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Eigenvalue Inequalities Associated with the Cartesian Decomposition

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Let $T = A + iB$ where A, B are Hermitian matrices. We obtain several inequalities relating the l_p distance between the eigenvalues of A and those of iB with the Schatten p-norm of T. The majorization results which lead to these inequalities are also used to get simple proofs of some known lower and upper bounds for the determinant of T.

1. INTRODUCTION

Let T be an n by n complex matrix. By the *Cartesian decomposition* of T we mean the decomposition $T = A + iB$, where A and B are Hermitian matrices, called the real and imaginary parts of T, and defined as

$$A = \frac{T + T^*}{2}, \qquad B = \frac{T - T^*}{2i}.$$

One major theme of this paper is the study of relations between the eigenvalues of A, those of B and the singular values of T. The history of

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this subject goes back to H. Weyl [19] who obtained inequalities between the eigenvalues and the singular values of T. (Weyl called these numbers "two kinds of eigenvalues" of T. Here we will study "three kinds of eigenvalues".) Following Weyl more inequalities of this type were obtained by several mathematicians. An account of these results may be found in the book [12] by A. W. Marshall and I. Olkin and in the paper [1] by T. Ando.

Some of the inequalities we obtain can be interpreted as bounds for the distance between the eigenvalues of matrices. The history of this subject goes back to another paper of Weyl [18]. There, he showed that if A and B are two Hermitian matrices with eigenvalues arranged as $\lambda_1 \geqslant \cdots \geqslant \lambda_n$ and $\mu_1 \geqslant \cdots \geqslant \mu_n$, respectively, then

$$\max_j |\lambda_j - \mu_j| \leqslant \|A - B\|. \tag{1}$$

Here $\|A\|$ denotes the usual operator bound norm of A as an operator on the Hilbert space $\mathbf{C}^n$. A powerful theorem of V. B. Lidskii [11] and H. Wielandt [20] led to a very considerable generalization of this result. Let $|||\cdot|||$ be any unitarily invariant norm on matrices. (See [12] for definitions.) Let $\mathrm{Eig}_{\downarrow}(A)$ denote the diagonal matrix whose diagonal entries are the eigenvalues of A listed in decreasing order as above. Then the Lidskii-Wielandt Theorem implies that for any two Hermitian matrices A and B we have

$$|||\mathrm{Eig}_{\downarrow}(A) - \mathrm{Eig}_{\downarrow}(B)||| \leqslant |||A - B||| \tag{2}$$

for every unitarily invariant norm. The inequality (1) is the special case of (2) for the operator norm.

The inequality (2) in this form was noted by L. Mirsky [14] who conjectured that an analogous result should be true when A and B are normal. More precisely, let $\mathrm{Eig}\,A$ denote a diagonal matrix whose diagonal consists of the eigenvalues of a given matrix A in any order. Given a permutation σ on n symbols let $\mathrm{Eig}_\sigma A$ denote the diagonal matrix whose diagonal is obtained from that of $\mathrm{Eig}\,A$ by applying to it the permutation σ. Mirsky conjectured that if A and B are any two normal matrices then

$$\min |||\mathrm{Eig}\,A - \mathrm{Eig}_\sigma B||| \leqslant |||A - B||| \tag{3}$$

for every unitarily invariant norm. A. J. Hoffman and H. W. Wielandt

[9] had already established this result for the special case of the Frobenius norm. It turns out that (3) is false for some unitarily invariant norms. For several known results on this problem the reader is referred to the papers by V. S. Sunder [17], R. Bhatia, Ch. Davis, J. A. R. Holbrook and A. McIntosh [3], [4], [5], [6], [7].

