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NOTES

THE DETERMINATION OF INDIFFERENCE QUALITY LEVEL SINGLE SAMPLING ATTRIBUTE PLANS WITH GIVEN RELATIVE SLOPE

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SUMMARY. The problem proposed by Hamaker (1950) of determining single sampling attribute plans satisfying a given indifference quality level and the relative slope for the operating characteristic curve of the plan as well as a weaker version of the above problem are considered. A solution method is developed for the weaker plan. Both the problems have been modelled as nonlinear mixed integer goal programming problems and solutions derived. Tables and examples are provided.

1. Introduction

In industrial applications of sampling inspection, one of the many ways the decision maker (DM) could specify the performance level he requires from a sampling plan is to choose a quality level denoted by 'indifference point', p_0 and h_0, the relative slope at the point p_0. For the fraction defective p, the operating characteristic (OC) function of the sampling plan and the relative slope of the OC at p are denoted by $P(p)$ and $-2p\,P'(p)$ respectively.

Hamaker (1950) introduced a class of sampling plans specified by the requirements $P(p_0) = \frac{1}{2}$ and $-2p_0\,P'(p_0) = 2R(p_0) = h_0$ and presented an approximate solution procedure. For single sampling plan (SSP), since n and c have to be integers, no plan exists for the Hamaker's problem which satisfies the conditions exactly. So there are four types of plans according to the different combinations of the requirements of $P(p_0)\ (\leqslant, \geqslant)\ \frac{1}{2}$ and $R(p_0)\ (\leqslant, \geqslant)\ \frac{h_0}{2}$. To attain the requirements exactly, one has to operate a 'group SSP' at random with specified proportions, (see Chakraborty, 1989). We shall consider a weaker Hamaker SSP given by one of the 4 types indicated above, defined as the SSP satisfying $P(p_0) \geqslant \frac{1}{2}$ and $R(p_0) \geqslant \frac{h_0}{2}$ as nearly as possible and c is as small as possible (see Chakraborty, 1988).

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Chakraborty (1986, 1988) modelled the problem of designing SSP of given strength as a Goal (Fuzzy Goal) programming problem and obtained solutions by optimisation technique.

In section 2, we present some preliminary results. In section 3, a solution method for the weaker Hamaker's problem is given. A preemptive solution method for Goal programming (GP) model for the weaker Hamaker's (Hamaker's) problem is presented in section 4 (5). In section 6, we make comparisons of the procedures and concluding remarks.

2. Preliminary results

We shall follow the notations from Hald's (1981) book and Chakraborty's (1986, 1988) papers. We shall restrict our discussions 'under Poisson conditions', so that for a SSP, the relative slope is given by $-2pG'(c, m) = 2mg(c, m) = 2R(c, m)$. We shall often refer to $R(p_0)$ as the relative slope.

Theorem 2.1 : *For the Hamaker's problem, the acceptance number c is given by*

$$c \simeq \tfrac{1}{2}\, \pi\, h_0^2 - 0.73 \quad \ldots \quad (1)$$

and the sample size n is given by

$$n \simeq (c+0.67)/p_0. \quad \ldots \quad (2)$$

Proof: See Hald (1981).

Definition : For a given relative slope $\frac{h_0}{2}$, the inverse relative slope $m_{\frac{h_0}{2}}$ is defined as the solution to the equation

$$R\left(c,\ m_{\frac{h_0}{2}}\right) = \frac{h_0}{2}. \quad \ldots \quad (3)$$

3. Solution for weaker Hamaker's problem

The problem is to find the minimum integer $c = c_0$ and an integer n satisfying the following conditions as nearly as possible

$$P(p_0) = G(c, np_0) \geqslant \frac{1}{2} \quad \ldots \quad (4)$$

and

$$R(p_0) = np_0\, g(c, np_0) \geqslant \frac{h_0}{2}. \quad \ldots \quad (5)$$

Expressed in terms of OC fractile and inverse relative slope, the two inequalities are identical to

$$np_0 \leqslant m_{0\cdot 5}(c) \quad \ldots \quad (6)$$

and
$$np_0 \geqslant m_{\frac{h_0}{2}}(c). \qquad \ldots \quad (7)$$

We define an auxiliary function $r\left(c, \frac{h_0}{2}\right)$ as

$$m_{0\cdot 5}(c) - m_{\frac{h_0}{2}}(c) = r\left(c, \frac{h_0}{2}\right). \qquad \ldots \quad (8)$$

