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TAX EVASION: A MODEL

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1. Introduction

Understatement of income in individual income tax returns is a widespread phenomenon. No doubt there will be some income earners in all countries who will indeed state their true income in their income tax returns as a matter of principle. There will also be others for whom an honest income tax return is only one of the set of possible returns. Such a person will submit an honest return only if it is 'optimal' (in some suitably defined sense) for him to do so.

The purpose of this paper is to provide a formal analysis of this problem. [1] First, the optimum (in a sense to be made precise in the next section) proportion of income to be understated will be derived as a function of true income, probability of detection of understatement and the properties of the tax function. Second, it will be shown that given the income distribution, a proportionate tax function yielding the same total revenue as a progressive tax function in the absence of understatement of income, will yield *larger* expected revenue in the presence of optimal understatement of income. Third, the problem of optimal allocation of resources towards detection of tax avoidance will be considered.

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This paper was submitted in 1971 before this journal was officially established, but as a result of an oversight only reached the editor in November 1972. The author was therefore unaware of the parallel work by M.G. Allingham and A. Sandmo published in the November 1972 issue of this journal.

[1] After this paper was written, I came across a paper by Emmanuel Sharon (1967) which presents a similar analysis. However, Sharon confines his attention mainly to a penalty function that depends on evaded tax. He does not consider comparison of tax structures and the problem of optimal allocation of resources to detect tax evasion.

2. The model

Let us consider an individual whose true income is y. Suppose he knows that the probability that he will be detected if he understates in his income is π. This probability could depend on the income level y. Let the tax to be paid as a function of income be $T(y)$. Let us denote by λ the proportion by which income is understated. Let $P(\lambda)$ be the penalty multiplier i.e. $P(\lambda)\lambda y$ is the penalty on the understated income λy. Let us assume that the individual chooses λ so as to maximize his expected income after taxes and penalties. Let $A(y)$ denote this. Since his after tax (and penalty, if any) income is $y - T\{(1-\lambda)y\}$ if he gets away with his understatement (the probability of which event is $1-\pi$) and it is $y - T(y) - \lambda P(\lambda)y$ if he is detected (the probability of which event is π), we get:

$$A(y) = \pi[y - T(y) - \lambda P(\lambda)y] + (1-\pi)[y - T\{(1-\lambda)y\}]. \quad (1)$$

Differentiating $A(y)$ with respect to λ we get (denoting derivatives by primes):

$$\frac{\partial A}{\partial \lambda} = -\pi[P(\lambda) + P'(\lambda)]y + (1-\pi)yT'\{(1-\lambda)y\} \quad (2)$$

$$\equiv \phi(\lambda, y, \pi) \qquad \text{(say)}. \quad (3)$$

It can be easily evaluated that:

$$\frac{\partial \phi}{\partial \lambda} = -\pi[2P'(\lambda) + \lambda P''(\lambda)]y - (1-\pi)y^2 T''\{(1-\lambda)y\} \quad (4)$$

$$\frac{\partial \phi}{\partial \pi} = -[P(\lambda) + \lambda P'(\lambda)]y - yT'\{(1-\lambda)y\} \quad (5)$$

$$\frac{\partial \phi}{\partial y} = -\pi[P(\lambda) + \lambda P'(\lambda)] + (1-\pi)T'\{(1-\lambda)y\}$$

$$+ y(1-\pi)(1-\lambda)T''\{(1-\lambda)y\}. \quad (6)$$

Let us assume that $T(y) > 0$, $1 > T'(y) > 0$, $T''(y) \geqq 0$ for all $y > 0$. In other words the tax $T(y)$ on income y is a positive, increasing, convex function of y. Marginal rate of tax $T'(y)$ is assumed to be strictly less than unity. If $T''(y) = 0$ for all y we get a constant marginal tax rate, which together with $T(0) = 0$ will yield a proportionate rate of tax. $T''(y) > 0$ will correspond to a progressive tax structure. Let us also assume that for all $\lambda > 0$, $P(\lambda) \geq 0$, $P'(\lambda) > 0$, $P''(\lambda) \gtreqless 0$. This means that the penalty multiplier is a positive, increasing and convex function of λ.

Under these assumptions it is clear that $\partial\phi/\partial\lambda < 0$, and $\partial\phi/\partial\pi < 0$. Also $\partial\phi/\partial y \gtreqless 0$ when $\phi = 0$. With the reasonable additional assumptions that $P(0) = 0$ i.e. penalty multiplier when there is no understatement of income and $T'(0) = 0$ i.e. that the marginal rate of tax is zero at zero income, it can be seen from (3) that $\phi(0, y, \pi) > 0$ and $\phi(1, y, \pi) < 0$ for all $y > 0$ and $0 < \pi < 1$. Thus:

Proposition 1: Given (a) $T'(y) \gtreqless 0$, $T'(0) = 0$, $T''(y) \gtreqless 0$ for $y > 0$, and (b) $P(0) = 0$, $P'(\lambda) > 0$, $P''(\lambda) > 0$, for all λ in (0, 1), there exists an unique λ^*, in the interior of (0, 1) which maximizes expected income after taxes and penalties.