In particular, Sunder [17] proved that in the special case when A is Hermitian and B is skew-Hermitian then (3) is true when the operator norm is used but false in all Schatten p-norms for $1 \leqslant p < 2$. In this paper we will show that if A is Hermitian and B is skew-Hermitian, then

$$\min \|\operatorname{Eig} A - \operatorname{Eig}_\sigma B\|_p \leqslant \|A - B\|_p \qquad \text{for} \quad 2 \leqslant p \leqslant \infty \tag{4}$$

and

$$\min \|\operatorname{Eig} A - \operatorname{Eig}_\sigma B\|_p \leqslant 2^{1/p - 1/2} \|A - B\|_p \qquad \text{for} \quad 1 \leqslant p \leqslant 2. \tag{5}$$

Here $\|A\|_p$ denotes the Schatten p-norm defined as

$$\|A\|_p = \left(\sum_{j=1}^{n} s_j^p \right)^{1/p}, \qquad 1 \leqslant p < \infty \tag{6}$$

where $s_1, \ldots, s_n$ are the singular values of A, and by the usual convention

$$\|A\|_\infty = \max_j s_j = \|A\|. \tag{7}$$

Sunder's result is thus a special case of (4).

Several inequalities complementary to (4) and (5) are also obtained in this paper.

The methods used here also lead to some results on a third problem going back to the work of A. M. Ostrowski and O. Taussky [15]. They showed that if in the Cartesian decomposition $T = A + iB$ the matrix A is positive semidefinite then $|\det T| \geqslant \det A$. Some other authors have obtained upper and lower bounds for $|\det T|$. Our results give refinements of some of these results and easier proofs of some others.

The unifying theme in deriving all these results is the concept of majorization. We refer the reader to [1] or [12] for a detailed study of this notion. However, for convenience, we very briefly give the basic definitions and some propositions which we use in the next section.

2. PRELIMINARIES

Let $x = (x_1, \ldots, x_n)$ be a vector in $\mathbf{R}^n$. Denote by $x_{[1]} \geqslant \cdots \geqslant x_{[n]}$ the coordinates of x rearranged in decreasing order. If x and y are two vectors such that

$$\sum_{j=1}^{k} x_{[j]} \leqslant \sum_{j=1}^{k} y_{[j]}, \qquad k = 1, 2, \ldots, n \tag{8}$$

we say that $x \prec_w y$; in words, x is *weakly submajorized* by y. If $-x \prec_w -y$ we say that $x \prec^w y$; in words, x is *weakly supermajorized* by y. If both the relations $x \prec_w y$ and $x \prec^w y$ hold we say that x is *majorized* by y and denote this relation as $x \prec y$. Equivalently, $x \prec y$ iff the inequalities (8) hold and the last of them is an equality.

The following well known result ([1], [12, p. 115]) will be used repeatedly in this paper:

PROPOSITION 2.1 *Let $x \prec y$. Let g be a convex function on* $\mathbf{R}$ *Then the vector $(g(x_1), \ldots, g(x_n))$ in $\mathbf{R}^n$ is weakly submajorized by the vector $(g(y_1), \ldots, g(y_n))$.*

Hence, if g is a concave function and if $x \prec y$ then $(g(x_1), \ldots, g(x_n)$ is weakly supermajorized by $(g(y_1), \ldots, g(y_n))$.

For convenience let us adopt the notation $\{g(x_j)\}_j$ for an n-vector whose j-th coordinate is $g(x_j)$.

Let X be a Hermitian matrix. Its eigenvalues arranged in decreasing order are labelled as $\lambda_{[1]}(X) \geqslant \cdots \geqslant \lambda_{[n]}(X)$. We denote by $\mathrm{Eig}\downarrow(X)$ the vector $\{\lambda_{[j]}(X\}_j$ and by $\mathrm{Eig}\uparrow(X)$ the vector $\{\lambda_{[n-j+1]}(X)\}_j$. That is these two vectors consist of the eigenvalues of X arranged in decreasing order and in increasing order, respectively. In Section 1 we used $\mathrm{Eig}_\downarrow(X)$ also to denote the diagonal matrix with $\lambda_{[j]}(X)$ as the j-th entry on the diagonal. It is convenient to retain this duplicity of notation, it being clear from the context whether we are talking of a vector or of a diagonal matrix whose diagonal coincides with this vector.

With these notations the Lidskii–Wielandt Theorem mentioned in Section 1 can be stated as:

THEOREM 2.2 (Lidskii–Wielandt) *Let X, Y be two Hermitian matrices. Then the following majorization relations hold*

$$\mathrm{Eig}_\downarrow(X) + \mathrm{Eig}\uparrow(Y) \prec \mathrm{Eig}\downarrow(X + Y) \prec \mathrm{Eig}\downarrow(X) + \mathrm{Eig}_\downarrow(Y).$$

The second of these relations also follows from an earlier theorem of Ky Fan. (See [12, pp. 241–242].)

