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PROBABILITIES OF MODERATE DEVIATIONS FOR SOME STATIONARY ϕ-MIXING PROCESSES

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Probabilities of moderate deviations of the sample mean from the population mean are calculated for certain strictly stationary ϕ-mixing processes.

1. Introduction. Consider a strictly stationary sequence $\{X_n, n \geqq 1\}$ of random variables (rv's) defined on a probability space $(\Omega, \mathscr{A}, P)$. Define M_1^a as the σ-field generated by $X_1, \cdots, X_a$ and M_{a+n}^{∞} as the σ-field generated by X_{a+n}, $X_{a+n+1}, \cdots$. It is assumed that

(1.1) for the events $A \in M_1^k$ with $P(A) > 0$ and $B \in M_{k+n}^{\infty}$, $|P(B|A) - P(B)| \leqq \phi(n)$, where $\phi(i)$'s are nonnegative numbers satisfying

$$1 \geqq \phi(1) \geqq \phi(2) \geqq \cdots, \lim_{n\to\infty} \phi(n) = 0 .$$

The condition (1.1) is usually referred to in the literature as the ϕ-mixing condition.

Central limit theorems (CLT's) for sequences of rv's satisfying (1.1) are proved by Ibragimov (1962) under certain conditions on the moments of the X_i's and on the $\phi(i)$'s. These results ensure under certain conditions convergence in law of the sample sum $S_n = \sum_1^n X_i$ (when suitably normalized) to the normal (0, 1) variable. Berry–Esseen type results for such sums are available in Reznik (1968) and Philipp (1971), and have been used by them in proving laws of the iterated logarithm.

In this paper, we obtain probabilities of moderate deviations (PMD) for sample means. We shall observe that our results generalize the PMD findings of Ghosh (1974) under m-dependence. Included also are the PMD results of Rubin and Sethuraman (R–S) (1965) in the i.i.d. case. R–S, however, consider also the case when the X_i's are independent but not identically distributed.

We adopt the convention of denoting by $K_1\,(> 0)$, $K_2\,(> 0)$, $\gamma\,(> 0)$, etc., generic constants. The word "strictly" will henceforth be omitted before "stationary" for brevity. Also, in what follows, $a_n \sim b_n$ will mean $a_n/b_n \to 1$ as $n \to \infty$.

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The main result of the paper is as follows.

THEOREM. *Suppose the stationary process* $\{X_n, n \geqq 1\}$ *satisfies* (1.1). *Then, for any* $c > 0$,

$$P(n^{-1}S_n - \mu > c\sigma(\log n/n)^{\frac{1}{2}}) \sim (2\pi c^2 \log n)^{-\frac{1}{2}} n^{-\frac{1}{2}c^2}, \tag{1.2}$$

and

$$P(|n^{-1}S_n - \mu| > c\sigma(\log n/n)^{\frac{1}{2}}) \sim 2(2\pi c^2 \log n)^{-\frac{1}{2}} n^{-\frac{1}{2}c^2} \tag{1.3}$$

hold under the conditions

$$E|X_1|^{c^2+2+\delta} < \infty, \qquad \textit{for some} \quad \delta > 0; \tag{1.4}$$

$$\textstyle\sum_{n=1}^{\infty} \phi^{\frac{1}{2}}(n) < \infty; \tag{1.5}$$

$$0 \neq \sigma^2 = V(X_1) + 2 \textstyle\sum_{j=1}^{\infty} \operatorname{Cov}(X_1, X_{1+j}). \tag{1.6}$$

The proof of the theorem is postponed to Section 3. Certain basic lemmas pertaining to the proof of the theorem are proved in Section 2. The results (1.2) and (1.3) are the PMD results. Noting that $x(1 - \Phi(x))/N(x) \to 1$ as $x \to \infty$, where $N(x) = (2\pi)^{-\frac{1}{2}} \exp(-\frac{1}{2}x^2)$, $\Phi(x) = \int_{-\infty}^{x} N(y)\,dy$, alternate representations of (1.2) and (1.3) are as follows:

$$P(n^{-\frac{1}{2}}(S_n - n\mu)/\sigma > c(\log n)^{\frac{1}{2}}) \sim 1 - \Phi(c(\log n)^{\frac{1}{2}}); \tag{1.7}$$

$$P(n^{-\frac{1}{2}}|S_n - n\mu|/\sigma > c(\log n)^{\frac{1}{2}}) \sim 2[1 - \Phi(c(\log n)^{\frac{1}{2}})]. \tag{1.8}$$

The calculations of Ibragimov (1962) show that $\sigma^2 < \infty$ under our conditions. Our conditions (1.4) and (1.5) are stronger than the ones required in proving the corresponding CLT's. Ibragimov (1962) has proved a CLT (see his Theorem 1.5) assuming (1.5) and the weaker condition $EX_1^2 < \infty$. For proving a Berry–Esseen type result Reznik (1968) needs the stronger assumption $E|X_1|^{2+\delta} < \infty$ for some $\delta > 0$ in place of $EX_1^2 < \infty$. It is not out of place to mention here that Reznik's (1968) paper contains a gap. He applies Esseen's lemma to two distribution functions, F_1 and F_2, without verifying that at least one of them satisfies some smoothness condition (see Feller (1968), page 512, Lemma 2). The gap has been closed by Stout (1974).

The moment condition (1.4) is assumed by R–S (1965) and Ghosh (1974). If $\phi(n) = 0$ for $n \geqq m + 1$, where m is some fixed nonnegative integer, we obtain the case considered by Ghosh (1974). Thus the present theorem includes the m-dependent as well as the i.i.d. cases earlier considered.

2. Some basic lemmas. We develop here some lemmas to be used subsequently in proving the theorem. From now on, it is assumed without any loss of generality that $\mu = 0$ and $\sigma = 1$. Define

$$\begin{aligned} X_i' = X'_{n,i} &= X_i \qquad \text{if} \quad |X_i| \leqq n^{\frac{1}{2}} \\ &= 0 \qquad \text{otherwise}, \qquad 1 \leqq i \leqq n,\ n \geqq 1. \end{aligned} \tag{2.1}$$

Then, if $S_n' = \sum_{i=1}^{n} X_i'$ $(n \geqq 1)$, one gets the following lemma.

LEMMA 1. *Under* (1.4),

$$|P(S_n' > c(n\log n)^{\frac{1}{2}}) - P(S_n > c(n\log n)^{\frac{1}{2}})| = O(^{-\frac{1}{2}c^2-\frac{1}{2}\delta}) . \tag{2.2}$$

PROOF.

$$\begin{aligned}\text{LHS of (2.2)} &\leq P(\textstyle\bigcup_{i=1}^{n} |X_i| > n^{\frac{1}{2}}) \leq nP(|X_1| > n^{\frac{1}{2}}) \\ &\leq nn^{-\frac{1}{2}(c^2+2+\delta)}\, E|X_1|^{c^2+2+\delta} = O(n^{-\frac{1}{2}c^2-\frac{1}{2}\delta}) ,\end{aligned}$$

using the definition of X_i''s, the stationarity of the X_i's, Markov's inequality and (1.4). This proves the lemma.

In view of the above lemma, to prove (1.2), it suffices to show that

$$P(S_n' > c(n\log n)^{\frac{1}{2}}) \sim (2\pi c^2 \log n)^{-\frac{1}{2}} n^{-\frac{1}{2}c^2} . \tag{2.3}$$

We prove one more lemma in this section which gives the order of $E|S_n'|^m$ for all positive integers $m \geq 2$. The present lemma generalizes Lemma 1.9 of Ibragimov (1962), but only for the truncated random variables X_i''s.

