.d.*, so is V^+, and that V^+ is an EP matrix. Thus

$$\begin{aligned} V^+[I-VV^+]=0 \quad &\Rightarrow \quad (V^+)^*[I-VV^+]=0 \\ &\Rightarrow \quad (V^+)^*=(V^+)^*VV^+ \in \mathscr{C}_1 \\ &\Rightarrow \quad V^+ \in \mathscr{C}_2 \quad \text{by Lemma 3.3.} \end{aligned}$$

4. SATURATED CLASSES OF PARALLEL SUMMABLE MATRICES

DEFINITION. A subclass Ω_0 (of a class Ω) of objects with a property $\mathbf{P}$ is said to be saturated with respect to this property if no further members from Ω can be added to Ω_0 without destroying the property.

The class Ω in our context is $\mathscr{C}^{n\times n}$, the vector space of complex matrices of order $n \times n$, and the property $\mathbf{P}$ is that of pairwise parallel summability.

Lemma 3.1 shows that the class $\mathscr{C}$ of *q.p.d.* matrices is not a saturated subclass of the class of *a.p.d.* matrices. In this section we shall describe two saturated subclasses of Ω when Ω is the class of *a.p.d.* matrices. Determining a saturated subclass in the wide context of $\mathscr{C}^{n\times n}$ is a problem still wide open.

We now prove the main theorem of this section.

THEOREM 4.1.

(a) *A pair of a.p.d. matrices of the same type are p.s., and the parallel sum is a.p.d. of the same type as the summands.*

(b) *$\mathscr{C}_1$ and $\mathscr{C}_2$ are saturated subclasses of a.p.d. matrices.*

Proof. (a): Let V and $W \in \mathscr{C}_1$, and the corresponding Q matrices be denoted by Q_1 and Q_2 respectively. Then

$$(V+W)_{re} = V_{re}^{+} + W_{re}^{+},$$

and the corresponding Q matrix is given by $2P(Q_1, Q_2) = Q_0$ say, using Anderson and Duffin's theorem on the minimum of two projections reported in Section 1. Hence

$$\begin{aligned}(V+W)x=0 \quad &\Rightarrow \quad \operatorname{Re} x^*(V+W)x = x^*(V_{re}^{+} + W_{re}^{+})x = 0\\ &\Rightarrow \quad x = Q_0 y \quad \text{for some } y.\end{aligned}$$

Hence also

$$\begin{aligned}(V+W)x=0 \quad &\Rightarrow \quad [(V_{im}^{+} + W_{im}^{+}) - (V_{im}^{-} + W_{im}^{-})]Q_0 y = 0\\ &\Rightarrow \quad y^*Q_0[(V_{im}^{+} - V_{im}^{-}) + (W_{im}^{+} - W_{im}^{-})]Q_0 y = 0\\ &\Rightarrow \quad y^*Q_0(V_{im}^{+} - V_{im}^{-})Q_0 y = 0, \quad y^*Q_0(W_{im}^{+} - W_{im}^{-})Q_0 y = 0\\ &\Rightarrow \quad Q_0(V_{im}^{+} - V_{im}^{-})Q_0 y = 0, \quad Q_0(W_{im}^{+} - W_{im}^{-})Q_0 y = 0,\end{aligned} \tag{4.1}$$

since $Q_0(V_{im}^{+} - V_{im}^{-})Q_0 = 4Q_2(Q_1+Q_2)^{+}Q_1(V_{im}^{+} - V_{im}^{-})Q_1(Q_1+Q_2)^{+}Q$ and $Q_0(W_{im}^{+} - W_{im}^{-})Q_0 = 4Q_1(Q_1+Q_2)^{+}Q_2(W_{im}^{+} - W_{im}^{-})Q_2(Q_1+Q_2)^{+}Q_1$ are both *h.n.n.d.* matrices. Also, since $Q_1(V_{im}^{+} - V_{im}^{-})Q_1$ and $Q_2(W_{im}^{+} - W_{im}^{-})Q_2$ are both *h.n.n.d.* matrices and both satisfy the rank condition (3.5), $Q_0(V_{im}^{+} - V_{im}^{-})Q_0$ and $Q_0(W_{im}^{+} - W_{im}^{-})Q_0$ not only are *h.n.n.d.*, but also satisfy the rank condition (3.5). Hence (4.1) implies