Now $\partial\lambda^*/\partial\pi = -(\partial\phi/\partial\pi)/(\partial\phi/\partial\lambda) < 0$. Hence:

Corollary 1: Ceteris paribus, the optimal proportion λ^* by which income is understated decreases as the probability of detection π increases.

It is clear that $\partial\lambda^*/\partial y = -(\partial\phi/\partial y)/(\partial\phi/\partial\lambda) > 0$ if T'' is positive. Hence:

Corollary 2: Given a progressive tax function, and a probability of detection π *independent* of income y, the richer a person, the larger is the optimal proportion by which he will understate his income.

It should be noted that while Proposition 1 and Corollary 1 hold true even if π is a function of income, Corollary 2 need not hold in the case where π is an increasing function of income. For in such a case $d\lambda^*/dy = \partial\lambda^*/\partial y + (\partial\lambda^*/\partial\pi)/(d\pi/dy)$. While $\partial\lambda^*/\partial y > 0$, $\partial\lambda^*/\partial\pi < 0$, since $d\pi/dy > 0$ the sign of $d\lambda^*/d\pi$ is indeterminate without additional assumptions. For instance, if we assume that the marginal rate of tax is constant, then λ^* decreases as y increases if π is an increasing function of income, leading to:

Corollary 3: If the marginal rate of tax is constant and π is an increasing function of income, then the optimal proportion λ^* of understatement of income decreases as income increases.

It can be shown that if the taxpayer wishes to maximize the expected value of a strictly concave function of his after tax and penalty income, rather than the expected value of after tax and penalty income, Proposition 1 and Corollary 1 still hold true.

Let us now compute the expected revenue and penalties accruing to the government under a given tax function $T(y)$. Let us assume that there are in all N taxpayers in our economy, distributed according to the density function $I(y)$ in the interval $(0, y_{max})$. In other words, there are $NI(y)\mathrm{d}y$ income earners in an income interval of width $\mathrm{d}y$ around y. By definition $\int_0^{y_{max}} I(y)\mathrm{d}y = 1$. Without loss of generality let us normalize units by setting $N = 1$. From here on let us treat the limits of integration of y as understood. It is easily seen that the expected revenue and penalties paid by a person whose true income is y is $\pi[\lambda^* P(\lambda^*)y + T(y)] + (1-\pi)T\{(1-\lambda^*)y\}$ where λ^* is the optimum proportion by which income is understated. Hence the expected revenue (and penalties) for the economy as a whole is given by:

$$R = \int [\pi\{\lambda^* P(\lambda^*)y + T(y)\} + (1-\pi)T\{(1-\lambda^*)\}]\, I(y)\, \mathrm{d}y. \quad (7)$$

Let us now compare the expected revenues and penalties from a proportionate tax function $T_1(y) = \theta y$ with those from a progressive tax function $T_2(y)$. To make the comparison meaningful let us postulate that these two tax functions would have yielded the same revenue in the absence of income understatement. That is

$$\int T_1(y)\, I(y)\, \mathrm{d}y = \int T_2(y)\, I(y)\, \mathrm{d}y\,. \quad (8)$$

The expected revenues and penalties associated with $T_1(y)$ are

$$R_1 = \int [\pi\{\lambda_1^* P(\lambda_1^*) + \theta\} + (1-\pi)(1-\lambda_1^*)\theta]\, y I(y)\, \mathrm{d}y\,, \quad (9)$$

where λ_1^* is the optimal proportion of understatement of income given the tax function $T_1(y)$.[2]

It is easily demonstrated that $R_1 < \int T_1(y) I(y)\, \mathrm{d}y$. This is an obvious result. It states that the expected revenue and penalties in the presence of understatement of income will be less than the revenue in its absence. The simplest way of demonstrating this is to show that the integrand in the case of R_1 is less than $T_1(y)I(y)$. That is

[2] It is to be noted that since $T_1'(y) = \theta \neq 0$ for all y, we cannot conclude that $\phi(1, y, \pi) < 0$ even though $\phi(0, y, \pi) < 0$ given $P(0) = 0$. Hence we cannot be sure that an interior solution $\lambda_1^* \in (0, 1)$ exists, unless we *assume* that $\phi(1, y, \pi) = -\pi[P'(1) + P(1)] + (1-\pi)\theta < 0$ (for all y in case π depends on y). Without this assumption, it is possible that for some (or even for all) y, $\lambda_1^* = 1$. For the following discussion it does not matter even if $\lambda_1^* = 1$ and hence we do not make the assumption that $\phi(1, y, \pi) < 0$ for all y.