LEMMA 2. *For* $1 \leq u \leq n$, *and for any positive integer* $m \geq 2$,

$$E|S_u'|^m \leq K_1(uR(m)(\log n)^{K_2} + u^{\frac{1}{2}m}) , \tag{2.4}$$

where $R(m) = R_n(m) = n^{\frac{1}{2}(m-c^2-2-\delta)}$ *or* 0 *according as* $m >$ *or* $\leq c^2 + 2 + \delta$; K_1 *and* K_2 *are constants which may depend on* m *but not on* n *and* u.

PROOF. We prove the lemma by induction. First observe that

$$E|S_u'|^2 = uE(X_1'^2) + 2\sum_{i=1}^{u-1}\sum_{j=1}^{u-i} E(X_1'X_{1+j}') , \tag{2.5}$$

using the stationarity of the X_j''s. Using now a result of Ibragimov (1962), (see, e.g., Lemma 1, page 170 of Billingsley (1968)), one gets

$$\begin{aligned}|E(X_1'X_{1+j}')| &\leq |E(X_1'X_{1+j}') - E(X_1')E(X_{1+j}')| + |E(X_1')E(X_{1+j}')| \\ &\leq 2\phi^{\frac{1}{2}}(j)E(X_1'^2) + |EX_1'|^2 .\end{aligned} \tag{2.6}$$

Now, from the definition of the X_i''s in (2.1), the stationarity of the X_i''s, Markov's inequality, and (1.4) one gets,

$$\begin{aligned}|EX_1'| &\leq E[|X_1|I_{(|X_1|>n^{\frac{1}{2}})}] \leq n^{-\frac{1}{2}(c^2+1+\delta)}E[|X_1|^{c^2+2+\delta}I_{(|X_1|\geq n^{\frac{1}{2}})}] \\ &= o(n^{-\frac{1}{2}c^2-\frac{1}{2}-\frac{1}{2}\delta}) ,\end{aligned} \tag{2.7}$$

where I is the usual indicator function. Also, $EX_1'^2 \leq EX_1^2 = A$ (say) $< \infty$. It follows now from (2.5)—(2.7) that

$$E|S_u'|^2 \leq Au + \sum_{i=1}^{u-1}\sum_{j=1}^{u-i}[2A\phi^{\frac{1}{2}}(j) + o(n^{-c^2-1-\delta})] \leq K_1 u , \tag{2.8}$$

where one uses $u \leq n$ and (1.5). This proves the lemma for $m = 2$.

To prove the lemma by induction, assume next that

$$C(u, m) = E|S_u'|^m \leq K_1(uR(m)(\log n)^{K_2} + u^{\frac{1}{2}m}) = D(u, m) \quad \text{(say)} , \tag{2.9}$$

and prove a similar inequality for $C(u, m+1)$. Towards this end, define $S_{u,t}' = \sum_{i=1}^{u} X_{u+t+i}'$ and $S_{u,t}'' = \sum_{i=1}^{t} X_{u+i}'$. Using the definition of $C(u, m)$ and the stationarity of the X_i's, one gets

$$\begin{aligned}E(|S_u' + S_{u,t}'|)^{m+1} &\leq E(|S_u'| + |S_{u,t}'|)^{m+1} = \sum_{j=0}^{m+1}\binom{m+1}{j}E(|S_u'|^{m+1-j}|S_{u,t}'|^j) \\ &= 2C(u, m+1) + \sum_{j=1}^{m}\binom{m+1}{j}E(|S_u'|^{m+1-j}|S_{u,t}'|^j) .\end{aligned} \tag{2.10}$$

Using the stationarity of the X_i's, and Lemma 1, page 170 of Billingsley (1968) with $r = (m+1)/(m+1-j)$, and $s = (m+1)/j$, one gets

$$
\begin{aligned}
&|E(|S_u'|^{m+1-j}|S'_{u,t}|^j) - E|S_u'|^{m+1-j}E|S'_{u,t}|^j| \\
(2.11)\qquad &\leqq 2\phi^{(m+1-j)/(m+1)}(t)(E|S_u'|^{m+1})^{(m+1-j)/(m+1)}(E|S_u'|^{m+1})^{j/(m+1)} \\
&\leqq 2\phi^{1/(m+1)}(t)C(u, m+1)\,, \qquad 1 \leqq j \leqq m\,.
\end{aligned}
$$

It follows now from (2.10) and (2.11) that

$$
\begin{aligned}
E|S_u' + S'_{u,t}|^{m+1} &\leqq (2 + 2\phi^{1/(m+1)}(t)2^{m+1})C(u, m+1) \\
&\quad + \textstyle\sum_{j=1}^{m} \binom{m+1}{j} E|S_u'|^{m+1-j}E|S_u'|^j \\
(2.12)\qquad &\leqq (2 + 2^{m+2}\phi^{1/(m+1)}(t))C(u, m+1) \\
&\quad + \textstyle\sum_{j=1}^{m} \binom{m+1}{j} (E|S_u'|^m)^{(m+1-j)/m}(E|S_u'|^m)^{j/m} \\
&\leqq (2 + 2^{m+2}\phi^{1/(m+1)}(t))C(u, m+1) \\
&\quad + 2^{m+1}C^{(m+1)/m}(u, m)\,.
\end{aligned}
$$

Again from (2.9), $C^{(m+1)/m}(u,m) \leqq K_1(u^{(m+1)/m}R^{(m+1)/m}(m)(\log n)^{K_2} + u^{\frac{1}{2}(m+1)})$. Now, for $m+1 \leqq c^2 + 2 + \delta$, $R(m) = R(m+1) = 0$. For $m+1 > c^2 + 2 + \delta$, $(uR(m))^{(m+1)/m} \leqq un^{1/m}n^{((m+1)/2m)(m-c^2-2-\delta)} \leqq un^{\frac{1}{2}(m+1-c^2-2-\delta)} = uR(m+1)$. It follows now from (2.12) that

$$
(2.13)\qquad E|S_u' + S'_{u,t}|^{m+1} \leqq [2 + 2^{m+2}\phi^{1/(m+1)}(t)]C(u, m+1) + D(u, m+1)\,.
$$

Using the Minkowski inequality, one gets

$$
\begin{aligned}
C(2u, m+1) &= E|S_u' + S'_{u,t} + S''_{u,t} - S''_{2u,t}|^{m+1} \\
(2.14)\qquad &\leqq \{E^{1/(m+1)}|S_u' + S'_{u,t}|^{m+1} \\
&\quad + (\textstyle\sum_{u+1}^{u+t} + \sum_{2u+1}^{2u+t})E^{1/(m+1)}|X_j'|^{m+1}\}^{m+1}\,.
\end{aligned}
$$

Note now that for $m+1 \leqq c^2 + 2 + \delta$,

$$
E|X_1'|^{m+1} \leqq E|X_1'|^{c^2+2+\delta} \leqq E|X_1|^{c^2+2+\delta} \leqq K_1\,,
$$

and for $m+1 > c^2 + 2 + \delta$,

$$
E|X_1'|^{m+1} \leqq n^{\frac{1}{2}(m+1-c^2-2-\delta)}E|X_1'|^{c^2+2+\delta} \leqq K_1 R(m+1)\,,
$$

using (1.4), (2.1) and the definition of $R(m)$. The above leads to

$$
(2.15)\qquad E|X_1'|^{m+1} \leqq K_1(1 + R(m+1))\,.
$$

Combining (2.13) and (2.15), one gets from (2.14),

$$
\begin{aligned}
(2.16)\qquad C(2u, m+1) &\leqq [\{(2 + 2^{m+2}\phi^{1/(m+1)}(t))C(u, m+1) + D(u, m+1)\}^{1/(m+1)} \\
&\quad + K_1 t(1 + R(m+1))^{1/(m+1)}]^{m+1}\,.
\end{aligned}
$$

Consider now the two separate cases (i) $m+1 \leqq c^2 + 2 + \delta$ and (ii) $m+1 > c^2 + 2 + \delta$.