$$(V_{\rm im}^{+} - V_{\rm im}^{-})x = (V_{\rm im}^{+} - V_{\rm im}^{-})Q_0 y = 0,$$

$$(W_{\rm im}^{+} - W_{\rm im}^{-})x = (W_{\rm im}^{+} - W_{\rm im}^{-})Q_0 y = 0.$$

Since $(V_{\rm re}^{+} + W_{\rm re}^{+})x = 0 \Rightarrow V_{\rm re}^{+}x = 0,\ W_{\rm re}^{+}x = 0$, it is seen that $(V + W)x = 0 \Rightarrow Vx = 0,\quad Wx = 0 \quad \Rightarrow \quad \mathscr{M}(V')\subset \mathscr{M}(V'+W'),\quad \mathscr{M}(W')\subset \mathscr{M}(V'+W')$. Similarly, arguing with V^* and W^* in place of V and W, it is seen that $\mathscr{M}(V) = \mathscr{M}(V+W)$, $\mathscr{M}(W)\subset \mathscr{M}(V+W)$. Hence V and W satisfy the conditions of Theorem 1.1 and are therefore *p.s.*

Since V and W are *a.p.d.*, by Lemma 2.1 of Mitra and Puri [7] both are EP matrices. Here by Theorem 1.2(b) and (e), $P(V, W) = P_a$ is also an EP matrix. Theorem 1.2(d) therefore implies

$$P_a = P_a(V^+ + W^+)P_a \quad \Rightarrow \quad P_a^* = P_a^*(V^+ + W^+)P_a$$

$$\Rightarrow \quad P_a^* \text{ is a.p.d. of type II} \quad \text{(using Lemmas 3.7 and 3.6)}$$

$$\Rightarrow \quad P_a \text{ is a.p.d. of type I} \quad \text{(using Lemma 3.3).}$$

(b): To show that $\mathscr{C}_1$ is a saturated subclass of the class of *a.p.d.* matrices, we show that no *a.p.d.* matrix which is outside $\mathscr{C}_1$ can be added to $\mathscr{C}_1$ without destroying the property of pairwise parallel summability. Let

$$V = V_{\rm re}^{+} + i(V_{\rm im}^{+} - V_{\rm im}^{-})$$

be an *a.p.d.* matrix not in $\mathscr{C}_1$. Since $V \notin \mathscr{C} = \mathscr{C}_1 \cap \mathscr{C}_2$, $(V_{\rm im}^{+} - V_{\rm im}^{-})Q \neq 0 \Rightarrow \mathscr{M}(V_{\rm im}) \not\subset \mathscr{M}(V_{\rm re}^{+})$. Check that

$$V^* = V_{\rm re}^{+} + i(V_{\rm im}^{-} - V_{\rm im}^{+}) \in \mathscr{C}_1$$

and V and V^* are not *p.s.*

The following counterexample will show that there could be non-a.p.d. matrices which are *p.s.* with each member of $\mathscr{C}_1$. Hence $\mathscr{C}_1$ is no longer a saturated class in the wider context.