$$[\pi\{P(\lambda_1^*)\lambda_1^* + \theta\} + (1-\pi)(1-\lambda_1^*)\theta]\, y\, I(y) < \theta y\, I(y) \quad \text{or}$$

$$\pi\{P(\lambda_1^*)\,\lambda_1^* + \theta\} + (1-\pi)(1-\lambda_1^*)\theta < \theta \quad \text{or}$$

$$\lambda_1^*\{\pi P(\lambda_1^*) - (1-\pi)\theta\} < 0 \quad \text{or}$$

$$\pi P(\lambda_1^*) < (1-\pi)\theta\,. \tag{10}$$

Now, if $-\pi[P'(1) + P(1)] + (1-\pi)\theta \geq 0$ then $\lambda_1^* = 1$ and (10) is satisfied. If $-\pi[P'(1) + P(1)] + (1-\pi)\theta < 0$ then $-\pi[\lambda_1^* P'(\lambda_1^*) + P(\lambda_1^*)] + (1-\pi)\theta = 0$ and once again (10) is satisfied.

Given the tax function $T_2(y)$, the expected revenue and penalty R_2 is given by:

$$R_2 = \int [\pi\{\lambda_2^* P(\lambda_2^*)y + T_2(y)\} + (1-\pi)T_2\{(1-\lambda_2^*)y\}]\, I(y)\, \mathrm{d}y\ ,$$

where λ_2^* is the optimal proportion of understatement of income.

It is to be remembered that λ_2^* will in general depend on y, $\pi(y)$ etc. Given his income y before taxes, an individual chooses his λ so as to *maximize* his expected income after taxes and penalties. This is equivalent to his *minimizing* the expected taxes and penalties since pre-tax income is given. Hence, given $T_2(y)$, the expected value of taxes and penalties that results from his choice of λ other than λ_2^* will be larger. In particular if the individual sets $\lambda = \lambda_1^*$ (the optimal value for his income y and the probability π, had the tax function been $T_1(y)$) rather than set $\lambda = \lambda_2^*$, his expected value of taxes and penalties will be higher. Thus

$$\pi[\lambda_2^* P(\lambda_2^*)y + T_2(y)] + (1-\pi)T_2\{(1-\lambda_2^*)y\}$$
$$< \pi[\lambda_1^* P(\lambda_1^*)y + T_2(y)] + (1-\lambda_1^*)T_2\{y(1-\lambda_1^*)\}. \tag{11}$$

Hence integrating both sides of (11) after multiplication by $I(y)$ we get:

$$R_2 < \int [\pi\{\lambda_1^* P(\lambda_1^*)y + T_2(y)\} + (1-\pi)T_2\{y(1-\lambda_1^*)\}]\, I(y)\, \mathrm{d}y$$
$$< R_1 + \int [\pi\{T_2(y) - \theta y\} + (1-\pi)$$
$$\times\ \{T_2\{y(1-\lambda_1^*)\} - \theta y(1-\lambda_1^*)\}]\, I(y)\, \mathrm{d}y\,.$$

Now, if we assume π is independent of y then λ_1^* is independent of y and

$$\int \{T_2(y) - \theta y\} I(y)\,\mathrm{d}y = 0 \quad \text{in view of (8). Also,}$$

$$\int (1-\pi)[T_2\{y(1-\lambda_1^*)\} - \theta y(1-\lambda_1^*)] I(y)\,\mathrm{d}y$$

$$= (1-\pi)(1-\lambda_1^*)\int [T_2\{y(1-\lambda_1^*)\}/y(1-\lambda_1^*) - \theta]\, yI(y)\mathrm{d}y$$

$$= (1-\pi)(1-\lambda_1^*)\int [T_2\{y(1-\lambda_1^*)\}/y(1-\lambda_1^*)$$

$$- T_2(y)/y]\, yI(y)\,\mathrm{d}y\,. \tag{12}$$

It is seen that in deriving (12) we have used (8). Now, the average tax rate $T_2(y)/y$ increases with y given a progressive $T_2(y)$ resulting in the right-hand side of (11) becoming negative. Hence $R_2 < R_1$. We can thus state: [3]

Proposition 2: Given a progressively increasing penalty multiplier $P(\lambda)$ with $P(0) = 0$, and a probability of detection π independent of the level of income, y, a progressive tax function (with a zero marginal and average rate of tax at zero income) that yields the *same* total revenue as a proportionate tax function in the absence of understatement of income, will yield *less* expected revenue and penalties in the *presence* of optimal understatement of income.