CASE (i). Here $R(m+1) = 0$. Given any ε_1 (> 0), choose t to be sufficiently large (not dependent on n) that $2^{m+2}\phi^{1/(m+1)}(t) < \varepsilon_1$ (by 1.5)). Then, one gets from (2.9) and (2.16),

$$\begin{aligned} C(2u, m+1) &\leqq [\{(2+\varepsilon_1)C(u, m+1) + K_1 u^{\frac{1}{2}(m+1)}\}^{1/(m+1)} + K_1]^{m+1} \\ &\leqq [(1+\varepsilon_1)\{(2+\varepsilon_1)C(u, m+1) + K_1 u^{\frac{1}{2}(m+1)}\}^{1/(m+1)}]^{m+1} \end{aligned}$$

for u sufficiently large, say $u \geqq u_0 = u_0(m, \varepsilon_1)$. Also, given any $\varepsilon > 0$, choose $\varepsilon_1\ (>0)$ such that $(1+\varepsilon_1)^{m+1}(2+\varepsilon_1) \leqq 2+\varepsilon$. This leads to

$$C(2u, m+1) \leqq (2+\varepsilon)C(u, m+1) + K_1 u^{\frac{1}{2}(m+1)}, \tag{2.17}$$

for $u \geqq u_0$. This is also valid for $u \leqq u_0$ by using the inequality $E|S_u'|^m \leqq u^m E|X_1'|^m \leqq u_0^m E|X_1|^{\epsilon^2+2+\delta}$ and by suitable readjustment of K_1 if necessary. Repeating (2.17) r times and using $C(1, m+1) \leqq E|X_1|^{\epsilon^2+2+\delta} \leqq K_1$ one gets

$$\begin{aligned} C(2^r, m+1) &\leqq (2+\varepsilon)^r C(1, m+1) \\ &\qquad + K_1(2^r/2)^{\frac{1}{2}(m+1)}(1 + (2+\varepsilon)2^{-\frac{1}{2}(m+1)} + \cdots \\ &\qquad + \{(2+\varepsilon)2^{-\frac{1}{2}(m+1)}\}^{r-1}) \\ &\leqq K_1[(2+\varepsilon)^r + (2^r)^{\frac{1}{2}(m+1)}] \leqq K_1(2^r)^{\frac{1}{2}(m+1)} = D(2^r, m+1), \end{aligned} \tag{2.18}$$

choosing $\varepsilon\ (>0)$ so small that $2+\varepsilon < 2^{\frac{3}{2}} \leqq 2^{\frac{1}{2}(m+1)}$ for $m \geqq 2$.

CASE (ii). Here $R(m+1) \neq 0$. Take $t = [(\log n)^{(m+1)/2}]$, the integer part of $(\log n)^{(m+1)/2}$. Then (2.16) and (1.5) lead to

$$\begin{aligned} &C(2u, m+1) \\ &\quad \leqq [\{(2 + K_1(\log n)^{-1})C(u, m+1) + D(u, m+1)\}^{1/(m+1)} \\ &\qquad + K_1(\log n)^{K_2} R^{1/(m+1)}(m+1)]^{m+1} \\ &\quad \leqq [(1 + K_1(\log n)^{-1})\{(2 + K_1(\log n)^{-1})C(u, m+1) \\ &\qquad + D(u, m+1)\}^{1/(m+1)}]^{m+1} \\ &\quad = (1 + K_1(\log n)^{-1})^{m+1}\{(2 + K_1(\log n)^{-1})C(u, m+1) + D(u, m+1)\} \\ &\quad \leqq 2(1 + K_1(\log n)^{-1})C(u, m+1) + K_1(uR(m+1)(\log n)^{K_2} + u^{\frac{1}{2}(m+1)}). \end{aligned} \tag{2.19}$$

When $2^r \leqq n$, i.e., $r \leqq (\log 2)^{-1}(\log n)$, repeating the inequality (2.19) r times and recalling $C(1, m+1) = E|X_1'|^{m+1} \leqq K_1 R(m+1)$, one gets

$$\begin{aligned} C(2^r, m+1) &\leqq 2^r(1 + K_1(\log n)^{-1})^r C(1, m+1) \\ &\qquad + K_1 2^r R(m+1)(\log n)^{K_2} \textstyle\sum_{j=0}^{r-1} \{(2 + K_1(\log n)^{-1})2^{-1}\}^j \\ &\qquad + K_1(2^r)^{\frac{1}{2}(m+1)} \textstyle\sum_{j=0}^{r-1} \{(2 + K_1(\log n)^{-1})2^{-(m+1)/2}\}^j \\ &\leqq K_1 2^r R(m+1) + K_1 2^r R(m+1)(\log n)^{K_2} + K_1(2^r)^{\frac{1}{2}(m+1)} \\ &\leqq D(2^r, m+1). \end{aligned} \tag{2.20}$$

Combining (2.18) and (2.20) one gets for any $u = 2^r\ (\leqq n)$,

$$C(2^r, m) \leqq K_1[2^r R(m)(\log n)^{K_2} + (2^r)^{\frac{1}{2}m}]. \tag{2.21}$$

Now suppose $2^r < u < 2^{r+1}$. Use the binary decomposition $u = \sum_{j=0}^{r} \nu_j 2^j$ with $\nu_r = 1$, $\nu_j = 0$ or 1 for $0 \leqq j \leqq r-1$. Then,

$$E|\textstyle\sum_1^u X_i'|^m = E|(\sum_1^{i_1} + \sum_{i_1+1}^{i_1+i_2} + \cdots + \sum_{i_1+\cdots+i_r+1}^{i_1+\cdots+i_{r+1}})X_j'|^m, \tag{2.22}$$

where $i_j = \nu_{r+1-j}2^{r+1-j}$ $(1 \leqq j \leqq r+1)$, $\sum_{m_1}^{m_2} X_j'$ is interpreted as zero if $m_2 < m_1$. Using now the Minkowski inequality, the stationarity of the X_i's and (2.21), one gets

$$
\begin{aligned}
E|\textstyle\sum_1^u X_i'|^m &\leqq \{E^{1/m}|\textstyle\sum_1^{i_1} X_j'|^m + \cdots + E^{1/m}|\sum_{i_1+\cdots+i_r+1}^{i_1+\cdots+i_r+i_{r+1}} X_j'|^m\}^m \\
&\leqq K_1[\textstyle\sum_{j=1}^{r+1}\{\nu_{r+1-j}(2^{r+1-j}R(m)(\log n)^{K_2} + 2^{\frac12 m(r+1-j)})^{1/m}\}]^m \\
&\leqq K_1[\textstyle\sum_{j=1}^{r+1}\nu_{r+1-j}\{2^{(r+1-j)/m}R^{1/m}(m)(\log n)^{K_2} + 2^{\frac12(r+1-j)}\}]^m \\
&\leqq K_1[(\textstyle\sum_{j=1}^{r+1}2^{(r+1-j)/m})R^{1/m}(m)(\log n)^{K_2} + \sum_{j=1}^{r+1}2^{\frac12(r+1-j)}]^m \qquad (2.23)\\
&\leqq K_1[2^{(r+1)/m}R^{1/m}(m)(\log n)^{K_2} + 2^{\frac12(r+1)}]^m \\
&\leqq K_1[2^{r+1}R(m)(\log n)^{K_2} + 2^{\frac12 m(r+1)}] \\
&\leqq K_1[2^rR(m)(\log n)^{K_2} + (2^r)^{\frac12 m}] \\
&\leqq K_1[uR(m)(\log n)^{K_2} + u^{\frac12 m}]\,.
\end{aligned}
$$