For an arbitrary matrix T we denote its singular values as $s_1(T) \geqslant \cdots \geqslant s_n(T) \geqslant 0$. We denote by $s_\downarrow(T)$ and $s_\uparrow(T)$, respectively the vectors $\{s_j(T)\}_j$ and $\{s_{n-j+1}(T)\}_j$.

Notice that if X is positive semidefinite, then $\lambda_{[j]}(X) = s_j(X)$. So for positive semidefinite X and Y the Lidskii–Wielandt Theorem gives

$$s_\downarrow(X) + s\uparrow(Y) \prec s\downarrow(X + Y) \prec s\downarrow(X) + s_\downarrow(Y). \tag{9}$$

In Section 4 we will use a "multiplicative version" of the Lidskii–Wielandt Theorem:

THEOREM 2.3 *Let X, Y be two positive definite matrices. Then the eigenvalues of XY are positive and the following majorization relations are satisfied*

$$\log \operatorname{Eig}_\downarrow(X) + \log \operatorname{Eig}\uparrow(Y) \prec \log \operatorname{Eig} XY$$
$$\prec \log \operatorname{Eig}\downarrow(X) + \log \operatorname{Eig}_\downarrow(Y). \tag{10}$$

Here $\log \operatorname{Eig}_\downarrow(X)$ stands for the vector whose j-th component is $\log\lambda_{[j]}(X)$ and so on. (For a proof of this result see, e.g., [1].)

3. THE MAIN RESULTS ON EIGENVALUES

To avoid repetition, let us fix the following notations for use throughout this section. T will denote an arbitrary matrix with the Cartesian decomposition $T = A + iB$. The eigenvalues of A and B will be denoted by α_j and β_j, respectively, ordered in such a way that $|\alpha_1| \geqslant \cdots \geqslant |\alpha_n|$ and $|\beta_1| \geqslant \cdots \geqslant |\beta_n|$. The symbol s_j will always mean $s_j(T)$, i.e. the j-th singular value of T. When some other matrix X is under consideration we will denote its j-th singular value by $s_j(X)$.

THEOREM 3.1 *The following majorization relations are satisfied*

$$\{|\alpha_j + i\beta_{n-j+1}|^2\}_j \prec \{s_j^2\}_j, \tag{11}$$

$$\left\{\frac{1}{2}(s_j^2 + s_{n-j+1}^2)\right\}_j \prec \{|\alpha_j + i\beta_j|^2\}_j. \tag{12}$$

Proof Let $X = A^2$, $Y = B^2$ and use (9) to get

$$\{|\alpha_j + i\beta_{n-j+1}|^2\}_j \prec \{s_j(A^2 + B^2)\}_j \prec \{|\alpha_j + i\beta_j|^2\}_j. \tag{13}$$

Next note that

$$A^2 + B^2 = \frac{1}{2}(T^*T + TT^*)$$

and

$$s_j(T^*T) = s_j(TT^*) = s_j^2.$$

Let $X = \frac{T^*T}{2}$, $Y = \frac{TT^*}{2}$, use (9) and the above observations to get

$$\left\{\frac{1}{2}(s_j^2 + s_{n-j+1}^2)\right\}_j \prec \{s_j(A^2 + B^2)\}_j \prec \{s_j^2\}_j. \tag{14}$$

The relations (11) and (12) now follow from (13) and (14). ∎

The relation (11) has been noted already by C.-K. Li [10]. We thank Professor J. F. Queiró for bringing this to our notice.