Consider the following complex matrix V of order 2×2:

$$V = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Here

$$Q=\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix},\qquad V_{\text{im}}Q=\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ -\frac{1}{2} & -\frac{1}{2}\end{pmatrix}.$$

Hence the rank condition (3.5) is not satisfied though $QV_{\text{im}}Q=0$ is *h.n.n.d.* This shows V is not *a.p.d.*. For a *p.s.d.* W to be not *p.s.* with V, it is necessary that W^{+}_{re} be a multiple of V^{+}_{re} (null matrix included), since if W^{+}_{re} were linearly independent of V^{+}_{re}, then $(V+W)_{\text{re}}$ would be *p.d.*, and of course V and W would be *p.s.* Also W_{im} and V_{im} must add up to a multiple (not necessarily a nonzero one) of V^{+}_{re}. Hence

$$(V_{\text{im}}+W_{\text{im}})Q=0.$$

This implies W is not *a.d.* Thus the only *p.s.d.* W's that are not *p.s.* with V are those that are not *a.p.d.* This establishes the counterexample.

5. AN APPLICATION—DETERMINING OTHER PROJECTIONS

Let $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$, and $\mathcal{M}_4$ be four n dimensional subspaces of a $2n$ dimensional vector space such that $\mathcal{M}_i$ is virtually disjoint with $\mathcal{M}_j$ if $i\neq j$. For example $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$, and $\mathcal{M}_4$ could be subspaces $\mathcal{M}$, $\mathcal{N}$ and their orthogonal complements $\mathcal{M}^{\perp}$, $\mathcal{N}^{\perp}$ in a $2n$ dimensional complex vector space, though not necessarily in the same order, with $\mathcal{M}$ and $\mathcal{N}$ in generic position (Halmos [3]). For $i\neq j$ let P_{ij} denote the projection of $\mathcal{M}_i$ along $\mathcal{M}_j$. Given P_{12} and P_{34}, we show how the remaining P_{ij}'s can be determined. Clearly

$$P_{21}=I-P_{12},\qquad P_{43}=I-P_{34}.$$

We show $P_{13}=P_{12}(P_{12}+P_{34})^{-1}$, but first we establish the invertibility of $P_{12}+P_{34}$. If there exists a nonnull vector x such that $(P_{12}+P_{34})x=0$, then either $P_{12}x$ is nonnull, in which case $P_{12}x=P_{34}(-x)$ is a nonnull vector in $\mathcal{M}_1\cap\mathcal{M}_3$, or $P_{12}x$ is null, in which case $x=P_{21}x=P_{43}x$ is a nonnull vector in $\mathcal{M}_2\cap\mathcal{M}_4$—both of which are impossible. Since P_{12} and P_{34} are seen to be disjoint, by Lemma 1.2, we have $(P_{12}+P_{34})^{-1}\in\{P_{12}^{-}\}$. Also $P_{12}(P_{12}+P_{34})^{-1}P_{34}=P(P_{12},P_{34})=0$ and $P_{12}(P_{12}+P_{34})^{-1}P_{12}(P_{12}+P_{34})^{-1}=P_{12}(P_{12}+P_{34})^{-1}$. Further, the range and null space of $P_{12}(P_{12}+P_{34})^{-1}$ are seen to be $\mathcal{M}_1$ and $\mathcal{M}_3$. Hence $P_{13} = P_{12}(P_{12} + P_{34})^{-1}$. Similarly

$$P_{24} = P_{21}(P_{21} + P_{43})^{-1}, \qquad P_{14} = P_{12}(P_{12} + P_{43})^{-1},$$
$$P_{23} = P_{21}(P_{21} + P_{34})^{-1}, \qquad P_{31} = I - P_{13}, \qquad P_{42} = I - P_{24},$$
$$P_{41} = I - P_{14}, \qquad P_{32} = I - P_{23}.$$

6. CONCLUDING REMARKS

We conclude this paper raising a few open problems:

(1) Determine a saturated class of pairwise *p.s.* matrices (complex or otherwise).

(2) Let T denote a linear transformation which maps matrices of order $m \times n$ onto matrices of order $p \times q$ defined on the same field. Characterize such T which also preserve parallel summability. Theorem 1.2(b) and (c) provide examples of linear transformations with this property.

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Received 6 August 1984; revised 27 December 1984