3. Proof of the theorem. We need only prove (1.2) as (1.3) is then immediate. Recall we have already assumed $\mu = 0$ and $\sigma = 1$. Let $p = p_n = [n^\alpha]$, $q = q_n = [n^\beta]$, $k = k_n = [n/(p+q)]$; α and β $(0 < \beta < \alpha < 1)$ will be chosen later; $[x]$ denotes the largest integer $\leqq x$. Then, $n = k(p+q) + r$, $0 \leqq r < p+q$. The sample sum S_n' is then partitioned into

$$S_n' = U_n + R_n + T_n\,, \tag{3.1}$$

where

$$U_n = \textstyle\sum_1^k \xi_i;\ \xi_i = \xi_{n,i} = \sum_{(i-1)(p+q)+1}^{ip+(i-1)q} X_j' \qquad 1 \leqq i \leqq k\,; \tag{3.2}$$

$$R_n = \textstyle\sum_1^k \eta_i;\ \eta_i = \eta_{n,i} = \sum_{ip+(i-1)q+1}^{i(p+q)} X_j' \qquad 1 \leqq i \leqq k\,; \tag{3.3}$$

$$T_n = T_{n,r} = \textstyle\sum_{k(p+q)+1}^{k(p+q)+r} X_j' \quad \text{if} \quad r \geqq 1\,; \qquad T_n = 0 \quad \text{if} \quad r = 0\,. \tag{3.4}$$

We now prove three lemmas which imply (2.3), and thus prove the theorem. Define $\zeta_n = (\log n)^{-1-\nu}$, $\nu > 0$ for $n \geqq 2$.

LEMMA 3. *For any $c > 0$, under* (1.4)—(1.6),

$$P\{U_n > (c \pm 2\zeta_n)(n\log n)^{\frac12}\} \sim (2\pi c^2 \log n)^{-\frac12} n^{-\frac12 c^2} \tag{3.5}$$

LEMMA 4. *For any $c > 0$, under* (1.4)—(1.6),

$$P\{R_n > \zeta_n(n\log n)^{\frac12}\} = o(n^{-\frac12 c^2}(\log n)^{-\frac12}) = P\{R_n < -\zeta_n(n\log n)^{\frac12}\}\,. \tag{3.6}$$

LEMMA 5. *For any $c > 0$, under* (1.4)—(1.6),

$$P\{T_n > \zeta_n(n\log n)^{\frac12}\} = o(n^{-\frac12 c^2}(\log n)^{-\frac12}) = P\{T_n < -\zeta_n(n\log n)^{\frac12}\}\,. \tag{3.7}$$

Further, for $\zeta_n < \frac12 c$, n large, one has

$$
\begin{aligned}
&P\{U_n > (c + 2\zeta_n)(n\log n)^{\frac12}\} - P\{R_n < -\zeta_n(n\log n)^{\frac12}\} \\
&\qquad - P\{T_n < -\zeta(n\log n)^{\frac12}\} \\
&\qquad\quad \leqq P\{S_n' > c(n\log n)^{\frac12}\} \qquad (3.8)\\
&\qquad\quad \leqq P\{U_n > (c - 2\zeta_n)(n\log n)^{\frac12}\} + P\{R_n > \zeta_n(n\log n)^{\frac12}\} \\
&\qquad\qquad + P\{T_n > \zeta_n(n\log n)^{\frac12}\}\,.
\end{aligned}
$$

Dividing both sides of (3.8) by $(2\pi^2 \log n)^{-\frac{1}{2}} n^{-\frac{1}{2}c^2}$, and then letting $n \to \infty$, one gets (2.3).

PROOF OF LEMMA 3. For later reference, note that

$$\log [n^{-\frac{1}{2}(c \pm 2\zeta_n)^2} n^{\frac{1}{2}c^2}] = \tfrac{1}{2}[c^2 - (c^2 \pm 2\zeta_n)^2] \log n = O_e(\log n)^{-\nu}, \qquad \nu > 0, \tag{3.9}$$

where O_e denotes exact order. By $M_n = O_e(a_n)$, we mean that there exist constants $K_1\,(>0)$ and $K_2\,(>0)$, and a positive integer n_0 such that $K_1 \leqq |M_n|/a_n \leqq K_2$ for $n \geqq n_0$. We have used (3.9) in deriving (3.30).

Defining $\xi_i' = p^{-\frac{1}{2}}\xi_i (1 \leqq i \leqq k)$ and $c_n = (c \pm 2\zeta_n)(n/(kp))^{\frac{1}{2}}(\log n/\log k)^{\frac{1}{2}}$,

$$P(U_n > (c \pm 2\zeta_n)(n \log n)^{\frac{1}{2}}) = P(\textstyle\sum_1^k \xi_i' > c_n(k \log k)^{\frac{1}{2}}). \tag{3.10}$$

Define now

$$\begin{aligned} \xi_{i0} &= \xi_i' \qquad \text{if} \quad |\xi_i'| \leqq (k/\log k)^{\frac{1}{2}}, \\ &= 0 \qquad \text{otherwise}; \end{aligned} \qquad 1 \leqq i \leqq k. \tag{3.11}$$

Using Lemma 2, one has

$$\begin{aligned} &|P(\textstyle\sum_1^k \xi_i' > c_n(k/\log k)^{\frac{1}{2}}) - P(\sum_1^k \xi_{i0} > c_n(k/\log k)^{\frac{1}{2}})| \\ &\quad \leqq kP(|\xi_1'| > (k/\log k)^{\frac{1}{2}}) \leqq k(k/\log k)^{-\frac{1}{2}m} E|\xi_1'|^m \\ &\quad \leqq k(k/\log k)^{-\frac{1}{2}m} p^{-\frac{1}{2}m} K_1(pn^{(m-c^2-2-\delta)/2}(\log n)^{K_2} + p^{m/2}) \\ &\quad \leqq K_1[n(kp/n)^{-m/2} n^{-1-\frac{1}{2}c^2-\frac{1}{2}\delta}(\log n)^{K_2} + k^{1-\frac{1}{2}m}(\log k)^{m/2}] \\ &\quad \leqq K_1\left[n^{-\frac{1}{2}c^2-\frac{1}{2}\delta}\left(1 + \frac{kq+r}{kp}\right)^{m/2}(\log n)^{K_2} + n^{(1-\alpha)(1-\frac{1}{2}m)}(\log n)^{K_2}\right] \\ &\quad \leqq K_1 n^{-\frac{1}{2}c^2}(\log n)^{-\frac{1}{2}-\eta}, \end{aligned} \tag{3.12}$$

for some $\eta > 0$, and for large n, if $(1-\alpha)(m-2) > c^2$, i.e.,

$$\alpha < 1 - c^2/(m-2). \tag{3.13}$$

Thus, to prove the lemma, it suffices to show that

$$P(\textstyle\sum_1^k \xi_{i0} > c_n(k \log k)^{\frac{1}{2}}) \sim (2\pi c^2 \log n)^{-\frac{1}{2}} n^{-\frac{1}{2}c^2} \tag{3.14}$$