Now note that the function $g(t) = t^{p/2}$ on $[0, \infty)$ is convex when $p \geqslant 2$ and concave when $1 \leqslant p \leqslant 2$. So, using Proposition 2.1 we get from (11)

$$\{|\alpha_j + i\beta_{n-j+1}|^p\}_j \prec_w \{s_j^p\}_j \quad \text{for} \quad 2 \leqslant p, \tag{15}$$

$$\{|\alpha_j + i\beta_{n-j+1}|^p\}_j \prec^w \{s_j^p\}_j \quad \text{for} \quad 1 \leqslant p \leqslant 2. \tag{16}$$

In particular, we have

$$\sum_{j=1}^n |\alpha_j + i\beta_{n-j+1}|^p \leqslant \sum_{j=1}^n s_j^p \quad \text{for} \quad 2 \leqslant p, \tag{17}$$

$$\sum_{j=1}^n |\alpha_j + i\beta_{n-j+1}|^p \geqslant \sum_{j=1}^n s_j^p \quad \text{for} \quad 1 \leqslant p \leqslant 2. \tag{18}$$

Repeat the same arguments using the relation (12) instead of (11) to get

$$\frac{1}{2^{p/2}}\{(s_j^2 + s_{n-j+1}^2)^{p/2}\}_j \prec_w \{|\alpha_j + i\beta_j|^p\}_j \quad \text{for} \quad 2 \leqslant p, \tag{19}$$

$$\frac{1}{2^{p/2}}\{(s_j^2 + s_{n-j+1}^2)^{p/2}\}_j \prec^w \{|\alpha_j + i\beta_j|^p\}_j \quad \text{for} \quad 1 \leqslant p \leqslant 2. \tag{20}$$

In particular, these give

$$\frac{1}{2^{p/2}}\sum_{j=1}^n (s_j^2 + s_{n-j+1}^2)^{p/2} \leqslant \sum_{j=1}^n |\alpha_j + i\beta_j|^p \quad \text{for} \quad 2 \leqslant p, \tag{21}$$

$$\frac{1}{2^{p/2}}\sum_{j=1}^{n}(s_j^2+s_{n-j+1}^2)^{p/2}\geqslant\sum_{j=1}^{n}|\alpha_j+i\beta_j|^p \quad \text{for} \quad 1\leqslant p\leqslant 2. \quad (22)$$

Next recall that for fixed nonnegative real numbers a_1, a_2 the function $(a_1^t+a_2^t)^{1/t}$ is a monotonically decreasing function of t. It follows that $s_j^p+s_{n-j+1}^p$ is dominated by $(s_j^2+s_{n-j+1}^2)^{p/2}$ for $p\geqslant 2$. For $1\leqslant p\leqslant 2$ this domination is reversed. Hence, we obtain from (21) and (22) the inequalities

$$2^{1-p/2}\sum_{j=1}^{n}s_j^p\leqslant\sum_{j=1}^{n}|\alpha_j+i\beta_j|^p \quad \text{for} \quad 2\leqslant p, \quad (23)$$

$$2^{1-p/2}\sum_{j=1}^{n}s_j^p\geqslant\sum_{j=1}^{n}|\alpha_j+i\beta_j|^p \quad \text{for} \quad 1\leqslant p\leqslant 2. \quad (24)$$

The inequalities (17), (18), (23) and (24) can be summarized as

THEOREM 3.2 *Let $T=A+iB$ where A and B are Hermitian. Then*
(a) *for $2\leqslant p\leqslant\infty$ we have*

$$\min_\sigma\|\mathrm{Eig}\,A+\mathrm{Eig}_\sigma(iB)\|_p\leqslant\|T\|_p, \quad (25)$$

$$\|T\|_p\leqslant 2^{1/2-1/p}\max_\sigma\|\mathrm{Eig}\,A+\mathrm{Eig}_\sigma(iB)\|_p; \quad (26)$$

(b) *for $1\leqslant p\leqslant 2$ we have*

$$\min_\sigma\|\mathrm{Eig}\,A+\mathrm{Eig}_\sigma(iB)\|_p\leqslant 2^{1/p-1/2}\|T\|_p, \quad (27)$$

$$\|T\|_p\leqslant\max_\sigma\|\mathrm{Eig}\,A+\mathrm{Eig}_\sigma(iB)\|_p. \quad (28)$$

If the diagonal matrices Eig A *and* Eig(iB) *have on their diagonals the respective eigenvalues of A and iB both arranged in decreasing order of modulus then the minimum in* (25) *and the maximum in* (28) *are both attained for the permutation σ which sends* $(1,2,\dots,n)$ *to* $(n,n-1,\dots,1)$ *while the minimum in* (27) *and the maximum in* (26) *are both attained for the identity permutation.*