Since $r\left(c, \frac{h_0}{2}\right)$ is an increasing function of c, we obtain the required c as the minimum $c = c_0$ satisfying

$$r\left(c, \frac{h_0}{2}\right) \geqslant 0 \qquad \ldots \quad (9)$$

$$\Rightarrow \quad m_{0\cdot 5}(c) \geqslant m_{\frac{h_0}{2}}(c). \qquad \ldots \quad (10)$$

In terms of the relative slope, (10) is equivalent to

$$R(c, m_{0\cdot 5}(c)) \geqslant \frac{h_0}{2}. \qquad \ldots \quad (11)$$

Having obtained c from (11), we obtain the interval for n from (6) and (7). We have proved :

Theorem 3.1 : *The smallest value of c, $c = c_0$ say, for the weaker Hamaker's problem is uniquely determined from*

$$R(c_0-1, m_{0\cdot 5}(c_0-1)) < \frac{h_0}{2} \leqslant R(c_0, m_{0\cdot 5}(c_0)) \qquad \ldots \quad (12)$$

and the corresponding sample sizes are obtained from

$$m_{\frac{h_0}{2}}(c_0)/p_0 \leqslant n \leqslant m_{0\cdot 5}(c_0)/p_0. \qquad \ldots \quad (13)$$

Table 1 provides the values of $R(c, m_{0\cdot 5})(c))$ from which the required c_0 can be easily found out.

TABLE 1. THE VALUES OF $R(c, m_{0\cdot 5}(c))$

c	0	1	2	3	4
$R(c, m_{0\cdot 5}(c))$	0.347	0.530	0.659	0.770	0.867
c	5	6	7	8	9
$R(c, m_{0\cdot 5}(c))$	0.955	1.035	1.109	1.178	1.244

To calculate the value of n, we require $m_{0\cdot 5}(c)$ which is given in Table 1 in Hald (1981) and also $m_{\frac{h_0}{2}}(c)$ which is provided in Table 2 for selected values of $\frac{h_0}{2}$ for the practical range of h_0 (see Sherman, 1965).

TABLE 2. THE VALUE OF $m_{\frac{h_0}{2}}(c)$ FOR SOME $\frac{h_0}{2}$

$c = 0$	$\frac{h_0}{2}$	0.10	0.15	0.20	0.25	0.30	0.343
	$m_{\frac{h_0}{2}}$	0.11	0.18	0.26	0.35	0.49	0.67
$c = 1$	$\frac{h_0}{2}$	0.343	0.35	0.40	0.45	0.50	0.525
	$m_{\frac{h_0}{2}}$	0.93	0.95	1.09	1.26	1.49	1.67
$c = 2$	$\frac{h_0}{2}$	0.525	0.55	0.60	0.65	0.659	
	$m_{\frac{h_0}{2}}$	1.86	2.03	2.25	2.57	2.67	
$c = 3$	$\frac{h_0}{2}$	0.659	0.70	0.75	0.770		
	$m_{\frac{h_0}{2}}$	2.95	3.13	3.45	3.67		
$c = 4$	$\frac{h_0}{2}$	0.770	0.80	0.85	0.867		
	$m_{\frac{h_0}{2}}$	3.94	4.10	4.46	4.67		
$c = 5$	$\frac{h_0}{2}$	0.867	0.90	0.95	0.955		
	$m_{\frac{h_0}{2}}$	4.95	5.14	5.60	5.67		
$c = 6$	$\frac{h_0}{2}$	0.955	1.00	1.035			
	$m_{\frac{h_0}{2}}$	5.95	6.27	6.67			
$c = 7$	$\frac{h_0}{2}$	1.035	1.05	1.10	1.109		
	$m_{\frac{h_0}{2}}$	6.95	7.05	7.52	7.67		
$c = 8$	$\frac{h_0}{2}$	1.109	1.15	1.178			
	$m_{\frac{h_0}{2}}$	7.95	8.28	8.67			
$c = 9$	$\frac{h_0}{2}$	1.178	1.20	1.244			
	$m_{\frac{h_0}{2}}$	8.95	9.12	9.67			