3. Allocation of resource for detection of income understatement

We now turn to the question of the determination of π, the probability of detecting the understatement of income in an income tax return. For simplicity let us assume that π is an increasing, concave function of x, the amount spent on the scrutiny of a return. Let us further assume that the amount x to be spent on a return is the same for all returns. Given a tax function $T(y)$ the government is interested in maximizing the difference between the cost of scrutiny and the expected revenue and penalties. Thus the maximand is

[3] I thank A.B. Atkinson for suggesting the simple proof given above.

$$Z = \pi(x)\int\{\lambda^*P(\lambda^*)y + T(y)\}\, I(y)\,\mathrm{d}y + \{1-\pi(x)\}\int T\{y(1-\lambda^*)\}\, I(y)\,\mathrm{d}y - x$$

where λ^* is, as earlier, the optimal proportion of understatement of income. Taking the derivative of Z with respect to x we get (using the fact that λ^* is optimal implies the partial of Z with respect to λ^* is zero):

$$\frac{\mathrm{d}Z}{\mathrm{d}x} = \pi'(x)\int[\lambda^*P(\lambda^*)y + T(y) - T\{(1-\lambda^*)y\}]\, I(y)\,\mathrm{d}y - 1$$

and

$$\frac{\mathrm{d}^2Z}{\mathrm{d}x^2} = \pi''(x)\int[\lambda^*P(\lambda^*)y + T(y) - T\{y(1-\lambda^*)\}]\, I(y)\,\mathrm{d}y + \{\pi'(x)\}^2\int[\{\lambda^*P'(\lambda^*) + P(\lambda^*)y\} + T'\{y(1-\lambda^*)\}]\,\frac{\partial\lambda^*}{\partial\pi}\, I(y)\,\mathrm{d}y$$

From the concavity of $\pi(x)$ and the fact that $\partial\lambda^*/\partial\pi < 0$, we get $\mathrm{d}^2Z/\mathrm{d}x^2 < 0$. This implies that if a solution exists for $\mathrm{d}Z/\mathrm{d}x = 0$ it is unique. A set of sufficient conditions for the existence of a solution to $\mathrm{d}Z/\mathrm{d}x = 0$ are $\lim_{x\to 0}\pi'(x) = \infty$ and $\lim_{x\to\infty}\pi'(x) = 0$. We shall assume these conditions to hold. Thus the optimum x is determined by

$$\pi'(x)\int[\lambda^*P(\lambda^*)y + T(y) - T\{y(1-\lambda^*)\}]\, I(y)\,\mathrm{d}y = 1\,.$$

It is easy to interpret this equation. The left-hand side represents the marginal product in terms of expected revenue and penalties (per return) per unit increase of the expenditure per return on scrutiny. This is the product of the marginal increase $\pi'(x)$ in the probability of detection of understatement of income and the increase and average revenue and penalties per return per unit increase in the probability $\pi(x)$.

4. Concluding remarks

It is very obvious that the model of this paper is rather over-simplified. It views the tax structure as a purely revenue collecting device ignoring completely its role in altering income distribution. A rather simple penalty function for understatement of income has been assumed. It may be worthwhile investigating a more general penalty function $P(y, y_d)$ defined for $0 \leqq y_d \leqq y$, where y is the true income and y_d is the income declared in the tax return. As plausible conditions in P, one would explore $P_1 > 0$, $P_{11} > 0$, $P_2 < 0$, $P_{22} > 0$, $P_{12} < 0$ where P_i represent the partial derivatives of P with respect to its ith argument and P_{ij} are the second partial derivatives. Of course, one would impose the condition $P(y, y) \equiv 0$. The penalty function assumed in this paper satisfies these conditions. So would a convex function of income understated $y - y_d$. A convex function of tax avoided $T(y) - T(y_d)$ with $T(y)$ convex will satisfy all conditions except possibly $P_{22} < 0$. The behaviour of the individual taxpayer need not be as simple as assumed here – for instance, he may be certain that he is likely to arouse the suspicion of the tax authority unless he declares a certain minimum income. This case has been discussed by Sharon (1967). It is feasible in the real world for a taxpayer to get the penalty on the act of understatement reduced at some cost through legal representation. A fuller analysis of the tax avoidance problem will have to take into account all these factors and more. It is hoped that the formal analysis presented in this paper, even though it is based on a simple model, will serve some useful purpose in comparing the performance of alternative tax structure.

Reference

Sharon, E., 1967, Income tax evasion as rational behaviour under risk, Working Paper No. 217, Center for Research in Management Science, University of California, Berkeley.