Proof The inequalities have already been proved. Only the statement about the extremal permutations needs proof. Recall from [1] or [12, pp. 146–155] the following facts about *lattice super-additive* functions. These are functions $\phi(u, v)$ defined for positive real arguments u, v such that

$$\phi(u_1, v_1) + \phi(u_2, v_2) \leqslant \phi(u_1 \vee u_2, v_1 \vee v_2) + \phi(u_1 \wedge u_2, v_1 \wedge v_2),$$

for any u_1, u_2, v_1, v_2, where $\vee$ and $\wedge$ stand for "maximum" and "minimum" as usual. If $u_1 \geqslant u_2 \geqslant \cdots \geqslant u_n \geqslant 0$ and $v_1 \geqslant v_2 \geqslant \cdots \geqslant v_n \geqslant 0$ and σ is any permutation on n symbols then any such function satisfies the inequalities

$$\sum_{j=1}^{n} \phi(u_j, v_{n-j+1}) \leqslant \sum_{j=1}^{n} \phi(u_j, v_{\sigma(j)}) \leqslant \sum_{j=1}^{n} \phi(u_j, v_j).$$

The function $\phi(u, v) = (u + v)^t$ is lattice super-additive for all $t \geqslant 1$ and for $0 < t \leqslant 1$ it is lattice sub-additive (i.e., $-\phi$ is lattice super-additive). From these known facts one easily gets:

$$\sum_{j=1}^{n} |\alpha_j + i\beta_{n-j+1}|^p \leqslant \sum_{j=1}^{n} |\alpha_j + i\beta_{\sigma(j)}|^p \leqslant \sum_{j=1}^{n} |\alpha_j + i\beta_j|^p$$

for $2 \leqslant p < \infty$, for every permutation σ. Both the inequalities are reversed for $1 \leqslant p \leqslant 2$. ∎

Remarks 1. All the statements for $1 \leqslant p < \infty$ have already been proved. The $p = \infty$ case follows by a limiting argument.

2. The following example shows that the above inequalities are sharp:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

3. It is interesting to note that while in the case of Hermitian matrices A, B the minimum in (3) is attained for the same permutation for every unitarily invariant norm because of inequality (2), the situation is quite different when we consider A and iB. Here the "best matching" of the eigenvalues of A and iB becomes the "worst matching" when we change from $p \geqslant 2$ to $p \leqslant 2$.

4. Let us say that a unitarily invariant norm $|||\cdot|||$ is a Q-norm if there exists another unitarily invariant norm $|||\cdot|||'$ such that $|||T||| = (|||T^*T|||')^{1/2}$. A Schatten p-norm is a Q-norm iff $p \geqslant 2$. For $1 \leqslant k \leqslant n$ the norms defined by $(s_1^2(T) + \cdots + s_k^2(T))^{1/2}$ are also Q-norms. The majorization (11), in fact, yields the inequality

$$\min_{\sigma} \|\operatorname{Eig} A + \operatorname{Eig}_{\sigma}(iB)\|_Q \leqslant \|T\|_Q \tag{29}$$

for every Q-norm $\|\cdot\|_Q$.

Another family of inequalities can be obtained from the above relations by the following considerations. Let $p \geqslant 2$. Using the convexity of the function $g(t) = t^{p/2}$ on $[0, \infty)$ and the inequalities (23) and (17) we get

$$\begin{aligned}
2^{1-p/2} \sum_{j=1}^{n} s_j^p &\leqslant \sum_{j=1}^{n} |\alpha_j + i\beta_j|^p \\
&= \sum_{j=1}^{n} (|\alpha_j|^2 + |\beta_j|^2)^{p/2} \\
&\leqslant 2^{p/2-1} \sum_{j=1}^{n} (|\alpha_j|^p + |\beta_j|^p) \\
&= 2^{p/2-1} \sum_{j=1}^{n} (|\alpha_j|^p + |\beta_{n-j+1}|^p) \\
&\leqslant 2^{p/2-1} \sum_{j=1}^{n} (|\alpha_j|^2 + |\beta_{n-j+1}|^2)^{p/2} \\
&= 2^{p/2-1} \sum_{j=1}^{n} |\alpha_j + i\beta_{n-j+1}|^p \\
&\leqslant 2^{p/2-1} \sum_{j=1}^{n} s_j^p.
\end{aligned}$$