To prove (3.14), one proceeds in analogy with Rubin and Sethuraman (1965). Let $F_k(x) = P(\sum_1^k \xi_{i0} \leqq x)$, $g_k(\theta) = \int_{-\infty}^{\infty} \exp(\theta x)\, dF_k(x)$, and $dG_{k,\theta}(x) = \exp(\theta x)\, dF_k(x)/g_k(\theta)$, $\theta\,(>0)$ being some real positive number. Then it is possible to express the LHS of (3.14) as

$$g_k(c_n(\log k/k)^{\frac{1}{2}}) \int_{c_n(k\log k)^{\frac{1}{2}}}^{\infty} \exp(-c_n(\log k/k)^{\frac{1}{2}}x)\, dG_{k,c_n(\log k/k)^{\frac{1}{2}}}(x).$$

Define now

$$f_n = E[\exp(c_n(\log k/k)^{\frac{1}{2}}\xi_{10})]; \tag{3.15}$$

$$m_n = f_n^{-1}E[\xi_{10}\exp(c_n(\log k/k)^{\frac{1}{2}}\xi_{10})]; \tag{3.16}$$

$$m_n^2 + \sigma_n^2 = f_n^{-1}E[\xi_{10}^2\exp(c_n(\log k/k)^{\frac{1}{2}}\xi_{10})]. \tag{3.17}$$

Then after some simple manipulations, it is possible to write

$$P(\textstyle\sum_1^k \xi_{i0} > c_n(k \log k)^{\frac{1}{2}}) = A_k \int_{B_k}^{\infty} \exp(-C_k z)\, d\Pi_{k,c_n}(z), \tag{3.18}$$

where $A_k = A_k(c_n) = E[\exp\{c_n(\log k/k)^{1/2} \sum_1^k \xi_{i0}\}]\exp(-c_n(k\log k)^{1/2}m_n)$, $B_k = B_k(c_n) = k^{-1/2}\sigma_n^{-1}(c_n(k\log k)^{1/2} - km_n)$, $C_k = c_n\sigma_n(\log k)^{1/2}$, $\Pi_{k,c_n}(z) = G_{k,c_n(\log k/k)^{1/2}}(km_n + k^{1/2}\sigma_n z)$. Next we obtain some useful estimates of A_k, B_k and C_k. Towards this end, first note that using Lemma 2

$$E|\xi_{i0}|^3 \leqq E|\xi_1'|^3 \leqq K_1 p^{-3/2}(p^{3/2} + pn^{(3-2-c^2-\delta)/2}(\log\log n)^{K_2}) \leqq K_1(1 + (n/p)^{1/2}n^{-\frac{1}{2}c^2-\frac{1}{2}\delta}) \leqq K_1 k^{1/2-\gamma}, \tag{3.19}$$

$1 \leqq i \leqq k$, for some $\gamma > 0$. Also,

$$E\xi_{10}^2 \leqq E\xi_1'^2 \leqq K_1 . \tag{3.20}$$

Further, using (3.11) and (3.19),

$$|E(\xi_{10})| \leqq (k/\log k)^{-1}E|\xi_1'|^3 \leqq K_1 k^{-1/2-\gamma}, \tag{3.21}$$

for some $\gamma > 0$. Next we show that

$$|E\xi_{10}^2 - 1| \leqq K_1 k^{-\gamma} \qquad \text{for some} \quad \gamma > 0 . \tag{3.22}$$

This is accomplished in several steps. First note that using (3.19)

$$|E\xi_1'^2 - E\xi_{10}^2| = E[\xi_1'^2 I_{(|\xi_1'| > (k/\log k)^{1/2})}] \leqq (k/\log k)^{-1/2}E|\xi_1'|^3 \leqq K_1 k^{-\gamma} \qquad \text{for some} \quad \gamma > 0 . \tag{3.23}$$

Also, using the stationarity of the X_i's, Lemma 1, page 170 of Billingsley (1968), (1.4), and $\sum_{j=1}^{p-1}\phi^{1/2}(j) \leqq K_1$ (which follows from (1.5)), one gets

$$\begin{aligned}
&|p^{-1}E(\textstyle\sum_1^p X_i)^2 - E\xi_1'^2| \\
&\quad = p^{-1}|pE(X_1^2 - X_1'^2) + 2\textstyle\sum_{j=1}^{p-1}(p-j)E(X_1X_{1+j} - X_1'X_{1+j}')| \\
&\quad \leqq E[X_1^2 I_{(|X_1| > n^{1/2})}] + 4\textstyle\sum_{j=1}^{p-1}|E(X_1 I_{(|X_1|>n^{1/2})}X_{1+j}I_{(|X_{1+j}|\leq n^{1/2})})| \\
&\qquad + 2\textstyle\sum_{j=1}^{p-1}|E(X_1 I_{(|X_1|>n^{1/2})}X_{1+j}I_{(|X_{1+j}|>n^{1/2})})| \\
&\quad \leqq n^{-\frac{1}{2}(c^2+\delta)}E|X_1|^{c^2+2+\delta} + 8\textstyle\sum_{j=1}^{p-1}\phi^{1/2}(j)E^{1/2}(X_1^2 I_{(|X_1|>n^{1/2})})E^{1/2}(X_1^2 I_{(|X_1|\leq n^{1/2})}) \\
&\qquad + 4\textstyle\sum_{j=1}^{p-1}\phi^{1/2}(j)E(X_1^2 I_{(|X_1|>n^{1/2})}) + 6p\{E[|X_1|I_{(|X_1|>n^{1/2})}]\}^2 \\
&\quad \leqq K_1 n^{-\frac{1}{2}(c^2+\delta)} + K_1 n^{-\frac{1}{2}(c^2+\delta)} + K_1 n^{-\frac{1}{2}(c^2+\delta)} + K_1 pn^{-(c^2+1+\delta)} \leqq K_1 n^{-\gamma}
\end{aligned} \tag{3.24}$$

for some $\gamma > 0$.

In view of (3.23) and (3.24), $EX_1 = 0$ and $\sigma^2 = 1$, (3.22) will now follow from the following lemma.

LEMMA 6. *Under* (1.4)—(1.6), $|p^{-1}V(\sum_1^p X_i) - \sigma^2| \leqq K_1 n^{-\gamma}$ *for some* $\gamma > 0$.