Example 1. Let $p_0 = 0.02$ and $h_0 = 1.40$, to find the weaker Hamaker's SSP.

From Table 1, $R(2, 2.674) < 0.7 \leqslant R(3, 3.674)$, so $c_0 = 3$. From Table 2 and Table 1 in Hald (1981), we have

$$n \geqslant \frac{3.13}{0.02} = 156.5 \text{ and } n \leqslant \frac{3.672}{0.02} = 183.6$$

which implies that all integral values of n in the interval [157, 183] satisfy the two requirements for $c = 3$.

4. Goal programming formulations and solutions for weaker Hamaker's problem

We have conflicting objectives (goals) of reduced inspection (smaller value of n and equivalently smaller c) and satisfying the conditions (4) and (5) as closely as possible. Following Chakraborty (1986) this can be achieved by modelling the problem as a GP (see also Lee, 1972 and Ignizio, 1976). The GP model (for notation and explanation, see Chakraborty, 1986) is the following :

$$\min z = P_1 c + P_2 w_1 d_1^2 + P_2 w_2 d_2^2 \qquad \ldots (14)$$

subject to

$$P(p_0) - d_1^2 = \frac{1}{2} \qquad \ldots (15)$$

$$R(p_0) - d_2^2 = \frac{h_0}{2}. \qquad \ldots (16)$$

Following Chakraborty (1986) we can prove the following theorems :

Theorem 4.1 : *For the weaker Hamaker's problem, the decision number c_0 is given by* (12) *and a necessary condition for the optimal sample size n_0 is given by*

$$n_0 = \frac{c_0 + \left(1 - \frac{w_1}{w_2}\right)}{p_0}. \qquad \ldots (17)$$

Remarks : This is the necessary condition for the extremum. For the minimum solution, this should be compared with the boundary solutions and the minimum among the three should be taken as the solution.

Theorem 4.2 : *For the weaker Hamaker's problem, with the preemptive priority that $P(p_0) \geqslant \frac{1}{2}\left(R(p_0) \geqslant \frac{h_0}{2}\right)$ must be held as nearly as possible, the decision number c_0 for both the cases is given by* (12) *and n_0 is given by*

$$n_0 = \lceil m_{0\cdot 5}(c_0)/p_0 \rceil \qquad \ldots (18)$$

$$= \left\lfloor m_{\frac{h_0}{2}}(c_0)/p_0 \right\rfloor \qquad \ldots (19)$$

Example 2 : Let $p_0 = 0.02$, $h_0 = 1.40$.

(a) Find weaker plan with $w_1 = 1$ and $w_2 = 2$. Here $c_0 = 3$ and from (17), $n_0 = 175$ and $z = 0.1470$. However, the boundary solutions are $n = 157$, $z = 0.1185$ and $n = 183$, $z = 0.1418$; hence the required solution is $c_0 = 3$, $n_0 = 157$ providing $P(p_0) = 0.6159$ and $R(p_0) = 0.7013$.

(b) Preemptive plan $P(p_0) \geqslant \frac{1}{2}$; here $c_0 = 3$ and $n_0 = 183$ with $P(p_0) = 0.5025$ and $R(p_0) = 0.7696$.

(c) Preemptive plan $R(p_0) \geqslant \frac{h_0}{2}$, here $c_0 = 3$ and $n_0 = 157$ with $P(p_0) = 0.6159$ and $R(p_0) = 0.7013$.