Thus we have, for $p \geqslant 2$

$$\sum_{j=1}^{n} s_j^p \leqslant 2^{p-2} \sum_{j=1}^{n} |\alpha_j + i\beta_{n-j+1}|^p, \tag{30}$$

$$\sum_{j=1}^{n} |\alpha_j + i\beta_j|^p \leqslant 2^{p/2-1} \sum_{j=1}^{n} s_j^p. \tag{31}$$

Same way, if $1 \leqslant p \leqslant 2$ then using the concavity of the function $g(t) = t^{p/2}$ on $[0, \infty)$ and the inequalities (24) and (18) we obtain for all $1 \leqslant p \leqslant 2$

$$\sum_{j=1}^{n} s_j^p \leqslant 2^{1-p/2} \sum_{j=1}^{n} |\alpha_j + i\beta_j|^p, \tag{32}$$

$$\sum_{j=1}^{n} |\alpha_j + i\beta_{n-j+1}|^p \leqslant 2^{2-p} \sum_{j=1}^{n} s_j^p. \tag{33}$$

We can reformulate these inequalities to get a theorem complementary to Theorem 3.2.

THEOREM 3.3 *Let $T = A + iB$ where A and B are Hermitian. Then*

(a) *for $2 \leqslant p \leqslant \infty$ we have*

$$\|T\|_p \leqslant 2^{1-2/p} \min_\sigma \|\text{Eig}\, A + \text{Eig}_\sigma(iB)\|_p, \tag{34}$$

$$\max_\sigma \|\text{Eig}\, A + \text{Eig}_\sigma(iB)\|_p \leqslant 2^{1/2-1/p} \|T\|_p; \tag{35}$$

(b) *for $1 \leqslant p \leqslant 2$ we have*

$$\|T\|_p \leqslant 2^{1/p-1/2} \min_\sigma \|\text{Eig}\, A + \text{Eig}_\sigma(iB)\|_p, \tag{36}$$

$$\max_\sigma \|\text{Eig}\, A + \text{Eig}_\sigma(iB)\|_p \leqslant 2^{2/p-1} \|T\|_p. \tag{37}$$

Remarks 1. Since the function $g(t) = t^{p/2}$ on $[0, \infty)$ is concave even for $0 < p < 1$ the majorization results (16) and (20) and their consequences are valid for $0 < p \leqslant 2$. However, only the results for $1 \leqslant p \leqslant 2$ can be translated to norm inequalities.

2. The inequalities derived above give a complete set of bounds from above and below for $\|T\|_p$ in terms of the eigenvalues of A and B. It still remains open to find a sharp bound like (2) for all unitarily invariant norms. It is reasonable to make the following

CONJECTURE
$$\min_\sigma |||\text{Eig}\, A + \text{Eig}_\sigma(iB)||| \leqslant \sqrt{2}|||T||| \tag{38}$$

for every unitarily invariant norm. Further, in view of (24) and (31) one may expect that in fact

$$|||\text{diag}(\alpha_1 + i\beta_1, \ldots, \alpha_n + i\beta_n)||| \leqslant \sqrt{2}|||T|||. \tag{39}$$

Notice that since $|||\mathrm{Eig}\, A||| = |||A||| \leqslant |||T|||$ and same for B the triangular inequality for norms implies that the left-hand side of (39) is dominated by $2|||T|||$.

3. Fan and Hoffman [8] (see also [12], p. 240) have proved that if $\alpha_{[j]}$, $1 \leqslant j \leqslant n$, are the eigenvalues of A arranged in decreasing order $\alpha_{[1]} \geqslant \cdots \geqslant \alpha_{[n]}$, then $\alpha_{[j]} \leqslant s_j$ for all j. Same way $\beta_{[j]} \leqslant s_j$ for all j. If A and B are both positive semidefinite then $\alpha_j = \alpha_{[j]}$ and $\beta_j = \beta_{[j]}$. So in this case $|\alpha_j + i\beta_j| \leqslant \sqrt{2} s_j$ for all j and hence (39) is true for such A and B.