PROOF.

$$\begin{aligned}
|\sigma^2 - p^{-1}V(\textstyle\sum_1^p X_i)| &\leqq 2\{\textstyle\sum_{j=1}^{p-1} jp^{-1}|\mathrm{Cov}(X_1, X_{1+j})| + \sum_{j=p}^{\infty}|\mathrm{Cov}(X_1, X_{1+j})|\} \\
&\leqq 2\textstyle\sum_{j=1}^{p-1} jp^{-1}\phi^{(c^2+1+\delta)/(c^2+2+\delta)}(j)(E|X_1|^{(c^2+2+\delta)/(c^2+1+\delta)})^{(c^2+1+\delta)/(c^2+2+\delta)} \\
&\qquad \times (E|X_1|^{c^2+2+\delta})^{1/(c^2+2+\delta)} + 2\textstyle\sum_{j=p}^{\infty}\phi^{(c^2+1+\delta)/(c^2+2+\delta)}(j) \\
&\qquad \times (E|X_1|^{(c^2+2+\delta)/(c^2+1+\delta)})^{(c^2+1+\delta)/(c^2+2+\delta)}(E|X_1|^{c^2+2+\delta})^{1/(c^2+2+\delta)} \\
&\leqq K_1[p^{-1}\textstyle\sum_{j=1}^{p-1} j^{-(c^2+\delta)/(c^2+2+\delta)} + \sum_{j=p}^{\infty} j^{-1-(c^2+\delta)/c^2+2+\delta}] \\
&\leqq K_1 p^{-\gamma} \leqq K_1 n^{-\gamma} ,
\end{aligned}$$

for some $\gamma > 0$, since (1.5) and the monotonicity of the $\phi(j)$'s give $\phi(j) \leq K_1 j^{-2}$.

With the usual expansion of $\exp(x)$ around $x = 0$, it follows from (3.20)—(3.22) that

(3.25) $$|f_n - 1 - \tfrac{1}{2}c_n^2(\log k/k)| \leq K_1 k^{-1-\gamma} \quad \text{for some} \quad \gamma > 0 .$$

Further, from (3.16)—(3.17) and (3.20)—(3.22),

(3.26) $$\begin{aligned} m_n &= c_n(\log k/k)^{1/2}E(\xi_{10}^2) + O(k^{-1/2-\gamma}) \\ &= c_n(\log k/k)^{1/2} + O(k^{-1/2-\gamma}) \quad \text{for some} \quad \gamma > 0 ; \end{aligned}$$

(3.27) $$m_n^2 + \sigma_n^2 = 1 + O(k^{-\gamma}) , \quad \text{for some} \quad \gamma > 0 .$$

Hence, $\sigma_n^2 = 1 + O(k^{-\gamma})$ for some $\gamma > 0$, and one has

(3.28) $$B_k = k^{-1/2}(1 + O(k^{-\gamma}))O(k^{1/2-\gamma}) = O(k^{-\gamma}) .$$

Also $n/(kp) = 1 + (kq + r)/(kp) = 1 + O(n^{-\gamma})$, for some $\gamma > 0$. Hence,

(3.29) $$C_k = c_n\sigma_n(\log k)^{1/2} = c(\log n)^{1/2}(1 + O(\log n)^{-\gamma}) .$$

We prove next

$$A_k = n^{-\frac{1}{2}c^2}(1 + O(\log n)^{-\gamma}) \quad \text{for some} \quad \gamma > 0 .$$

Since $-c_n(k \log k)^{1/2}m_n = -c_n^2 \log k + O(k^{-\gamma} \log k) = -c^2 \log n + O((\log n)^{-\gamma})$ for some $\gamma > 0$, it suffices to show that

(3.30) $$E[\exp(c_n(\log k/k)^{1/2} \textstyle\sum_1^k \xi_{i0})] = n^{\frac{1}{2}c^2}[1 + O(\log n)^{-\gamma}] ,$$

for some $\gamma > 0$. Note that in view of (3.25), $k \log f_n = \frac{1}{2}c_n^2 \log k + O(k^{-\gamma}) = \frac{1}{2}c^2 \log n + O((\log n)^{-\gamma})$. Hence,

(3.31) $$f_n^k = n^{\frac{1}{2}c^2}(1 + O(\log n)^{-\gamma}) , \quad \text{for some} \quad \gamma > 0 .$$

In view of (3.31), in proving (3.30), it suffices to show that

(3.32) $$|E(\exp(c_n(\log k/k)^{1/2} \textstyle\sum_1^k \xi_{i0})) - f_n^k| = O(n^{\frac{1}{2}c^2-\gamma}) , \quad \text{for some} \quad \gamma > 0 .$$

To prove (3.32), we make repeated use of the inequality (20.28) in Billingsley (1968), page 171. The inequality is due to Ibragimov (1962).

Using stationarity and $|\xi_{10}| \leq (k/\log k)^{1/2}$, one gets

(3.33) $$\begin{aligned} &|E(\exp(c_n(\log k/k)^{1/2} \textstyle\sum_1^k \xi_{i0})) - f_n^k| \\ &\quad \leq |E(\exp(c_n(\log k/k)^{1/2} \textstyle\sum_1^k \xi_{i0})) - E(\exp(c_n(\log k/k)^{1/2} \textstyle\sum_1^{k-1} \xi_{i0}))f_n| \\ &\qquad + f_n|E(\exp(c_n(\log k/k)^{1/2} \textstyle\sum_1^{k-1}\xi_{i0})) \\ &\qquad - E(\exp(c_n(\log k/k)^{1/2} \textstyle\sum_1^{k-2} \xi_{i0}))f_n| + \cdots \\ &\qquad + f_n^{k-2}|E(\exp(c_n(\log k/k)^{1/2} \textstyle\sum_1^2 \xi_{i0})) - f_n^2| \\ &\quad \leq 2\,\phi(q)\exp(2c_n) \\ &\qquad \{\textstyle\sum_{j=1}^{k-2} f_n^{k-1-j}E[\exp(c_n(\log k/k)^{1/2} \textstyle\sum_{i=1}^j \xi_{i0})] + f_n^{k-2}\} . \end{aligned}$$

Again, repeated application of (20.28) of Billingsley gives

(3.34) $$\begin{aligned} &E[\exp(c_n(\log k/k)^{1/2} \textstyle\sum_{i=1}^j \xi_{i0})] \\ &\quad \leq [f_n + 2\phi(q)\exp(c_n)]E[\exp(c_n(\log k/k)^{1/2} \textstyle\sum_1^{j-1} \xi_{i0})] \leq \cdots \\ &\quad \leq [f_n + 2\phi(q)\exp(c_n)]^j , \qquad j = 2, \cdots, k-1 . \end{aligned}$$

It follows now from (3.33) that

$$
\begin{aligned}
(3.35)\qquad & |E(\exp(c_n(\log k/k)^{\frac12} \textstyle\sum_1^k \xi_{i0})) - f_n{}^k| \\
& \quad \leqq 2k\phi(q)\exp(2c_n)[f_n + 2\phi(q)\exp(c_n)]^{k-1} \\
& \quad \leqq K_1 q^{-2}k(1 + \tfrac12 c_n{}^2(\log k/k) + K_1(k^{-1-\gamma} + q^{-2}))^{k-1} ,
\end{aligned}
$$

using (3.25) and (1.5). Note that for large n the RHS of (3.35) is majorized by $K_1 n^{-2\beta+1-\alpha}\exp[k\log(1 + \frac12 c_n{}^2(\log k/k) + K_1 k^{-1-\gamma})]$ for some $\gamma > 0$ provided $2\beta > 1 - \alpha$. The above is again majorized by $K_1 n^{-2\beta+1-\alpha}\exp(\frac12 c_n{}^2\log k + K_1 k^{-\gamma}) = O(n^{1-\alpha-2\beta+\frac12 c^2})$. Choose now $\alpha = 1 - (c^2 + \delta)/(m-2)$, $\beta = 1 - (c^2 + 2\delta)/(m-2)$, where $m > 3c^2 + 5\delta + 2$. Then, (3.13) is satisfied, $0 < \beta < \alpha < 1$ and $2\beta > 1 - \alpha$. Also, $1 - \alpha - 2\beta = -2 + (3c^2 + 5\delta)/(m-2) < -1$. This proves (3.32). Finally we show

$$
(3.36)\qquad \sup_z |\Pi_{k,c_n}(z) - \Phi(z)| = O((\log n)^{-\frac12}) .
$$

Assume for the moment that this is true. To see how the proof is concluded first note that in view of (3.18) and $A_k = n^{-\frac12 c^2}(1 + O(\log n)^{-\gamma})$, for some $\gamma > 0$, it sufficies to show that

$$
(3.37)\qquad \textstyle\int_{B_k}^\infty \exp(-C_k z)\, d\Pi_{k,c_n}(z) \sim (2\pi c^2 \log n)^{-\frac12} .
$$