5. GP formulation for Hamaker's problem

GP formulation (see Chakraborty, 1989) is

$$\min z = w_1 d_1^+ + w_2 d_1^- + w_3 d_2^+ + w_4 d_2^- \quad \ldots \quad (20)$$

subject to

$$P(p_0) + d_1^+ - d_1^- = \frac{1}{2} \quad \ldots \quad (21)$$

$$R(p_0) + d_2^+ - d_2^- = \frac{h_0}{2} \quad \ldots \quad (22)$$

$$d_1^+ d_1^- = 0, \; d_2^+ d_2^- = 0 \quad \ldots \quad (23)$$

$$d_1^+, d_1^-, d_2^+, d_2^-, n, c \geqslant 0. \quad \ldots \quad (24)$$

This is a nonlinear mixed integer programming problem and can be solved by some search or branch and bound method.

Example 3 : $p_0 = 0.02$, $h_0 = 1.40$, to find the SSP for

(a) $Z_1 = d_1^+ + d_1^- + d_2^+ + d_2^-$

and

(b) $Z_2 = d_2^- + d_2 + 5d^+ + 5d_2^-$.

By search method

(a) $n_0 = 134$, $c_0 = 2$, $Z_1 = 0.0416$ with $P(p_0) = 0.4985$ and $R(p_0) = 0.6599$.

(b) $n_0 = 157$, $c_0 = 3$, $Z_2 = 0.1220$ with $P(p_0) = 0.6159$ and $R(p_0) = 0.7013$.

6. Comparisons and conclusions

Comparisons. Since the decision variables (n, c) of the Hamaker's problem are integers, the approximate solution provided by Hamaker in (1) and (2) is true for a range of $\frac{h_0}{2}$. For example, if $0.596 \leqslant \frac{h_0}{2} \leqslant 0.717$, the value of c would be 2 and for any p_0 we shall get the SSP as $(n, 2)$ providing $P(p_0)$ very close to $\frac{1}{2}$ but $R(p_0)$ would be fixed at 0.6595. However, in the GP model, for a p_0, with a proper choice of w_1, w_2, w_3, and w_4, we can get a host of SSPs $(n, 2)$ and $(n, 3)$ which would provide various combinations of $P(p_0)$'s and $R(p_0)$'s giving the DM the required 'satisfactory' SSP. To illustrate, we provide a particular case, $p_0 = 0.02$ and $\frac{h_0}{2} = 0.70$. Hamaker's solution is $c = 2.347$, $n = 135.5$ i.e. $c = 2$ and $n = 134$ providing $P(p_0) = 0.4985$ and $R(p_0) = 0.6595$. In Table 3, four GP plans of different types $\left(P(p_0)\ (\leqslant, \geqslant)\ \frac{1}{2},\ R(p_0)\ (\leqslant, \geqslant)\ \frac{h_0}{2}\right)$ are given.

TABLE 3. FOUR TYPES OF GP PLANS

sl. no.	w_1	w_2	w_3	w_4	n	c	$P(p_0)$	$R(p_0)$
1	5	1	1	5	133	2	.5034	.6534
2	1	5	4	5	137	2	.4839	.6641
3	1	1	5	5	157	3	.6159	.7013
4	1	5	5	1	184	3	.4983	.7710

The DM may find a satisfactory SSP depending on his requirement.

Concluding remarks: It is wellknown that the approximation $m_\alpha(c) = c + \frac{2}{3}$ is very good for $c \geqslant 2$, so that in the Hamaker's solution, the goal $P(p_0) = \frac{1}{2}$ is always maintained very closely but the deviation in the other goal can not be controlled. In GP model, $P(p_0)$ can be allowed to deviate so that a closer $R(p_0)$ is arrived at so that the DM can obtain a 'satisfactory' SSP. Also various other goals such as $d_1^+, d_1^- \leqslant \gamma_1(\frac{1}{2})$ and $d_2^+, d_2^- \leqslant \gamma_2 \frac{h_0}{2}$ (where γ_1 and γ_2 are small real numbers) can be included in the model for obtaining 'satisfactory' SSP.

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