4. Note that for $k = 1, 2, \ldots, n$, we have from a well known theorem of Fan (see [12, p. 243])

$$\sum_{j=1}^{k} s_j \leqslant \sum_{j=1}^{k} (s_j(A) + s_j(B)) = \sum_{j=1}^{k} (|\alpha_j| + |\beta_j|)$$

$$\leqslant \sum_{j=1}^{k} \sqrt{2}(\alpha_j^2 + \beta_j^2)^{1/2} = \sum_{j=1}^{k} \sqrt{2}|\alpha_j + i\beta_j|.$$

In other words $\{s_j\}_j$ is weakly submajorized by $\{\sqrt{2}|\alpha_j + i\beta_j|\}_j$. Hence

$$|||T||| \leqslant \sqrt{2} \max_{\sigma} |||\mathrm{Eig}\, A + \mathrm{Eig}_\sigma(iB)|||. \tag{40}$$

Note that by the example in Remark 2 following Theorem 3.2 this inequality cannot be improved.

5. In [13] M. E. F. Miranda has proved that

$$\sum_{j=1}^{k} (\alpha_{n-j+1}^2 + \beta_{n-j+1}^2) \leqslant \sum_{j=1}^{k} s_j^2, \qquad 1 \leqslant k \leqslant n-1, \tag{41}$$

$$\sum_{j=1}^{2k-n} s_{j+2(n-k)}^2 \leqslant \sum_{j=1}^{k} (\alpha_j^2 + \beta_j^2), \qquad \left[\frac{n+1}{2}\right] \leqslant k \leqslant n. \tag{42}$$

Note that the majorization (11) actually implies

$$\sum_{j=1}^{k} (\alpha_j^2 + \beta_{n-j+1}^2) \leqslant \sum_{j=1}^{k} s_j^2, \qquad 1 \leqslant k \leqslant n. \tag{43}$$

Since

$$\sum_{j=1}^{k} \alpha_{n-j+1}^2 \leqslant \sum_{j=1}^{k} \alpha_j^2,$$

the inequality (43) is stronger than (41).

Also note that the majorization (12) implies

$$\frac{1}{2}\sum_{j=1}^{m}(s_j^2 + s_{n-j+1}^2) \leqslant \sum_{j=1}^{m}(\alpha_j^2 + \beta_j^2), \qquad 1 \leqslant m \leqslant n. \tag{44}$$

This, in turn, gives

$$\sum_{j=1}^{m} s_{n-j+1}^2 \leqslant \sum_{j=1}^{m}(\alpha_j^2 + \beta_j^2), \qquad 1 \leqslant m \leqslant n. \tag{45}$$

Now note that if $\left[\frac{n+1}{2}\right] \leqslant k \leqslant n$ then

$$\sum_{j=1}^{2k-n} s_{j+2(n-k)}^2 = \sum_{j=1}^{2k-n} s_{n-j+1}^2 \leqslant \sum_{j=1}^{k} s_{n-j+1}^2.$$

So the inequality (42) of Miranda is weaker than (45) which, in turn, is weaker than (44).

The referee has pointed out that the inequalities (41) and (42) are immediate consequences of the relation

$$\sum_{j=1}^{n} s_j^2 = \sum_{j=1}^{n}(\alpha_j^2 + \beta_j^2),$$

that is the easiest part of (11) and the assumed orderings.

4. SOME BOUNDS FOR DETERMINANTS

Some known results on determinants follow easily from our results in Section 3 and so are presented here.