From (3.28), (2.29) and (3.36) one has

$$
|\textstyle\int_{B_k}^\infty \exp(-C_k z)\, d\Pi_{k,c_n}(z) - \int_{B_k}^\infty \exp(-C_k z)\, d\Phi(z)| = O((\log n)^{-\frac12}) .
$$

Hence, one need only prove that

$$
(3.38)\qquad \textstyle\int_{B_k}^\infty \exp(-C_k z)\, d\Phi(z) \sim (2\pi c^2 \log n)^{-\frac12} .
$$

It is easy to see that $|\int_{B_k}^\infty \exp(-C_k z)\, d\Phi(z) - \int_0^\infty \exp(-C_k z)\, d\Phi(z)| \leqq K_1|B_k| = O(n^{-\gamma})$ for some $\gamma > 0$ (using (3.28)). Also,

$$
\begin{aligned}
(3.39)\qquad \textstyle\int_0^\infty \exp(-C_k z)\, d\Phi(z) &= \textstyle\int_0^\infty (2\pi)^{-\frac12}\exp(-\frac12(z + C_k)^2 + \frac12 C_k{}^2)\, dz \\
&= (2\pi)^{-\frac12}[1 - \Phi(C_k)]/N(C_k) \sim (2\pi)^{-\frac12} C_k{}^{-1} \sim (2\pi c^2 \log n)^{-\frac12} .
\end{aligned}
$$

It remains to prove (3.36). If $H(x) = P(\xi_{10} \leqq x)$ and $dH_n(x) = \exp(c_n(\log k/k)^{\frac12}x)\, dH(x)/\int_{-\infty}^\infty \exp(c_n(\log k/k)^{\frac12}x)\, dH(x)$, it follows that m_n and $\sigma_n{}^2$ are the mean and the variance of ξ_{10} wrt the conjugate distribution H_n. If ξ_{i0}^* $(i = 1, \cdots, k)$ are independent rv's each having the *same* conjugate distribution H_n as ξ_{10}, Lemma 2, page 171 of Billingsley (1968) gives

$$
\begin{aligned}
(3.40)\qquad & |E_{H_n}(\textstyle\prod_1^k \exp\{it(\xi_{i0} - m_n)k^{-\frac12}\sigma_n{}^{-1}\}) - \prod_1^k E_{H_n}[\exp\{it(\xi_{i0}^* - m_n)k^{-\frac12}\sigma_n{}^{-1}\}]| \\
& \quad \leqq 2k\phi(q) \leqq K_1 k q^{-2} = O(n^{1-\alpha-2\beta}) = O(n^{-\gamma}) \qquad \text{for some } \gamma > 0 ;
\end{aligned}
$$

in the above E_{H_n} denotes expectation wrt H_n. Let now $F_1(x)$ and $F_2(x)$ denote the respective df's of $\sum_1^k (\xi_{i0} - m_n)k^{-\frac12}\sigma_n{}^{-1}$ and $\sum_1^k (\xi_{i0}^* - m_n)k^{-\frac12}\sigma_n{}^{-1}$ when the ξ_{i0}'s and ξ_{i0}^*'s have df $H_n(x)$. Let f_0, f_1, and f_2 denote the characteristic functions corresponding to Φ, F_1 and F_2. Define $\beta_3 = E_{H_n}|\xi_{10}^*|^3$. It follows from (3.21) and

(3.25) that $\beta_3 = O(k^{\frac{1}{2}-\gamma})$ for some $\gamma > 0$. Define $T_n = \sigma_n^3\beta_3^{-1}k^{\frac{1}{2}}$ so that $T_n^{-1} = O(k^{-\gamma})$. Proceeding in analogy with Theorem 5.1 of Feller ((1966), pages 515–516), one gets for $|t| \leqq T_n$,

$$|f_2(t) - f_0(t)| \leqq K\beta_3\sigma_n^{-3}k^{-\frac{1}{2}}|t|^3 \exp(-\tfrac{1}{12}t^2) = O(n^{-\gamma})|t|^3 \exp(-\tfrac{1}{12}t^2)\,. \tag{3.41}$$

Using now Esseen's lemma (see, e.g., Feller (1966), page 512), one gets for all real z,

$$\begin{aligned}|F_1(z) - \Phi(z)| &\leqq K[\textstyle\int_{-T_n}^{T_n}|f_1(t) - f_0(t)||t|^{-1}\,dt + T_n^{-1}] \\ &\leqq K[\textstyle\int_{-T_n}^{T_n}|f_1(t) - f_2(t)||t|^{-1}\,dt \\ &\qquad + \textstyle\int_{-T_n}^{T_n}|f_2(t) - f_0(t)||t|^{-1}\,dt + T_n^{-1}]\,.\end{aligned} \tag{3.42}$$

Again $|f_1(t) - f_2(t)| < K(\log n)^{-\frac{1}{2}}|t|$ for $|t| < (\log n)^{-\frac{1}{2}}$ (since the corresponding variables admit moments of order higher than two). Using this and (3.40), one gets for large n,

$$\begin{aligned}\textstyle\int_{-T_n}^{T_n}|f_1(t) - f_2(t)||t|^{-1}\,dt &= \textstyle\int_{|t|<(\log n)^{-\frac{1}{2}}} + \int_{(\log n)^{-\frac{1}{2}}\leq t\leq T_n}|f_1(t) - f_2(t)||t|^{-1}\,dt \\ &\leqq K[(\log n)^{-\frac{1}{2}} + n^{-\gamma}\log n]\,,\end{aligned}$$

noting that $\log T_n = O(\log n)$; (3.41), (3.42) and the above now lead to $\sup_z|F_1(z) - \Phi(z)| = O((\log n)^{-\frac{1}{2}})$. But F_1 is the same as Π_{k,c_n}. This proves (3.36). The proof of Lemma 3 is now complete.

Proof of Lemma 4. We need only prove that $P(R_n > \zeta_n(n \log n)^{\frac{1}{2}} = o(n^{-\frac{1}{2}c^2}(\log n)^{-\frac{1}{2}})$ as the proof of the other quality in (3.6) is analogous. To this end, first define,

$$\begin{aligned}\eta_{i0} &= \eta_i\,, &&\text{if } |\eta_i| \leqq n^{\frac{1}{2}}/(\log n)^{\frac{1}{2}+\nu} \\ &= 0\,, &&\text{otherwise.}\end{aligned} \tag{3.43}$$

Then, using the stationarity of the η_i's, and Lemma 2,

$$\begin{aligned}&|P(R_n > \zeta_n(n \log n)^{\frac{1}{2}}) - P(\textstyle\sum_1^k \eta_{i0} > \zeta_n(n \log n)^{\frac{1}{2}})| \\ &\quad\leqq kP(|\eta_1| > n^{\frac{1}{2}}/(\log n)^{\frac{1}{2}+\nu}) \leqq kE|\eta_1|^m n^{-m/2}(\log n)^{m(\frac{1}{2}+\nu)} \\ &\quad\leqq K_1k[qn^{\frac{1}{2}(m-c^2-2-\delta)}(\log n)^{K_2} + q^{\frac{1}{2}m}]n^{-m/2}(\log n)^{m(\frac{1}{2}+\nu)} \\ &\quad\leqq K_1[n^{-c^2/2-\delta/2}(\log n)^{K_2} + n^{1-\alpha-(m/2)(1-\beta)}](\log n)^{m(\frac{1}{2}+\nu)}\,.\end{aligned} \tag{3.44}$$

But

$$\begin{aligned}1 - \alpha - \frac{m}{2}(1 - \beta) &= \frac{c^2 + \delta}{m - 2} - \frac{m}{2}\,\frac{c^2 + 2\delta}{m - 2} \\ &= \frac{-(m - 2)c^2 - 2(m - 1)\delta}{2(m - 2)} = -\tfrac{1}{2}c^2 - \frac{m - 1}{m - 2}\delta\,.\end{aligned}$$

Hence, it follows from (3.44) that

$$\begin{aligned}&|P(R_n > \zeta_n(n \log n)^{1/2}) - P(\textstyle\sum_1^k \eta_{i0} > \zeta_n(n \log n)^{\frac{1}{2}})| \\ &\qquad\leqq K_1 n^{-\frac{1}{2}c^2-\gamma} \qquad \text{for some} \quad \gamma > 0\,.\end{aligned} \tag{3.45}$$

Thus, it suffices to show that

$$P\left(\textstyle\sum_1^k \eta_{i0} > \zeta_n (n \log n)^{\frac12}\right) = o\left(n^{-\frac12 c^2}(\log n)^{-\frac12}\right) .$$

Note first that

$$(3.46) \qquad P\left(\textstyle\sum_1^k \eta_{i0} > \zeta_n(n \log n)^{\frac12}\right) \leqq \inf_{\theta>0} \left[\exp\left(-\theta n^{-\frac12}(\log n)^{\frac32+\nu}\zeta_n(n\log n)^{\frac12}\right) \times E\left(\exp\left(\theta n^{-\frac12}(\log n)^{\frac32+\nu} \textstyle\sum_1^k \eta_{i0}\right)\right)\right] .$$

Using arguments similar to (3.33)—(3.35), it follows that

$$(3.47) \qquad \left|E\left(\exp\left(\theta n^{-\frac12}(\log n)^{\frac32+\nu} \textstyle\sum_1^k \eta_{i0}\right)\right) - h_n^k\right| \leqq 2k\phi(p)\exp(2\theta)[h_n + 2\theta\phi(p)]^{k-1} ,$$

where $h_n = E[\exp(\theta n^{-\frac12}(\log n)^{\frac32+\nu}\eta_{10})]$. By a Taylor expansion we have

$$(3.48) \qquad h_n \leqq 1 + \theta n^{-\frac12}(\log n)^{\frac32+\nu}|E\eta_{10}| + \tfrac12\theta^2 n^{-1}(\log n)^{3+2\nu}E(\eta_{10}^2)\exp(\theta) .$$

But, using Lemma 2,

$$\begin{aligned} |E(\eta_1 - \eta_{10})| &= |E\eta_1 I_{\{|\eta_1| > n^{\frac12}/(\log n)^{\frac32+\nu}\}}| \\ &\leqq \{n^{-\frac12}(\log n)^{\frac32+\nu}\}E(\eta_1^2) \leqq K_1 q n^{-\frac12}(\log n)^{\frac32+\nu} . \end{aligned}$$

Also,

$$|E\eta_1| = q|EX_1'| \leqq qE[|X_1|I_{\{|X_1|>n^{\frac12}\}}] \leqq q(n^{-\frac12})^{c^2+1+\delta}E|X_1|^{c^2+2+\delta} \leqq K_1 q n^{-\frac12(c^2+1+\delta)} .$$

Thus,

$$(3.49) \qquad |E\eta_{10}| \leqq K_1 q n^{-\frac12}(\log n)^{\frac32+\nu} .$$

Also, using Lemma 2, $E\eta_{10}^2 \leqq E\eta_1^2 \leqq K_1 q$. Thus one has

$$\begin{aligned} &(k-1)\log(h_n + 2\phi(p)\exp(\theta)) \\ &\quad\leqq (k-1)\log(1 + qn^{-1}(\log n)^{3+2\nu} + K_1 p^{-2}) \\ (3.50) \qquad &\quad\leqq k[qn^{-1}(\log n)^{3+2\nu} + K_1 n^{-2\alpha}] \leqq K_1[n^{\beta-\alpha}(\log n)^{3+2\nu} + n^{1-3\alpha}] \\ &\quad= K_1[n^{\beta-\alpha}(\log n)^{3+2\nu} + n^{-2+3(c^2+\delta)/(m-2)}] \\ &\quad\leqq K_1 n^{-\gamma} \qquad \text{for some} \quad \gamma > 0 , \end{aligned}$$

since $m > 3c^2 + 5\delta + 2$. One has now from (3.47), for large n,

$$(3.51) \qquad \left|E\left(\exp\left(\theta n^{-\frac12}(\log n)^{\frac32+\nu}\textstyle\sum_1^k \eta_{i0}\right)\right) - h_n^k\right| \leqq K_1 n^{1-3\alpha}\exp(K_1 n^{-\gamma}) \leqq K_1 n^{-\gamma}$$

for some $\gamma > 0$. Also, (3.50) leads to $k \log h_n \leqq K_1 n^{-\gamma}$ for some $\gamma > 0$. So, $h_n^k \leqq 1 + K_1 n^{-\gamma}$ for some $\gamma > 0$. Thus, from (3.51) and the above,

$$(3.52) \qquad \text{the RHS of} \quad (3.46) \leqq \inf_{\theta>0} n^{-\theta}(1 + K_1 n^{-\gamma}) < K_1 n^{-c^2/2-\gamma} \quad \text{for some} \quad \gamma > 0 ;$$

(3.46), and hence the lemma follows.

Proof of Lemma 5. Using $r \leqq p + q$, Markov's inequality and Lemma 2,

$$\begin{aligned} &P\{|T_n| > \zeta_n(n\log n)^{\frac12}\} \\ (3.53) \qquad &\quad\leqq \{n^{\frac12}(\log n)^{-\frac12-\nu}\}^{-m}E|T_n|^m \\ &\quad\leqq n^{-m/2}(\log n)^{m(\frac12+\nu)}K_1[(p+q)n^{\frac12(m-c^2-2-\delta)}(\log n)^{K_2} + (p+q)^{m/2}] . \end{aligned}$$

Since $(p+q)/n = O(n^{-1+\alpha}) = O(n^{-(c^2+\delta)/(m-2)})$, it follows from (3.53) that $P\{|T_n| > \zeta_n(n \log n)^{\frac{1}{2}}\} \leqq K_1 n^{-\frac{1}{2}c^2-\gamma}$ for some $\gamma > 0$ by a suitable choice of m (> 0). This completes the proof of the theorem.

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