In [16], J. F. Queiró and A. L. Duarte have proved the following theorem of which we give a much simpler proof:

THEOREM 4.1 *Let $T = A + iB$ where A and B are Hermitian with eigenvalues $|\alpha_1| \geqslant \cdots \geqslant |\alpha_n|$ and $|\beta_1| \geqslant \cdots \geqslant |\beta_n|$ respectively. Then*

$$|\det T| \leqslant \prod_{j=1}^{n} |\alpha_j + i\beta_{n-j+1}|. \tag{46}$$

Proof The function $g(t) = \frac{1}{2}\log t$ is concave on $(0, \infty)$. So by (11) and Proposition 2.1 we have

$$\sum_{j=1}^{n} \log|\alpha_j + i\beta_{n-j+1}| \geqslant \sum_{j=1}^{n} \log s_j.$$

Hence,

$$\prod_{j=1}^{n} |\alpha_j + i\beta_{n-j+1}| \geqslant \prod_{j=1}^{n} s_j = |\det T|. \qquad \blacksquare$$

A lower bound for $|\det T|$ complementary to the above result is possible when A and B are both positive semidefinite. Such a bound was derived by N. Bebiano [2]. Before coming to this let us make the following observations.

Let $T = A + iB$ with A positive definite. Then we can write

$$T = A^{1/2}(I + iA^{-1/2}BA^{-1/2})A^{1/2},$$

$$\det T = \det A \cdot \det (I + iA^{-1/2}BA^{-1/2}). \tag{47}$$

Next note that

$$\begin{aligned}
&|\det(I + iA^{-1/2}BA^{-1/2})|^2 \\
&\quad = \det(I + iA^{-1/2}BA^{-1/2})(I - iA^{-1/2}BA^{-1/2}) \\
&\quad = \det(I + (A^{-1/2}BA^{-1/2})^2) \\
&\quad = \prod_{j=1}^{n} (1 + (s_j(A^{-1/2}BA^{-1/2}))^2).
\end{aligned} \tag{48}$$

So, we have

$$|\det T| = \det A \prod_{j=1}^{n} (1 + s_j(A^{-1/2}BA^{-1/2})^2)^{1/2}. \tag{49}$$

This is a refinement of the Ostrowski–Taussky bound $|\det T| \geqslant \det A$.

We next prove the following result of Bebiano using the above analysis:

THEOREM 4.2 *Let $T = A + iB$ where A, B are positive semidefinite with eigenvalues $\alpha_1 \geqslant \cdots \geqslant \alpha_n$ and $\beta_1 \geqslant \cdots \geqslant \beta_n$ respectively. Then*

$$|\det T| \geqslant \prod_{j=1}^{n} |\alpha_j + i\beta_j|. \tag{50}$$

Proof It suffices to prove the theorem when A, B are both positive definite. By the equations (47) and (48) the inequality (50) will be established if we show

$$\prod_{j=1}^{n} (1 + s_j(A^{-1/2}BA^{-1/2})^2) \geqslant \prod_{j=1}^{n} (1 + \alpha_j^{-2}\beta_j^2). \tag{51}$$

Since B is positive definite

$$s_j(A^{-1/2}BA^{-1/2}) = s_j(A^{-1/2}B^{1/2})^2, \qquad j = 1, 2, \ldots, n.$$

By Theorem 2.3 the sequence $\{\log s_{n-j+1}(A^{-1/2}) + \log s_j(B^{1/2})\}_j$ is majorized by $\{\log s_j(A^{-1/2}B^{1/2})\}_j$. But $s_{n-j+1}(A^{-1/2}) = \alpha_j^{-1/2}$ and $s_j(B^{1/2}) = \beta_j^{1/2}$. So we have

$$\{\log(\alpha_j^{-1/2}\beta_j^{1/2})\}_j \prec \{\log s_j(A^{-1/2}B^{1/2})\}_j. \tag{52}$$

Now note that the function $g(t) = \log(1 + e^{4t})$ is convex. So (52) and Proposition 2.1 imply

$$\log\{(1 + \alpha_j^{-2}\beta_j^2)\}_j \prec_w \{\log(1 + s_j(A^{-1/2}B^{1/2})^4)\}_j.$$

In particular, this gives

$$\sum_{j=1}^{n} \log(1 + \alpha_j^{-2}\beta_j^2) \leqslant \sum_{j=1}^{n} \log(1 + s_j(A^{-1/2}B^{1/2})^4)$$

$$= \sum_{j=1}^{n} \log(1 + s_j(A^{-1/2}BA^{-1/2})^2).$$

Hence, we have the inequality (51).

Remark If only A is positive semidefinite and B is an arbitrary Hermitian matrix then an analogue of the bound (50) need not hold. To see this choose

$$A = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where ε is any positive number other than 1.

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